Journal of Elasticity (2004) **75:** 267–289

Characterization of Elasticity-Tensor Symmetries Using $SU(2)$

ANDREJ BÓNA[1], IOAN BUCATARU[2] and MICHAEL A. SLAWINSKI[1]
[1]*Department of Earth Sciences, Memorial University, St. John's, NL A1B 3X5 Canada. E-mail: andrej@lagrange.esd.mun.ca, mslawins@mun.ca*
[2]*Faculty of Mathematics, Al. I. Cuza University, Iasi, Romania. E-mail: bucataru@uaic.ro*

Received: 1 April 2004; in revised form 6 September 2004

Abstract. A symmetry class of an elasticity tensor, c, is determined by the variance of this tensor with respect to a subgroup of the special orthogonal group, $SO(3)$. Using the double covering of $SO(3)$ by the special unitary group, $SU(2)$, we determine the subgroups of $SU(2)$ that correspond to each of the eight symmetry classes. A family of maps between $\mathbb{C}^2$ and $\mathbb{R}^3$ that preserve the action of the two groups is constructed. Using one of these maps and three associated polynomials, we derive new methods for characterizing the symmetry classes of elasticity tensors.

Mathematics Subject Classifications (2000): 74B05, 74E10.

Key words: elasticity tensor, symmetry classes, group theory, $SU(2)$.

1. Introduction

Several studies, notably, Chadwick et al. [5], Forte and Vianello [9], Ting [14], as well as Bóna et al. [3], show that there are eight symmetry classes of an elasticity tensor. These classes are characterized by eight subgroups of the orthogonal group $O(3)$. Since all of the symmetry groups of an elasticity tensor contain $-I_3$, the negative identity in $\mathbb{R}^3$, the symmetry classes are completely determined by eight corresponding subgroups of the special orthogonal group $SO(3)$. Herein, we use the special unitary group $SU(2)$ to study these classes.

In Section 2, we introduce the elasticity tensor as a fourth-rank tensor possessing certain intrinsic symmetries and consider two associated second-rank tensors. These second-rank tensors were also used to study symmetries of elasticity tensors by Baerheim [2], by Chadwick et al. [5], as well as by Cowin and Mehrabadi [7].

In Section 3, using the symmetry groups for each symmetry class, we use the double covering of $SO(3)$ by the special unitary group $SU(2)$ to determine the subgroups of $SU(2)$ that correspond to each symmetry class of an elasticity tensor.

In Section 4, we construct a family of maps between $\mathbb{C}^2$, and $\mathbb{R}^3$, that preserve the action of $SO(3)$ and $SU(2)$ on these two vector spaces. The symmetry class of an elasticity tensor is determined by the invariance of three associated polynomials under the action of $SO(3)$. Among the family of maps that preserve the action of the two groups $SO(3)$ and $SU(2)$, we select a map that also preserves the homogeneity of the polynomials. This map has been used also by Backus [1], Baerheim [2], and Zou and Zheng [16] to determine the Maxwell multipoles of an elasticity tensor. In Section 5, using this map, we consider the pull back of the three polynomials. Only the first polynomial and its pull back are the same as those derived by Backus [1] and by Baerheim [2]. These authors, however, do not use the elasticity tensor directly, as we do in this paper. Instead, they work with an associated fourth-rank harmonic tensor.

In Section 6, we prove that the eight subgroups of $SU(2)$ we derived in Section 3 completely determine all symmetry classes of elasticity tensors. In order to do this, we study the invariance of the polynomials obtained in Section 5 under the action of the eight subgroups of $SU(2)$. Then, unlike Backus [1] and Baerheim [2] who determine only the necessary conditons, we determine both the necessary and sufficient conditions for an elasticity tensor to belong to one of the symmetry classes. These conditions are concisely presented in a table in Section 7. One can use this table to discuss all possible routes of increasing symmetries for elasticity tensors, as we do in the last section.

Studying the symmetries of an elasticity tensor in the context of $SU(2)$ gives us a simpler way to characterize the symmetry classes than the standard approach in the context of $SO(3)$. As expected, in either case, the symmetry classes are the same.

2. Elasticity Tensors

In this section, we introduce the elasticity tensor and two associated second-rank tensors. Throughout this paper, we deal with tensors defined on $\mathbb{R}^3$ – the Euclidean three-dimensional space.

