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Examples of Concentrated Contact Interactions in Simple Bodies

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Abstract. The phenomenology of *concentrated contact interactions*, a rare but at times necessary occurrence to guarantee partwise equilibrium, is illustrated by means of examples taken from two-dimensional equilibrium problems where concentrated loads are applied to infinite bodies occupying either a half plane or the whole plane. Although the corresponding three-dimensional problems in linearly isotropic elasticity have been solved since long past, this phenomenology has remained latent so far.

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1. Introduction

In continuum mechanics, both a body and its environment and adjacent body parts are presumed to have *contact interactions*, which are thought of as accounting for the short-range forces between neighboring particles envisaged by discrete mechanics. In the classical case of *simple* continuous bodies, these interactions are modeled by a *stress vector* field: when evaluated at a point x of a common boundary surface oriented by the unit normal $\hat{\boldsymbol{n}}(x)$, such field is interpreted as delivering the force $\hat{\boldsymbol{c}}(x, \hat{\boldsymbol{n}}(x))$ per unit area exerted either by the environment over the body or by the part lying on the positive side of the boundary surface over the adjacent part. By virtue of Cauchy's tetrahedron theorem, given the contact force mapping $\hat{\boldsymbol{c}}(x, \cdot)$ at a typical body point x and three mutually orthogonal unit vectors $\boldsymbol{n}_i$, the construct

$$\widehat{\boldsymbol{S}}(x) = \sum_{i=1}^{3} \hat{\boldsymbol{c}}(x, \boldsymbol{n}_i) \otimes \boldsymbol{n}_i \tag{1.1}$$

defines the value at x of the *stress tensor* field; conversely, given the latter, its affine action over the sphere of unit vectors yields the stress vector on the oriented planes through x:

$$\hat{\boldsymbol{c}}(x, \boldsymbol{n}) = \widehat{\boldsymbol{S}}(x)\boldsymbol{n}. \tag{1.2}$$

Thus – and this is the main thrust of Cauchy's result – the information carried by the contact-force mapping $\hat{\boldsymbol{c}}$ and the stress-tensor mapping $\widehat{\boldsymbol{S}}$ are essentially equivalent.

This equivalence holds, though, for *regular* contact force and stress fields. It seems reasonable to ask what can be said about the matters in *singular* cases, if any. My concern here is to exhibit and discuss situations when singular contact forces and stress fields are indeed in order. Precisely, by inspection of certain problems of pure statics, which are two-dimensional counterparts of the problems in classical elasticity solved by Flamant, Boussinesq, Cerruti and Kelvin in the last decades of the XIX century, I demonstrate *per exempla* that *partwise equilibrium of a simple continuous body may require that adjacent body parts exchange concentrated contact forces*. Two concepts in this sentence, 'partwise equilibrium' and 'concentrated forces', both central to our present developments, need some discussion.

That whatever part of an equilibrated body must be in equilibrium as well would be hardly contended by anybody. The widespread and fruitful use of *free-body diagrams* in mechanics is based on this assumption, and on the accompanying presumption that a body part, when ideally isolated from the rest, would be in equilibrium if it were acted upon by external forces reproducing faithfully the forces, both external and internal, it directly experiences in reality. Usually, the subbodies whose equilibrium is characterized in this manner are imagined to have an everywhere smooth boundary. Not always so here, where consideration of sharp-cornered parts is at times necessary to exhibit the concentration effects of contact forces in which we are interested.

Concentrated forces, regarded as convenient idealizations of diffused loads applied to a small part of a body's boundary, are of common use in engineering mechanics. To quote from a popular textbook, "the free-body diagram is the most important single step in the solution of problems in mechanics" [21, p. 104]); "modeling the action of forces" "exerted *on* the body to be *isolated*, *by* the body to be *removed*" (*ibid.*, p. 105; italics as in the original text) is a mandatory, preliminary step; and those forces, especially but not exclusively in statics, are for most practical purposes modeled as concentrated.

Diffused contact loads are germane to contact interactions between adjacent body parts: in fact, when a contact-interaction mapping is specified for a body class, then it is customarily thought of as delivering both contact interactions, at *interior* points, and contact loads, at *boundary* points.⋆ Concentrated loads, applied at interior and boundary points, have been often considered in continuum mechan-

⋆ Interestingly, the concept of diffused contact interactions between internal adjacent body parts begun to condensate in Cauchy's mind on the basis of a similarity with standard examples of diffused contact loads exerted on a body by an environment of a different nature, such as the hydrostatic pressure of a fluid on an immersed solid [11]. Cauchy's model of internal contact interactions has been applied without changes to contact interactions of a body with its exterior, with the contact interaction mapping accounting for both. An implicit drawback of this practice is that no difference is made between geometrical surfaces obtained by ideal cutting and fabricated surfaces obtained by actual cutting [13]; moreover, the issue of boundary compatibility of a (body,environment) pair is completely overlooked [10, 9].

ics, and carefully modeled mathematically (see [16, Section 52], for the class of linearly elastic bodies). I see no reason why the germane notion of *concentrated contact interactions* should not be introduced. They are not ubiquitous; in fact, they are a rather rare necessity. Let us revert for a moment to engineering mechanics for guidance. A judicious practice there is to make sure that the free-body diagram features *all* possible forces applied to the isolated body; at times, we find out that balance and/or symmetry conditions require that some of those forces be null. Likewise, in continuum mechanics, we should contemplate concentrated contact interactions by default, because there are cases, no matter how few, when they turn out to be crucial to guarantee partwise equilibrium.

If concentrated contact interactions are considered, an interesting problem to tackle is, as pointed out in the beginning of this introduction, the conjectural equivalence in information of contact forces, regular and singular, and the accompanying, somewhere singular, stress field. Luckily, concentrated forces, be they idealizations of applied loads or of contact interactions, occur 'naturally' in weak formulations of continuum mechanics. In fact, in such formulations, concentrated loads are as 'natural' as edges and vertices in the domain where a boundary value problem is formulated. There is no need today to justify consideration of concentrated forces, as was done over a century ago, by thinking of them as limits of smooth distributions of volume or surface forces, just as there is no need to round off a domain's corners. In addition, weak formulations relieve us from dealing with a delicate issue arising when sequences of approximating problems are employed, namely, to investigate under what hypotheses an associated sequence of smooth solutions has a unique limit.

The phenomenology of concentrated contact interactions is here illustrated by means of examples taken from two-dimensional equilibrium problems where concentrated loads are applied to infinite bodies occupying either a half plane or the whole plane. This phenomenology has remained latent so far, so much so that the currently accepted mathematical models can only handle diffused contact interactions. Recent progresses in *geometric calculus* (a denomination coined in [17]) and related issues in continuum physics [22, 23] give grounds for hoping that a full-fledged theory of contact interactions, diffused and concentrated, will be available soon.

To minimize computations and facilitate visualization, all examples are worked out in two space dimensions; such choice of convenience entails no conceptual loss. We only deal with equilibrium problems in terms of stress; thus, while uniqueness is not to be expected in general, our conclusions apply to simple continuous bodies of whatever material constitution. Although the problems we consider where set originally in linear elasticity, the related symmetric-valued stress fields can be freely interpreted as divergenceless Piola stress fields over a reference shape chosen as to balance the external loads assigned at the boundary.

The contents of this paper can be summarized as follows. In Section 2, the statics of the two-dimensional *Flamant Problem* [14] is discussed, with a view toward interpreting the assignment of a concentrated force acting perpendicular to the Flamant half plane as a *weak boundary condition of traction* resulting from an appropriate *partwise balance of contact interactions*.⋆ In Section 3, it is shown that there are parts of the Flamant half plane, both finite and infinite, whose free-body diagram must feature a concentrated contact interaction to guarantee equilibrium. The rest of the paper is devoted to further substantiate this evidence: in Section 4, a class of balanced stress fields is characterized, which includes, in addition to Flamant, the stress fields in the two-dimensional counterparts of two other classical problems solved in the XIX century, *Cerruti*'s [12] and *Kelvin*'s [25]; in Section 5, related examples of concentrated contact interactions are given.

2. The Stress Field in the Flamant Problem

In 1892 [14], the French mechanist Alfred-Aimé Flamant (1839–1914) solved the equilibrium problem for a linearly elastic, isotropic body occupying a half space and being acted upon by a perpendicular *line load* of constant magnitude per unit length and infinitely long support. In 1878 [5], Joseph Valentin Boussinesq (1842–1929), another student of Saint-Venant's, had considered the case of a *concentrated load* perpendicular to a half space, a problem he was to return to repeatedly later on [6–8]. Both Boussinesq's and Flamant are relatively easy problems in three-dimensional elasticity, the first because of its inherent central symmetry, the second because it admits a plane-strain solution. The 'Flamant Problem' we here study can be regarded as a two-dimensional version of both, indeed, a partial version, because we only treat the related equilibrium stress field.

We let a concentrated force $\boldsymbol{f} = f\boldsymbol{e}_1$ be applied at point o of the half plane

$$\mathcal{H}^+ := \{x \mid (x-o)\cdot\boldsymbol{e}_3 = 0,\ (x-o)\cdot\boldsymbol{e}_1 \geqslant 0\} \tag{2.1}$$

(Figure 1; the triad of vectors $\boldsymbol{e}_i$, $i \in \{1,2,3\}$, is chosen to be left-handed and orthonormal). We denote by $\boldsymbol{r} := x - o$ the position vector of point $x \in \mathcal{H}^+$ with respect to o; moreover, we set $\rho := |\boldsymbol{r}|$, $\boldsymbol{e} := \rho^{-1}\boldsymbol{r}$, $\vartheta := \arcsin((\boldsymbol{e}_1 \times \boldsymbol{e})\cdot\boldsymbol{e}_3)$, whence, in particular, $\boldsymbol{e} = \hat{\boldsymbol{e}}(\vartheta) := \cos\vartheta\,\boldsymbol{e}_1 + \sin\vartheta\,\boldsymbol{e}_2$.

In terms of the curvilinear coordinates ρ and ϑ, the *Flamant stress field* in $\mathcal{H}^+$ has the form

$$\boldsymbol{S}_F = \widehat{\boldsymbol{S}}_F(\rho,\vartheta) = -\frac{2f}{\pi\rho}\cos\vartheta\,\hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta), \qquad \text{for } \rho \in (0,+\infty),\ \vartheta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right].^{\star\star} \tag{2.2}$$

⋆ The mechanical point of view that traction boundary conditions may be generally regarded as consequences of balancing distance and contact actions on boundary parts has been given a precise mathematical setting by Marzocchi and Musesti [20].

⋆⋆ Cf., e.g., [19, equation (8.3.35)]. Needless to say, the three-dimensional problems of the class to which Flamant belongs are expounded in a number of textbooks in linear elasticity, with variable

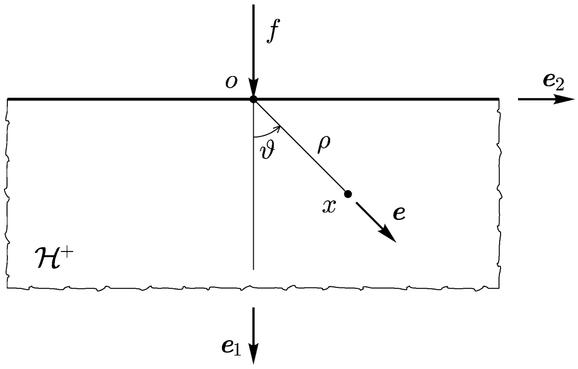

Figure 1. The Flamant half plane: coordinates, base vectors and applied load.

It is straightforward to show that

$$\operatorname{Div} \boldsymbol{S}_F = \boldsymbol{0} \quad \text{at every interior point of } \mathcal{H}^+.^\star \tag{2.3}$$

Thus, the Flamant stress satisfies the standard pointwise equation of force balance; we now consider in what sense it satisfies the accompanying traction boundary conditions, for the half space and for its parts.

For any fixed $\rho > 0$, we consider a half-disk $\mathcal{S}_\rho$ of radius ρ about o, whose oriented contour is the union of the half-circle

$$\mathcal{C}_\rho := \left\{ x \mid x - o = \rho \hat{\boldsymbol{e}}(\vartheta),\ \vartheta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2} \right] \right\}$$

and the segment

$$\mathcal{I}_\rho := \left\{ x \mid x - o = \sigma \boldsymbol{e}_2,\ \sigma \in [+\rho, -\rho] \right\}$$

(Figure 2). It follows from (2.2) that the stress vector field over $\partial \mathcal{S}_\rho$ is null at points of $\mathcal{I}_\rho \setminus \{o\}$, and is

$$\boldsymbol{S}_F \boldsymbol{e} = -\frac{2f}{\pi\rho} \cos\vartheta\, \boldsymbol{e} \tag{2.4}$$

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detail and accuracy; we here quote, in addition to Malvern's book [19], Barber's [3], Benvenuto's [4], Love's [18] and Sokolnikoff's [24].

* By definition,

$$\operatorname{Div} \boldsymbol{S} = \boldsymbol{S}_{,\rho}\, \boldsymbol{g}^\rho + \boldsymbol{S}_{,\vartheta}\, \boldsymbol{g}^\vartheta,$$

where $\boldsymbol{g}^\rho = \boldsymbol{e}$ and $\boldsymbol{g}^\vartheta = \rho^{-1}\boldsymbol{e}'$ are the contravariant base vectors, with $\boldsymbol{e}' = -\sin\vartheta\, \boldsymbol{e}_1 + \cos\vartheta\, \boldsymbol{e}_2$ the unit vector obtained by differentiating $\hat{\boldsymbol{e}}(\vartheta)$. Hence,

$$\boldsymbol{S}_{F,\rho}\, \boldsymbol{g}^\rho = \frac{2f}{\pi\rho^2} \cos\vartheta\, \boldsymbol{e} = -\boldsymbol{S}_{F,\vartheta}\, \boldsymbol{g}^\vartheta .$$

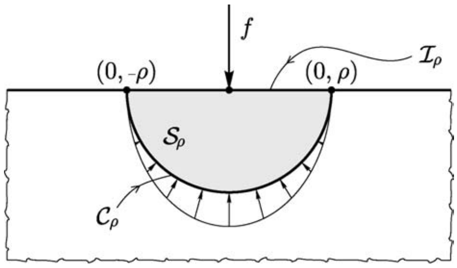

Figure 2. For a half-disk, diffused contact interactions are enough to balance the applied load.

at points of $\mathcal{C}_\rho$, so that

$$\int_{\mathcal{C}_\rho} \boldsymbol{S}_F \boldsymbol{e} = -\boldsymbol{f}. \tag{2.5}$$

In view of this result, we are led to interpret the force $\boldsymbol{f}$ concentrated at o as a *Dirac traction* $\boldsymbol{t}_\rho$ applied over the segment $\mathcal{I}_\rho$:

$$\boldsymbol{f} = \int_{\mathcal{I}_\rho} \boldsymbol{t}_\rho, \quad \boldsymbol{t}_\rho(x) := f\delta(x-o)\boldsymbol{e}_1,\ x \in \mathcal{I}_\rho. \tag{2.6}$$