An *Elasticity tensor c* is a fourth-rank tensor, namely, a four-linear map $c\colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ that satisfies the following intrinsic symmetries:

$$c(u, v, z, w) = c(v, u, z, w) = c(z, w, u, v), \quad \forall u, v, z, w \in \mathbb{R}^3. \tag{1}$$

An orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ allows us to express the components of the elasticity tensor with respect to this basis as

$$c_{ijkl} = c(e_i, e_j, e_k, e_l), \quad i, j, k, l \in \{1, 2, 3\}.$$

Thus, c has $3^4 = 81$ components c_{ijkl}. With respect to these components, we can write tensor c as

$$c(u, v, z, w) = c_{ijkl} u^i v^j z^k w^l,$$

where $u = u^i e_i, v = v^i e_i, z = z^i e_i$ and $w = w^i e_i$.⋆ Using this coordinate expression, conditions (1) can also be written as

$$c_{ijkl} = c_{jikl} = c_{klij}, \tag{2}$$

for $i, j, k, l \in \{1, 2, 3\}$.

⋆ Throughout this paper, we use the summation convention of repeated indices.

Due to the intrinsic symmetries of the elasticity tensor, stated by conditions (2), we conclude that c has only twenty-one independent components. We choose to represent these components by

$$\begin{array}{llllll} 1c_{1111}, & 1c_{2222}, & 1c_{3333}, & & & \\ 4c_{2323}, & 2c_{2233}, & 4c_{1212}, & 2c_{1122}, & 4c_{1313}, & 2c_{1133}, \\ 4c_{1123}, & 8c_{1213}, & 4c_{1233}, & 8c_{1323}, & 4c_{2213}, & 8c_{1223}, \\ 4c_{1222}, & 4c_{1112}, & 4c_{2223}, & 4c_{1113}, & 4c_{2333}, & 4c_{1333}. \end{array} \tag{3}$$

In this list, the number in front of each term corresponds to the number of components of c that the particular term represents; in other words, the number of times that a given component occurs in c. We will use this representation to study nonintrinsic symmetries of the elasticity tensor, which result from material properties.

Two symmetric second-rank tensors can be associated with an elasticity tensor, as discussed in several papers, for instance, by Cowin and Mehrabadi [6] or Forte and Vianello [9]. Their components with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ are

$$\begin{aligned} \mathcal{V}_{ij} &= c_{i1j1} + c_{i2j2} + c_{i3j3}, \\ \mathcal{D}_{ij} &= c_{11ij} + c_{22ij} + c_{33ij}. \end{aligned} \tag{4}$$

Tensor $\mathcal{V}$ is called the Voigt tensor, while tensor $\mathcal{D}$ is called the dilatation tensor.

3. Subgroups of $SU(2)$ and Symmetry Classes

Forte and Vianello [9] discuss two different ways of defining symmetries for an elasticity tensor. One way of defining the symmetry classes of an elasticity tensor has been introduced by Huo and del Piero [12]; according to their definition there are ten symmetry classes. In this paper we follow the definition used by Forte and Vianello [10] and Chadwick et al. [5]. According to this definition, there are eight symmetry classes for an elasticity tensor. Each of these classes corresponds to a subgroup of $O(3)$ that leaves the elasticity tensor invariant.

An elasticity tensor c is invariant under an orthogonal transformation A if

$$c(Au, Av, Aw, Az) = c(u, v, w, z), \quad \forall u, v, z, w \in \mathbb{R}^3.$$

The set of all orthogonal transformations G_c that leaves an elasticity tensor invariant is a subgroup of $O(3)$, the orthogonal group that is generated by rotations and reflections. For an elasticity tensor, its symmetry group G_c always contains $-I_3$. Consequently the elasticity tensor is invariant under an orthogonal transformation A if and only if it is invariant under $-A$. Since only one of A and $-A$ is a rotation, the symmetry class to which an elasticity tensor belongs is determined only by rotations. This means that a symmetry class is completely determined by a subgroup of $SO(3)$, which consists only of rotations. We refer to such a subgroup by $\widetilde{G}_c$.

We note that the order of group G_c is twice the order of group $\widetilde{G}_c$. The relations between these two groups are given by

$$\widetilde{G}_c = G_c \cap SO(3) \quad \text{and} \quad G_C = -\widetilde{G}_c \cup \widetilde{G}_c. \tag{5}$$

Using the double covering of $SO(3)$ by $SU(2)$, we determine the subgroups of $SU(2)$ that correspond to each symmetry class. The double covering of $SO(3)$ by $SU(2)$ is a group morphism $\psi\colon SU(2) \to SO(3)$ that for each rotation $A \in SO(3)$ assigns exactly two special unitary transformations $\pm B \in SU(2)$ such that $\psi(\pm B) = A$.

Now, we define ψ for particular unitary transformations, which we can extend to all transformations using the fact that ψ preserves the group multiplication. With respect to an orthonormal basis in $\mathbb{R}^3$ and $\mathbb{C}^2$, we can define ψ as

$$\begin{aligned}
\psi\colon \quad &\pm\begin{pmatrix} \cos\frac{\theta}{2} & \iota\sin\frac{\theta}{2} \\ \iota\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \\
\psi\colon \quad &\pm\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \\
\psi\colon \quad &\pm\begin{pmatrix} e^{\iota\theta/2} & 0 \\ 0 & e^{-\iota\theta/2} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{6}$$

where $\iota = \sqrt{-1}$.

For each of the eight symmetry classes of elasticity tensors, the corresponding symmetry groups are explicitely written in Bóna et al. [3]. Using these groups and morphism ψ, we will determine the subgroup of $SU(2)$ that corresponds to each symmetry class. We refer to this subgroup by H_c, which means that $\psi(H_c) = \widetilde{G}_c$. Group H_c has the same order as group G_c and twice the order of $\widetilde{G}_c$.

1. GENERALLY ANISOTROPIC. An elasticity tensor is said to be generally anisotropic if its symmetry group is $G_c = \{\pm I_3\}$. This means that $\widetilde{G}_c = \{I_3\}$ and $H_c = \{\pm I_2\}$. With respect to an orthonormal basis of $\mathbb{R}^3$, this tensor has twenty-one independent components given by expression (3).

2. MONOCLINIC. An elasticity tensor has monoclinic symmetry if its symmetry group is $G_c = \{\pm I_3, \pm R_{e_3}\}$, where R_{e_3} is a reflection about a plane that is orthogonal to e_3. The corresponding subgroup of rotations is $\widetilde{G}_c = \{I_3, -R_{e_3}\}$, while the corresponding subgroup of $SU(2)$ is

$$H_c = \left\{\pm I_2,\ \pm\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}\right\}. \tag{7}$$

With respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ – where e_3 is the unitary vector orthogonal to the reflection plane and $\{e_1, e_2\}$ is an arbitrary orthonormal basis for the reflection plane – this elasticity tensor has thirteen independent components. These components are

$$\begin{array}{llll} c_{1111}, & c_{2222}, & c_{3333}, & \\ c_{1122}, & c_{1133}, & c_{2233}, & \\ c_{1212}, & c_{1313}, & c_{2323}, & \\ c_{1112}, & c_{1222}, & c_{1233}, & c_{1323}. \end{array} \tag{8}$$

3. ORTHOTROPIC. An elasticity tensor has orthotropic symmetry if its symmetry group is $G_c = \{\pm I_3, \pm R_{e_i},\ i \in \{1, 2, 3\}\}$. The corresponding subgroup of rotations is $\widetilde{G}_c = \{I_3, -R_{e_i},\ i \in \{1, 2, 3\}\}$, while the corresponding subgroup of $SU(2)$ is

$$H_c = \left\{ \pm I_2,\ \pm\begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \right\}. \tag{9}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ – where e_1, e_2, e_3 are unitary vectors orthogonal to the three reflection planes – this elasticity tensor has nine independent components. These components are

$$\begin{array}{lll} c_{1111}, & c_{2222}, & c_{3333}, \\ c_{1122}, & c_{1133}, & c_{2233}, \\ c_{1212}, & c_{1313}, & c_{2323}. \end{array} \tag{10}$$