With this interpretation, the relation

$$\int_{\mathcal{C}_\rho} \boldsymbol{S}_F \boldsymbol{e} + \int_{\mathcal{I}_\rho} \boldsymbol{t}_\rho = \boldsymbol{0} \tag{2.7}$$

expresses the *balance of contact interactions*, internal and external, applied to the part $\mathcal{S}_\rho$, whatever $\rho > 0$. Moreover, it follows from a formally stated divergence theorem and (2.3) that

$$\int_{\mathcal{S}_\rho} \operatorname{Div} \boldsymbol{S}_F = \int_{\mathcal{C}_\rho} \boldsymbol{S}_F \boldsymbol{e} + \int_{\mathcal{I}_\rho} \boldsymbol{S}_F \boldsymbol{e}_1 = \int_{\mathcal{I}_\rho} (-\boldsymbol{t}_\rho + \boldsymbol{S}_F \boldsymbol{e}_1) = \boldsymbol{0}. \tag{2.8}$$

We regard the last relation as an appropriate weak version of the *traction boundary condition* prevailing over the segment $\mathcal{I}_\rho$ [20].

REMARK. The last equality in (2.8) shows that, over the straight line bounding $\mathcal{H}^+$, the surface traction $\boldsymbol{S}_F \boldsymbol{n} = -\boldsymbol{S}_F \boldsymbol{e}_1$ is a *measure*, concentrated at the point o where the external *contact force* $\boldsymbol{f}$ is applied [23]. Suppose now that the Flamant stress field is continuously extended to null to the upper half-plane $\mathcal{H}^-$, and let $\widetilde{\boldsymbol{S}}_F$ denote such extended field over the plane $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. Interestingly, the field $\widetilde{\boldsymbol{S}}_F$ has *divergence measure* in $\mathcal{H}$, equal to $-\delta(x-o)\boldsymbol{f},\ x \in \mathcal{H}$.⋆ To interpret the latter result along the same lines we have interpreted the former, i.e., as a consequence of a force balance, we may consider a disk-shaped part $\mathcal{D}_\rho$ of $\mathcal{H}$, of center o and radius ρ, and imagine it as subject to an external *distance*

⋆ A. Musesti, private communication, June 2004.

force $\boldsymbol{f}$ applied at o, balanced by diffused tractions being identically null over $\partial\mathcal{D}_\rho \cap \mathcal{H}^-$ and equal to $\boldsymbol{S}_F\boldsymbol{e}$ over $\partial\mathcal{D}_\rho \cap \mathcal{H}^+$. Continuum mechanics provides us with a unifying format for balance statements that allows for a further, precise interpretation of these two analytical findings:

A pair $((\boldsymbol{c}, \boldsymbol{d}), \boldsymbol{S})$, formed by contact and distance force fields $\boldsymbol{c}$ and $\boldsymbol{d}$ and a stress field $\boldsymbol{S}$ over a region Ω with boundary $\partial\Omega$, is *weakly balanced* whenever the stress working equals the distance working plus the contact working, i.e., whenever

$$\int_\Omega \boldsymbol{S} \cdot \nabla \boldsymbol{v} = \int_\Omega \boldsymbol{d} \cdot \boldsymbol{v} + \int_{\partial\Omega} \boldsymbol{c} \cdot \boldsymbol{v}, \quad \text{for all smooth test fields } \boldsymbol{v}; \tag{2.9}$$

moreover, in view of a standard differential identity,

$$\int_\Omega \boldsymbol{S} \cdot \nabla \boldsymbol{v} = \int_\Omega (-\mathrm{Div}\, \boldsymbol{S}) \cdot \boldsymbol{v} + \int_{\partial\Omega} (\boldsymbol{S}\boldsymbol{n}) \cdot \boldsymbol{v}, \tag{2.10}$$

so that we can regard $-\mathrm{Div}\, \boldsymbol{S}$ as the distance force, and $\boldsymbol{S}\boldsymbol{n}$ as the contact force, associated to a given stress field $\boldsymbol{S}$. If we apply this balance format to the parts $\mathcal{S}_\rho$ of $\mathcal{H}^+$ and $\mathcal{D}_\rho$ of $\mathcal{H}$, we find that

$$\int_{\mathcal{S}_\rho} \boldsymbol{S}_F \cdot \nabla \boldsymbol{v} - \int_{\mathcal{C}_\rho} \boldsymbol{S}_F\boldsymbol{e} \cdot \boldsymbol{v} = \boldsymbol{f} \cdot \boldsymbol{v}(o) = \int_{\mathcal{D}_\rho} \widetilde{\boldsymbol{S}}_F \cdot \nabla \boldsymbol{v} - \int_{\mathcal{C}_\rho} \boldsymbol{S}_F\boldsymbol{e} \cdot \boldsymbol{v}; \tag{2.11}$$

in plain words, that the working of the external force applied at o equals the difference between the stress working and the contact working over $\mathcal{C}_\rho$. In the case of $\mathcal{S}_\rho$, we may write

$$\boldsymbol{f} \cdot \boldsymbol{v}(o) = \int_{\mathcal{I}_\rho} \boldsymbol{S}_F\boldsymbol{n} \cdot \boldsymbol{v} = \text{the contact working over } \mathcal{I}_\rho; \tag{2.12}$$

in the case of $\mathcal{D}_\rho$,

$$\boldsymbol{f} \cdot \boldsymbol{v}(o) = \int_{\mathcal{D}_\rho} (-\mathrm{Div}\, \widetilde{\boldsymbol{S}}_F) \cdot \boldsymbol{v} = \text{the distance working over } \mathcal{D}_\rho. \tag{2.13}$$

3. Concentrated Contact Interactions in the Flamant Problem

That concentrated contact interactions are in order between adjacent parts of the Flamant half plane is easily demonstrated by the use of free-body diagrams.

3.1. EXAMPLE 1

Consider the quarter-disk $\mathcal{P}_\rho$ sketched in Figure 3. When part $\mathcal{P}_\rho$ is ideally cut away from the rest of $\mathcal{H}^+$, then it must be in equilibrium under the action of: (i) the concentrated force $(1/2)f\boldsymbol{e}_1$; (ii) the diffused contact force $\boldsymbol{S}_F\boldsymbol{e}$ exerted by

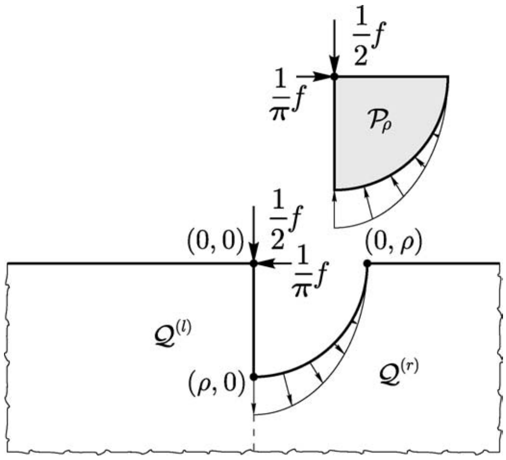

Figure 3. For a quarter-disk, a concentrated contact interaction is needed to balance the applied load.

the right adjacent part $\mathcal{Q}^{(r)}$ (the same as the internal contact interaction between the two parts before the cut!), which, in view of (2.4), is equipollent to the concentrated force

$$\int_{(1/2)\mathcal{C}_\rho} \boldsymbol{S}_F\boldsymbol{e} = -\frac{1}{2} f\boldsymbol{e}_1 - \frac{1}{\pi} f\boldsymbol{e}_2, \tag{3.1}$$

applied at o; (iii) the contact action exerted by the left adjacent part $\mathcal{Q}^{(l)}$. Now, the diffused contact force $-\boldsymbol{S}_F\boldsymbol{e}_2$ exerted by $\mathcal{Q}^{(l)}$ on $\mathcal{P}_\rho$ is everywhere null along their common boundary:

$$\widehat{\boldsymbol{S}}_F(\sigma, 0)\boldsymbol{e}_2 \equiv 0 \quad \text{for } \sigma \in [0, \rho], \tag{3.2}$$

just as their internal contact interaction is before the cut. Then, to guarantee the free-body equilibrium of $\mathcal{P}_\rho$, we are driven to admit that the cut operation brings into evidence an internal *concentrated contact interaction*

$$\hat{\boldsymbol{f}}(\mathcal{P}_\rho, \mathcal{Q}^{(l)}) = -\hat{\boldsymbol{f}}(\mathcal{Q}^{(l)}, \mathcal{P}_\rho) = \frac{1}{\pi} f\boldsymbol{e}_2 \tag{3.3}$$

at point o.*

REMARKS. 1. This result is neither dependent on the parameter ρ nor on whichever curve from point $(\rho, 0)$ to point $(0, \rho)$ we pick to bound part $\mathcal{P}_\rho$. Indeed, if $\boldsymbol{r} = r(\vartheta)\boldsymbol{e}(\vartheta)$ is the position vector of a typical point x on such a curve, denoted by $\mathcal{C}$ in Figure 4, then the vectors $\boldsymbol{r}' = r'\boldsymbol{e} + r\boldsymbol{e}'$ and $\boldsymbol{n} = |\boldsymbol{r}'|^{-1}\boldsymbol{r}' \times \boldsymbol{e}_3$ are, respectively, tangent and normal to $\mathcal{C}$; in particular, then, $\boldsymbol{e} \cdot \boldsymbol{n} = |\boldsymbol{r}'|^{-1}r$. The total

* Here $\hat{\boldsymbol{f}}(\mathcal{A}, \mathcal{B})$ denotes the total contact force exerted by part $\mathcal{B}$ over part $\mathcal{A}$ over their common boundary.

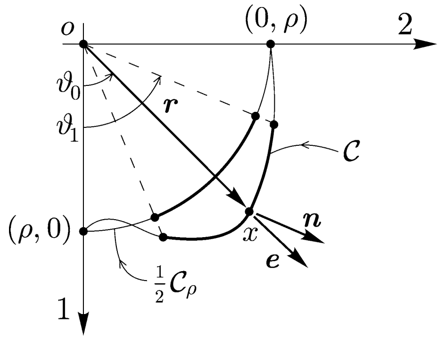

Figure 4. The concentrated contact interaction is the same whatever the curve $\mathcal{C}$ joining $(\rho, 0)$ to $(0, \rho)$.

contact action exerted along any chosen portion of the curve $\mathcal{C}$ is then the same as the contact action exerted on the corresponding portion of $\frac{1}{2}\mathcal{C}_\rho$:

$$\int_{[\mathcal{C}]_{\vartheta_0}^{\vartheta_1}} \boldsymbol{S}_F\boldsymbol{n} = \int_{\vartheta_0}^{\vartheta_1} (\boldsymbol{e}\cdot\boldsymbol{n})\boldsymbol{S}_F\boldsymbol{e}|\boldsymbol{r}'|\,\mathrm{d}\vartheta = \int_{\vartheta_0}^{\vartheta_1} \boldsymbol{S}_F\boldsymbol{e}\,r\,\mathrm{d}\vartheta = \int_{[(1/2)\mathcal{C}_\rho]_{\vartheta_0}^{\vartheta_1}} \boldsymbol{S}_F\boldsymbol{e}.$$

Thus, the concentrated contact force arising at point o is a *local effect*, in the sense that it is a manifestation (in this case, the only manifestation) of the interaction between any two adjacent body parts sharing the segment $\{x \mid x - o = \sigma\boldsymbol{e}_1,\ \sigma \in [+\rho, -\rho]\}$ as a common boundary, whatever $\rho > 0$. Note also that this effect concentrates at a point that belongs to the *topological boundary* of part $\mathcal{P}_\rho$, but not to its *reduced boundary*.⋆

2. Assuming that the vertical external force on part $\mathcal{P}_\rho$ be $(1/2)\boldsymbol{f}$ may seem arbitrary and ponderous, but in fact it is not. To see this, imagine to ideally cut the half-disk of Figure 2 into two identical quarter-disks, with a view toward sketching a free-body diagram for each of the latter: symmetry then requires that the concentrated external force is split equal.

3. That the concentrated interaction forces at the vertex of the right angle, both in part $\mathcal{P}_\rho$ and in its complement, should be those shown in Figure 3 can be seen also by a limit argument suggested by Roger Fosdick.⋆⋆ With reference to Figure 5, we let

$$\boldsymbol{g}(\bar{\rho}, \bar{\vartheta}) = -\int_{[\mathcal{C}_\rho]_{-\pi/2}^{\bar{\vartheta}}} \boldsymbol{S}_F\boldsymbol{e} \tag{3.4}$$

⋆ Roughly speaking, the reduced boundary of a set – a measure-theoretic notion carefully introduced in [2, p. 154] – is the subset of all points of the topological boundary where a (inner) normal is well defined.

⋆⋆ Private communication, August 2004.

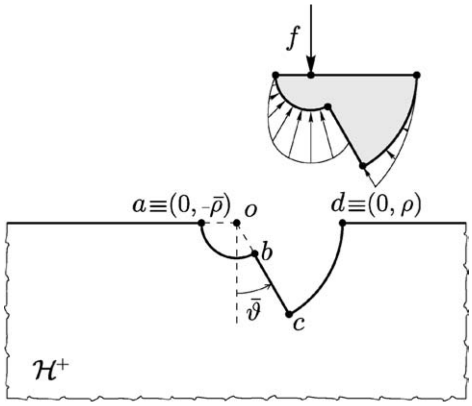

Figure 5. Free-body diagram for a part of the Flamant half-plane in the form of a "nosed piece of pie".

denote the total force equipollent with respect to point o to the diffused interaction force exerted by the "nosed" part on the right over its complement along the common boundary curve through points a, b, and c; and we compute

$$\boldsymbol{g}(\bar\rho, \bar\vartheta) = \frac{1}{2} f\left(1 + \frac{2}{\pi}\left(\bar\vartheta + \sin\bar\vartheta\cos\bar\vartheta\right)\right)\boldsymbol{e}_1 - \frac{1}{\pi} f\left(1 - \sin^2\bar\vartheta\right)\boldsymbol{e}_2. \tag{3.5}$$

Not surprisingly, in the light of Remark 1, this force is independent of the parameter $\bar\rho$, and reduces to the expected vector for $\bar\vartheta = 0$.