4. TRIGONAL. An elasticity tensor has trigonal symmetry if its symmetry group is $G_c = \{\pm I_3, \pm R_{u_\alpha}, \pm R_{\pm 2\pi/3, e_3},\ \alpha \in \{1, 2, 3\}\}$, where $R_{u_\alpha},\ \alpha \in \{1, 2, 3\}$, are reflections about three planes that contain e_3. The angle between any two planes of reflection is $2\pi/3$. Since vectors u_α are orthogonal to the reflection planes, it follows that they are orthogonal to e_3. One can choose e_2 to be one of these vectors u_α. Then, $u_\alpha = \sin(\theta_\alpha/2)e_1 - \cos(\theta_\alpha/2)e_2$, where $\theta_\alpha \in \{0, \pm 2\pi/3\}$. The corresponding subgroup of rotations is $\widetilde{G}_c = \{I, -R_{u_\alpha}, R_{\pm 2\pi/3, e_3},\ \alpha \in \{1, 2, 3\}\}$, while the corresponding subgroup of $SU(2)$ is

$$\begin{aligned} H_c = \Bigg\{ &\pm I_2, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & \mathrm{e}^{\pm\iota\pi/3} \\ -\mathrm{e}^{\mp\iota\pi/3} & 0 \end{pmatrix}, \\ &\pm\begin{pmatrix} \mathrm{e}^{\pm\iota\pi/3} & 0 \\ 0 & \mathrm{e}^{\mp\iota\pi/3} \end{pmatrix} \Bigg\}. \end{aligned} \tag{11}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, the elasticity tensor has six independent components. These components are

$$\begin{array}{l} c_{1111} = c_{2222},\ c_{3333}, \\ c_{1122},\ c_{1133} = c_{2233}, \\ c_{1212} = \dfrac{1}{2}(c_{1111} - c_{1122}), \quad c_{1313} = c_{2323}, \\ c_{1123} = -c_{2223} = c_{1213}. \end{array} \tag{12}$$

We note that if a trigonal elasticity tensor is also invariant under R_{e_3}, then it is invariant under any rotation around e_3. Consequently, the components $c_{1123} = -c_{2223} = c_{1213}$ vanish and the tensor has only five independent components. We shall see below that such a case corresponds to a transversely isotropic tensor whose components are given by expression (17).

5. TETRAGONAL. An elasticity tensor has tetragonal symmetry if its symmetry group is $G_c = \{\pm I_3, \pm R_{\pm\pi/2,e_3}, \pm R_{\pi,e_3}, \pm R_{u_\alpha},\ \alpha \in \{1,2,3,4\}\}$, where $R_{u_\alpha},\ \alpha \in \{1,2,3,4\}$, are reflections about four planes that contain e_3. The angle between the planes of reflection is $\pi/4$ and, since vectors u_α are orthogonal to the reflection planes, it follows that they are orthogonal to e_3. One can choose e_1 and e_2 to be two of these vectors. Then $u_\alpha = \sin(\theta_\alpha/2)e_1 - \cos(\theta_\alpha/2)e_2$, where $\theta_\alpha \in \{0, \pm\pi/2, \pi\}$. The corresponding subgroup of rotations is $\widetilde{G}_c = \{I_3, R_{\pm\pi/2,e_3}, R_{\pi,e_3}, -R_{u_\alpha},\ \alpha \in \{1,2,3,4\}\}$, while the corresponding subgroup of $SU(2)$ is

$$H_c = \left\{\pm I_2,\ \pm\begin{pmatrix} \mathrm{e}^{\pm\iota\pi/4} & 0 \\ 0 & \mathrm{e}^{\mp\iota\pi/4} \end{pmatrix},\ \pm\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix},\ \pm\begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix},\right. \\ \left. \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & \iota\mathrm{e}^{\pm\iota\pi/4} \\ -\mathrm{e}^{\mp\iota\pi/4} & 0 \end{pmatrix}\right\}. \tag{13}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, this elasticity tensor has six independent components. These components are

$$\begin{aligned} &c_{1111} = c_{2222},\ c_{3333}, \\ &c_{1122},\ c_{1133} = c_{2233}, \\ &c_{1212},\ c_{1313} = c_{2323}. \end{aligned} \tag{14}$$

6. TRANSVERSELY ISOTROPIC. An elasticity tensor has transversely isotropic symmetry if its symmetry group is

$$G_g = \left\{\pm\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \theta \in (-\pi, \pi]\right\}. \tag{15}$$

The first elements of the symmetry group are $\pm R_{\theta,e_3}$, namely, the rotations by angle θ about vector e_3. The last elements are $\pm R_u$, namely, the reflections about the plane that is orthogonal to $u(\theta) = \sin(\theta/2)e_1 - \cos(\theta/2)e_2$.