3.2. EXAMPLE 2

The magnitude of the Flamant stress tensor is constant at those points of $\mathcal{H}^+$ where

$$\rho^{-1}\cos\vartheta = (2c)^{-1} = \text{a positive constant;} \tag{3.6}$$

it is not difficult to see that those points lie on the circumference $\mathcal{C}_c$ of a circle of center $(c, 0)$ and radius c (Figure 6). At a point of $\mathcal{C}_c$, the outer unit normal is

$$\boldsymbol{n} = (1 + \tan^2\vartheta)^{-1/2}(\boldsymbol{e} + \tan\vartheta\,\boldsymbol{e}'); \tag{3.7}$$

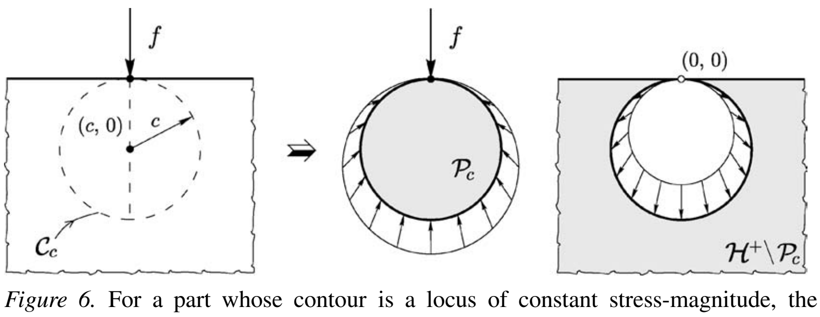

Figure 6. For a part whose contour is a locus of constant stress-magnitude, the contact interactions are exclusively diffused.

hence, the stress vector turns out to be

$$\boldsymbol{S}_F\boldsymbol{n} = -\frac{f}{\pi c}\frac{1}{(1+\tan^2\vartheta)^{1/2}}\boldsymbol{e}. \tag{3.8}$$

We think of the part $\mathcal{P}_c$ of $\mathcal{H}^+$ bounded by $\mathcal{C}_c$ as being balanced under the combined action of such diffused force and the concentrated load $\boldsymbol{f}$; consistently, we think of the complementary part $\mathcal{H}^+ \setminus \mathcal{P}_c$ as subject to diffused contact interactions only.

3.3. EXAMPLE 3

For another example of concentrated contact interaction, consider the stress field in the whole plane which results from 'suturing' two mirror-symmetric Flamant stress fields (Figure 7). The stress is then identically null along the suture line, and there are no concentrated loads. In particular, the disk $\mathcal{D}_\rho$, if isolated from the rest by means of an ideal cut, would be in equilibrium under a distribution of Flamant boundary tractions of type (2.4), as sketched in the figure. However, were the two half-disks composing $\mathcal{D}_\rho$ separated by a further ideal cut along the suture line, then their individual equilibrium would require concentrated contact interactions, in fact, the mirror-symmetric concentrated forces of the two Flamant problems we begun our construction with.* Interestingly, at variance with the previous example, the concentrated interaction now arises at an interior point, not at an end point, of the common boundary with the adjacent part applying it to the part of interest.

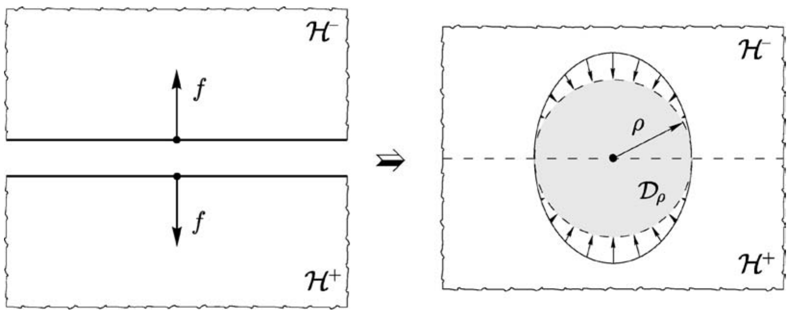

Figure 7. Contact interactions on a centered circular part of two sutured mirror-symmetric Flamant half-planes.

* More generally, the only contact force on the half plane

$$\{x \mid (x-o)\cdot\boldsymbol{h}(\vartheta_0) \leqslant 0, \boldsymbol{h}(\vartheta_0) = -\sin\vartheta_0\boldsymbol{e}_1 + \cos\vartheta_0\boldsymbol{e}_2\}$$

is the force

$$\frac{2f}{\pi}((\vartheta_0 + \sin\vartheta_0\cos\vartheta_0)\boldsymbol{e}_1 - (1-\sin^2\vartheta_0)\boldsymbol{e}_2),$$

concentrated at the origin.

3.4. DISCUSSION

The free-body diagram shown in Figure 3 can be thought of as obtained by combining two ideal cuts, the one rectilinear through points $(0, 0)$ and $(\rho, 0)$ of the Flamant half plane, the other semi-circular through points $(\rho, 0)$ and $(0, \rho)$. Example 1 suggests that, if an ideal cut passes through a point such as $(0, 0)$ where a concentrated force is applied to a body by its environment, then a contact interaction concentrated at that same point should be included in the free-body diagram, the most general concentrated interaction compatible with partwise equilibrium and symmetry conditions, if any. A need for concentrated interactions is also indicated by Example 3, whenever an ideal cut is drawn through a *focal point* for the given stress field.

Example 2 seems to contradict the expectation raised by Example 1, since the equilibrium of the part $\mathcal{P}_c$ isolated by the single circular cut of radius c centered at $(0, c)$ requires only diffused contact interactions (but a further vertical cut through $(0, 0)$ would entail horizontal concentrated interactions at that point). A difference between the two cases is that, in Example 2, the point of application of the concentrated force, when regarded as a point of the boundary of part $\mathcal{H}^+ \setminus \mathcal{P}_c$, does not possess the *cone property*.*

We shall have more examples to discuss after we review, in the next section, the two-dimensional static versions of Cerruti's and Kelvin's problems.

4. Plane Stress Fields, Balanced and Separable

A plane stress field

$$\begin{aligned}\widehat{\boldsymbol{S}}(\rho, \vartheta) = {}& \hat{\alpha}(\rho, \vartheta)\hat{\boldsymbol{e}}(\vartheta) \otimes \hat{\boldsymbol{e}}(\vartheta) + \hat{\beta}(\rho, \vartheta)\hat{\boldsymbol{e}}'(\vartheta) \otimes \hat{\boldsymbol{e}}'(\vartheta) \\ & + \hat{\gamma}(\rho, \vartheta)(\hat{\boldsymbol{e}}(\vartheta) \otimes \hat{\boldsymbol{e}}'(\vartheta) + \hat{\boldsymbol{e}}'(\vartheta) \otimes \hat{\boldsymbol{e}}(\vartheta))\end{aligned} \tag{4.1}$$

is termed *balanced* (for null distance loads) whenever it is divergenceless, i.e., granted smoothness, whenever the scalar fields entering the representation (4.1) satisfy the following system of PDE's:

$$\begin{aligned}\alpha,_\rho + \rho^{-1}(\alpha - \beta + \gamma,_\vartheta) &= 0, \\ \gamma,_\rho + \rho^{-1}(\beta,_\vartheta + 2\gamma) &= 0.\end{aligned} \tag{4.2}$$

All the two-dimensional, purely static counterparts of the 'named problems' in classical elasticity we here study fall within the following fully *separable* class:

$$\begin{aligned}&\hat{\alpha}(\rho, \vartheta) = \alpha_0\rho^{-1}\hat{a}(\vartheta), \qquad \hat{\beta}(\rho, \vartheta) = \beta_0\rho^{-1}\hat{b}(\vartheta), \\ &\hat{\gamma}(\rho, \vartheta) = \gamma_0\rho^{-1}\hat{c}(\vartheta),\end{aligned} \tag{4.3}$$

* Various types of cone conditions, detailed, e.g., in [1] and [15], find an use in the regularization at the boundary of elliptic equations and systems. Roughly speaking, a set Ω possesses the (interior) cone property whenever at each point of its boundary there is a finite right cone K with vertex in x which is contained in the closure of Ω.

for which system (4.2) reduces to

$$\begin{aligned} \beta_0 b - \gamma_0 c' &= 0, \\ \beta_0 b' + \gamma_0 c &= 0. \end{aligned} \tag{4.4}$$

There are three basic types of solutions to this system, namely,

$$\beta_0 = \gamma_0 = 0; \tag{4.5}$$
$$\beta_0 \hat{b}(\vartheta) = \gamma_0 \cos\vartheta, \quad \hat{c}(\vartheta) = \sin\vartheta; \tag{4.6}$$
$$\beta_0 \hat{b}(\vartheta) = -\gamma_0 \sin\vartheta, \quad \hat{c}(\vartheta) = \cos\vartheta. \tag{4.7}$$

If we pick (4.5), the Flamant stress field (2.2) follows from (4.1) and (4.3), provided we choose

$$\alpha_0\, \hat{a}(\vartheta) = -\frac{2f}{\pi}\cos\vartheta. \tag{4.8}$$

The two other named problems important to our present developments are considered in the next subsections.

4.1. THE STRESS FIELD IN THE CERRUTI PROBLEM

The two-dimensional version of the equilibrium problem solved by the Italian mechanist Valentino Cerruti (1850–1909), in a paper which appeared ten years before Flamant's [12], concerns a half-plane acted upon at point o by a *tangent* force $\boldsymbol{f} = f\boldsymbol{e}_2$. The related balanced stress field of type (4.1), (4.3) again follows from (4.5), together with

$$\alpha_0\, \hat{a}(\vartheta) = -\frac{2f}{\pi}\sin\vartheta \tag{4.9}$$

in the place of (4.8); it has the form

$$\widehat{\boldsymbol{S}}_C(\rho,\vartheta) = -\frac{2f}{\pi\rho}\sin\vartheta\, \hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta), \quad \text{for } \rho\in(0,+\infty),\ \vartheta\in\left[-\frac{\pi}{2},+\frac{\pi}{2}\right] \tag{4.10}$$

(Figure 8).

REMARKS. 1. Let the *wedge*

$$\mathcal{W}_{\vartheta_0} := \left\{x \mid x - o = \rho\hat{\boldsymbol{e}}(\vartheta),\ \hat{\boldsymbol{e}}(\vartheta)\cdot\boldsymbol{e}_3 = 0,\ |\hat{\boldsymbol{e}}(\vartheta)\cdot\boldsymbol{e}_1| \leqslant |\cos\vartheta_0|\right\}$$

be acted upon at its vertex by a concentrated force (Figure 9). Due to its focal symmetry, the Flamant stress field solves the equilibrium problem on the left, whatever the vertex angle $2\vartheta_0 \in (0, 2\pi)$; likewise, the Cerruti stress field solves the right

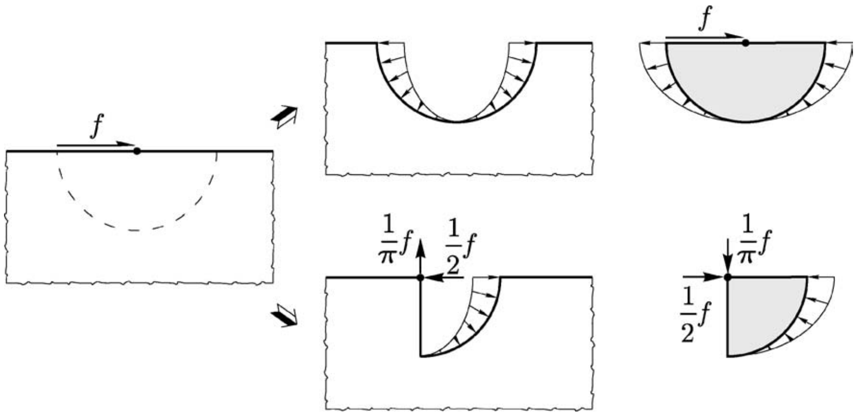

Figure 8. Contact interactions on half- and quarter-disk parts of the Cerruti half-plane.

equilibrium problem. The stress fields in the wedge turn out to be, respectively,

$$\widehat{\boldsymbol{S}}_{WF}(\rho, \vartheta) = -\frac{f}{\vartheta_0\, \rho} \cos\vartheta\, \hat{\boldsymbol{e}}(\vartheta) \otimes \hat{\boldsymbol{e}}(\vartheta) \tag{4.11}$$

and

$$\widehat{\boldsymbol{S}}_{WC}(\rho, \vartheta) = -\frac{f}{\vartheta_0\, \rho} \sin\vartheta\, \hat{\boldsymbol{e}}(\vartheta) \otimes \hat{\boldsymbol{e}}(\vartheta), \tag{4.12}$$

for $\rho \in (0, +\infty),\ \vartheta \in [-\vartheta_0, +\vartheta_0]$.

2. A linear combination of the Flamant and Cerruti stress fields yields a stress field balancing an *oblique* applied force, both for a half plane and for a wedge.

4.2. THE STRESS FIELD IN THE KELVIN PROBLEM

Lord Kelvin (William Thompson, 1824–1907) solved the problem that was later named after him in 1848 [25]. The problem consists of finding the equilibrium state of an elastic space subject to a concentrated load. Here, we are only concerned with the stress field(s) encountered in the two-dimensional version of this problem.

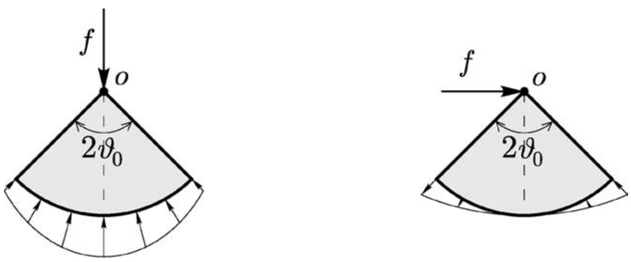

Figure 9. Flamant's and Cerruti's wedges.

Without any loss of generality, we suppose that a load $\boldsymbol{f} = f\boldsymbol{e}_1$ is applied at point o. Then, it is not difficult to see that each stress field of the one-parameter family

$$\begin{aligned}\boldsymbol{S}_K = \widehat{\boldsymbol{S}}_K(\rho, \vartheta; \boldsymbol{e}_1) &= \frac{\alpha_0}{\rho}\cos\vartheta\,\hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta) + \frac{\gamma_0}{\rho}\cos\vartheta\,\hat{\boldsymbol{e}}'(\vartheta)\otimes\hat{\boldsymbol{e}}'(\vartheta)\\ &\quad + \frac{\gamma_0}{\rho}\sin\vartheta\,(\hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}'(\vartheta) + \hat{\boldsymbol{e}}'(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta)), \quad \alpha_0 - \gamma_0 = -\frac{f}{\pi}, \end{aligned} \tag{4.13}$$

is balanced, and solves the Kelvin problem in the plane $\mathcal{H}$ in the sense that

$$\int_{\partial\mathcal{D}_\rho} \boldsymbol{S}_K\boldsymbol{e} = -f\boldsymbol{e}_1, \tag{4.14}$$

where

$$\boldsymbol{S}_K\boldsymbol{e} = \widehat{\boldsymbol{S}}_K(\rho, \vartheta; \boldsymbol{e}_1)\boldsymbol{e}(\vartheta) = \frac{\alpha_0}{\rho}\cos\vartheta\,\hat{\boldsymbol{e}}(\vartheta) + \frac{\gamma_0}{\rho}\sin\vartheta\,\hat{\boldsymbol{e}}'(\vartheta), \tag{4.15}$$

and where the integration path is the contour $\partial\mathcal{D}_\rho$ of the disk $\mathcal{D}_\rho$ of center o and radius ρ (cf., e.g., [24, Section 78]).

REMARKS. 1. With the use of the second of (4.13), it is not difficult to transform (4.15) into

$$\widehat{\boldsymbol{S}}_K(\rho, \vartheta; \boldsymbol{e}_1)\boldsymbol{e}(\vartheta) = \frac{\alpha_0}{\rho}\hat{\boldsymbol{e}}(2\vartheta) + \frac{f}{\pi\rho}\sin\vartheta\,\hat{\boldsymbol{e}}'(\vartheta), \tag{4.16}$$

which allows for an easier visualization of the stress vector at any point of $\partial\mathcal{D}_\rho$; note that the first addendum does not contribute to the integral in (4.14).

2. The stress fields (4.13) are derived from (4.1) and (4.3) when the scalar fields $\hat{\beta}$ and $\hat{\gamma}$ are chosen as in (4.6), with $\beta_0 = \gamma_0$. The choice specified by (4.7) leads instead to the Kelvin stress fields for the load $\boldsymbol{f} = f\boldsymbol{e}_2$, namely,

$$\begin{aligned}&\widehat{\boldsymbol{S}}_K(\rho, \vartheta; \boldsymbol{e}_2)\\ &\quad = \frac{\alpha_0}{\rho}\sin\vartheta\,\hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta) - \frac{\gamma_0}{\rho}\sin\vartheta\,\hat{\boldsymbol{e}}'(\vartheta)\otimes\hat{\boldsymbol{e}}'(\vartheta)\\ &\qquad + \frac{\gamma_0}{\rho}\cos\vartheta\,(\hat{\boldsymbol{e}}(\vartheta)\otimes\hat{\boldsymbol{e}}'(\vartheta) + \hat{\boldsymbol{e}}'(\vartheta)\otimes\hat{\boldsymbol{e}}(\vartheta)), \quad \alpha_0 + \gamma_0 = -\frac{f}{\pi}. \end{aligned} \tag{4.17}$$

5. Other Examples of Concentrated Contact Interactions

5.1. EXAMPLE 4

It is not difficult to construct a balanced and separable stress field, different from Kelvin's, induced in the whole plane by a force concentrated at the origin: one 'sutures' two antimirror-symmetric Flamant stress fields or, alternatively, two mirror-

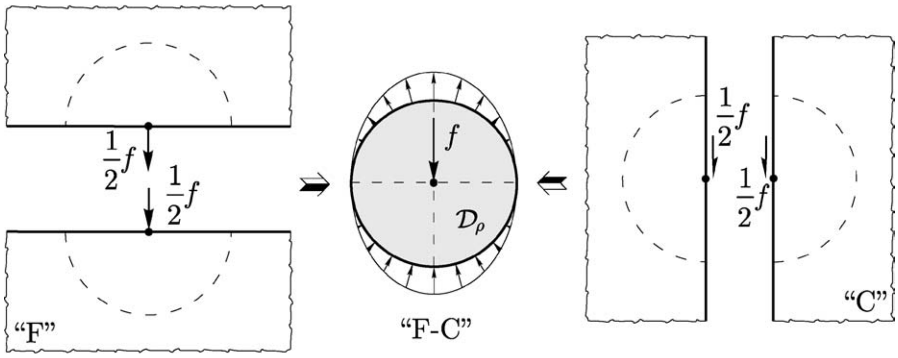

Figure 10. Contact interactions on a centered circular part of either two sutured antimirror-symmetric Flamant half-planes or two sutured mirror-symmetric Cerruti's half-planes.

symmetric Cerruti stress fields (Figure 10; note that, in both cases, the stress field is identically null along the suture line, and continuous through it).

Each of the two different traction fields, Kelvin's and Flamant–Cerruti's, over the contour of a disk $\mathcal{D}_\rho$ centered at the origin o is equipollent to the concentrated force $\boldsymbol{f} = f\boldsymbol{e}_1$ applied at o. However, a diametral Euler–Cauchy cut splitting $\mathcal{D}_\rho$ in two halves gives rise to different contact interactions (see Figure 11, where only diametral contact interactions are shown): whatever the diameter chosen for the cut, the contact interaction consists of the concentrated force $(1/2)\boldsymbol{f}$ in the Flamant–Cerruti case, of that concentrated force *plus* a diffused force $\boldsymbol{g}(\sigma) = -\gamma_0\sigma^{-1}\boldsymbol{e}_2$, $\sigma \in [-\rho, +\rho]$, in the Kelvin case. What makes for the interest of the latter is precisely the simultaneous occurrence of both concentrated and diffused contact interactions.

5.2. EXAMPLE 5

Suppose the half plane $\mathcal{H}^+$ in the Flamant problem is ideally split in a centered wedge $\mathcal{W}_{\vartheta_1}$, with $\vartheta_1 = \pi/2 - 2\vartheta_0$, and two complementing mirror-symmetric wedges of opening $2\vartheta_0$. One can think of the latter two wedges as acted upon by centered forces $\boldsymbol{g}^{(l)} = g(\sin\vartheta_0\boldsymbol{e}_1 - \cos\vartheta_0\boldsymbol{e}_2)$ and $\boldsymbol{g}^{(r)} = g(\sin\vartheta_0\boldsymbol{e}_1 + \cos\vartheta_0\boldsymbol{e}_2)$,

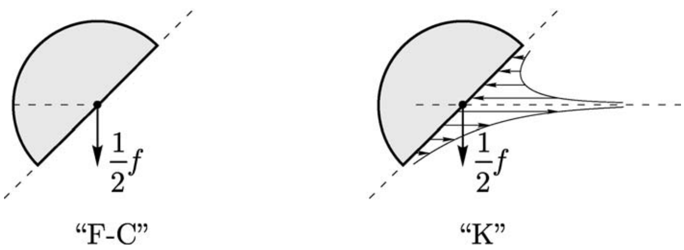

Figure 11. Diametral contact interactions for a centered circular part of a Flamant–Cerruti and a Kelvin plane.

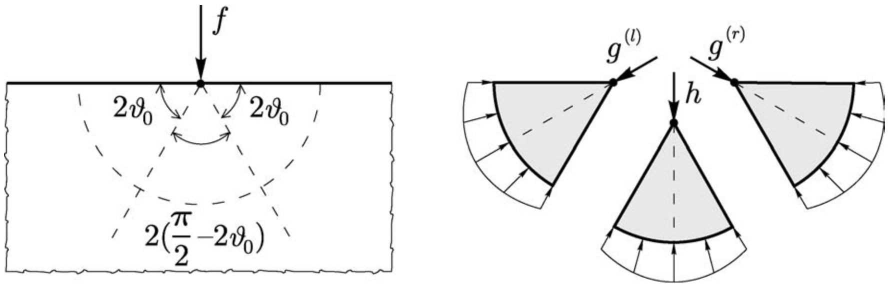

Figure 12. Contact interactions for a wedgewise symmetric decomposition of a half-circle part of the Flamant half-plane.

respectively; and of $\mathcal{W}_{\vartheta_1}$ as acted upon by a vertical force $\boldsymbol{h} = h\boldsymbol{e}_1$ (Figure 12). These forces are to be chosen equipollent to the Flamant force $\boldsymbol{f} = f\boldsymbol{e}_1$, that is to say, such that

$$2\sin\vartheta_0\, g + h = f, \tag{5.1}$$

and, in addition, such as to guarantee continuity of the stress field for $\vartheta = \pm(\pi/2 - 2\vartheta_0)$, which is tantamount as requiring that

$$\left(\frac{\pi}{2} - 2\vartheta_0\right) g - 2\vartheta_0 \sin\vartheta_0 h = 0. \tag{5.2}$$

Note that adjacent wedges do not exchange contact interactions, whatever the opening angle ϑ_0. In particular, for $\vartheta_0 = \pi/4$, we have that $g = (1/\sqrt{2})\, f$ and $h = 0$: see Figure 13, where the forces applied to the quarter-disk part $\mathcal{P}_\rho$ in the present case are compared with the forces on that part in the Flamant case.

6. Scholion: Euler–Cauchy Cuts and Free-Body Diagrams

The continuum mechanical roots of the engineering practice with free-body diagrams (*stress* or) *cut principle* of Euler and Cauchy. At variance with free-body diagrams, where both diffused and concentrated contact interactions have citizenship, this principle only concerns the mathematical description of diffused contact

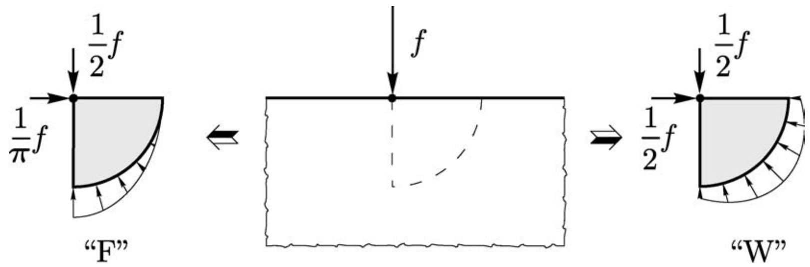

Figure 13. The concentrated interactions on a quarter-disk part are different for the Flamant and the wedgewise stress fields.

interactions: concentrated contact interactions are simply not contemplated. Here is how the principle is enunciated in two wellknown textbooks.

> "The *stress principle* specifies the nature of contact loads. It asserts that upon any imagined closed diaphragm $\mathcal{S}$ within a body there exists a field of *stress vectors* $\boldsymbol{t}_{(\boldsymbol{n})}$, where $\boldsymbol{n}$ is the unit normal; this field is equipollent to the loads exerted by the material outside upon the material inside, or, in case $\mathcal{S}$ is a boundary, to the applied loads." (From [27, Section 200]. The stress principle of Euler and Cauchy.)

> "According to $(3)_2$ the *resultant contact force* $\mathbf{f}_{\mathrm{C}}^{\mathrm{r}}$ is an absolutely continuous function of the area of the bounding surface $\partial\boldsymbol{\chi}(\mathcal{P})$ on which it acts.* The surface density $\mathbf{t}_{\partial\boldsymbol{\chi}(\mathcal{P})}$ is called the *traction field* on $\partial\boldsymbol{\chi}(\mathcal{P})$. If that field be known, the resultant contact force is determined and is independent of whatever may be occurring at places not laying upon $\partial\boldsymbol{\chi}(\mathcal{P})$. In this sense, the traction field is equipollent to the action upon $\mathcal{P}$ of the bodies outside $\mathcal{P}$ and adjacent to it. The assumption that the contact force is of this kind is the **cut principle** of Euler and Cauchy: *Within the shape of a body at any given time, conceive a smooth, closed diaphragm; then the action of the part of the body outside that diaphragm and adjacent to the part inside is equipollent to that of a field of vectors defined on the diaphragm.*" (From [26, p. 154]; italic and boldface types as in the original text.)

Apart for differences in wording and coverage,** if attention is restricted to interior parts of a material body, the two quotations are in remarkable agreement. I find them both vague about the concept, in what they say and, more important, in what they do not.

As to the first quotation, in the case when (a portion or, for what it matters, all of) an "imagined closed diaphragm $\mathcal{S}$" "is a boundary", why should we assume that there is upon $\mathcal{S}$ a "field equipollent to the loads exerted by the material outside upon the material inside" instead of a field coinciding *pointwise* with the given applied load? A remark – not a criticism – prompted by the second quotation is that in it the resultant force on a body part and its density with respect to the area measure are notions with a different statute: the former is regarded as a primitive element of the mechanics of continuous bodies, represented by the value of a surface integral whose integrand, the latter, is not regarded as primitive, in that it can be picked at will in the equipollency class of vector fields determined by the former. In fact,

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* Relation $(3)_2$ reads as follows:

$$\mathbf{f}_{\mathrm{C}}^{\mathrm{r}}(\mathcal{P}) = \int_{\partial\boldsymbol{\chi}(\mathcal{P})} \mathbf{t}_{\partial\boldsymbol{\chi}(\mathcal{P})}\,\mathrm{d}A.$$

** As to wording, the 'contact loads' in the first quotation are the same as the 'contact forces' in the second; likewise, 'field of stress vectors' $\equiv$ 'traction field' and 'loads exerted by the material outside upon the material inside' $\equiv$ 'action of the part of the body outside [the] diaphragm and adjacent to the part inside'.

As to coverage, the italicized statement in the second quotation appears to concern only interior body parts.

any chosen “smooth, closed diaphragm” “[w]ithin the shape of a body at any time” is there regarded as the support of a vector field that must be “equipollent” (but not necessarily pointwise equal) to “the action of the part of the body outside that diaphragm and adjacent to the part inside”. The resulting specification of such action – an information item of paramount interest – is looser than desirable and possible: strictly speaking, the equipollency condition could be met by an action whose mathematical description requires more than one vector field having the nature of force per unit area.

Ignore for simplicity distance forces, whatever their kind, and consider a body in *balanced* contact interaction with its environment. A cut principle based only on *partwise equipollency* is of scarce help to specify the actual contact action experienced by a body part. Not so if the requirement of *partwise balance* typical of free-body diagrams is introduced, because such requirement sets a direct quantitative restriction on the contact actions over the body part whose contour coincides with the chosen imaginary diaphragm. Such an ‘augmented’ cut principle is best envisaged as an operation of mental surgery, a *Gedanken Experiment* with the purpose of evaluating the contact interactions that the body part singled out by the cut exchanges with its complement with respect to the body itself. The evaluation is performed as follows: (i) an arbitrary part $\mathcal{P}$ of body $\mathcal{B}$ is distinguished from the rest by an ‘ideal’ cut, that is to say, a cut leaving all mechanical actions and interactions unaltered (something that no real cut, no matter how carefully executed, could achieve); (ii) the part’s environment $\mathcal{E}(\mathcal{P}) := (\mathcal{B} \setminus \mathcal{P}) \cup \mathcal{E}(\mathcal{B})$, where $\mathcal{E}(\mathcal{B})$ is the environment of $\mathcal{B}$, is thought of as applying to $\mathcal{P}$ over its boundary $\partial\mathcal{P}$ a load $\boldsymbol{t}(\mathcal{P}, \mathcal{E}(\mathcal{P}))$ pointwise identical to the contact interaction $\mathcal{P}$ was subjected to before the cut, be it either a known applied load, if $\partial\mathcal{P}$ and $\partial\mathcal{B}$ have a measurable intersection where external loads are assigned, or the unknown contact load $\boldsymbol{t}(\mathcal{P}, \mathcal{B} \setminus \mathcal{P})$ exerted by $\mathcal{B} \setminus \mathcal{P}$; (iii) the field $\boldsymbol{t}(\mathcal{P}, \mathcal{E}(\mathcal{P}))$ is required to be balanced, whence a generally partial specification of the field $\boldsymbol{t}(\mathcal{P}, \mathcal{B} \setminus \mathcal{P})$.

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