Consequently, the symmetry group contains all rotations around e_3 and all reflections about the planes that contain e_3 – the symmetry group coincides with $O(2)$, as a subgroup of $O(3)$.

The corresponding subgroup of rotations is $G_c = \{R_{\theta,e_3}, -R_{u(\theta)},\ \theta \in (-\pi, \pi]\}$, while the corresponding subgroup of $SU(2)$ is

$$H_c = \left\{\pm\begin{pmatrix} \mathrm{e}^{\iota\theta/2} & 0 \\ 0 & \mathrm{e}^{-\iota\theta/2} \end{pmatrix},\ \pm\begin{pmatrix} 0 & \mathrm{e}^{\iota\theta/2} \\ -\mathrm{e}^{-\iota\theta/2} & 0 \end{pmatrix},\ \theta \in (-\pi, \pi]\right\}. \tag{16}$$

With respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, where e_3 is the unitary axis of rotation, this elasticity tensor has five independent components. These components are

$$\begin{aligned}
&c_{1111} = c_{2222},\ c_{3333},\\
&c_{1122},\ c_{1133} = c_{2233},\\
&c_{1212} = \frac{1}{2}(c_{1111} - c_{1122}), \quad c_{1313} = c_{2323}.
\end{aligned} \tag{17}$$

7. CUBIC. An elasticity tensor has cubic symmetry if its symmetry group is $G_c = \{A \in O(3),\ A(e_i) = \pm e_j,\ i, j \in \{1, 2, 3\}\}$. The corresponding subgroup of rotations is $\widetilde{G}_c = \{A \in SO(3),\ A(e_i) = \pm e_j,\ i, j \in \{1, 2, 3\}\}$, while the corresponding subgroup of $SU(2)$ is

$$\begin{aligned}
H_c = \Bigg\{ &\pm I_2,\ \pm\begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix},\\
&\pm\begin{pmatrix} \mathrm{e}^{\pm\iota\pi/4} & 0 \\ 0 & \mathrm{e}^{\mp\iota\pi/4} \end{pmatrix},\ \pm\begin{pmatrix} 0 & \mathrm{e}^{\pm\iota\pi/4} \\ -\mathrm{e}^{\mp\iota\pi/4} & 0 \end{pmatrix},\\
&\pm\begin{pmatrix} \cos\frac{\pi}{4} & \pm\iota\sin\frac{\pi}{4} \\ \pm\iota\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix},\ \pm\begin{pmatrix} \cos\frac{\pi}{4} & \mp\sin\frac{\pi}{4} \\ \pm\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix},\\
&\pm\begin{pmatrix} \mp\sin\frac{\pi}{4} & \iota\cos\frac{\pi}{4} \\ \iota\cos\frac{\pi}{4} & \mp\sin\frac{\pi}{4} \end{pmatrix},\ \pm\begin{pmatrix} \pm\iota\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \\ -\cos\frac{\pi}{4} & \mp\iota\sin\frac{\pi}{4} \end{pmatrix},\\
&\pm\begin{pmatrix} \iota\cos\frac{\pi}{4} & \mp\sin\frac{\pi}{4} \\ \pm\sin\frac{\pi}{4} & -\iota\cos\frac{\pi}{4} \end{pmatrix},\ \pm\begin{pmatrix} \pm\iota\sin\frac{\pi}{4} & \iota\cos\frac{\pi}{4} \\ \iota\cos\frac{\pi}{4} & \mp\iota\sin\frac{\pi}{4} \end{pmatrix},\\
&\pm\begin{pmatrix} \pm\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \\ -\cos\frac{\pi}{4} & \pm\sin\frac{\pi}{4} \end{pmatrix},\ \pm\begin{pmatrix} \iota\cos\frac{\pi}{4} & \mp\iota\sin\frac{\pi}{4} \\ \mp\iota\sin\frac{\pi}{4} & -\iota\cos\frac{\pi}{4} \end{pmatrix} \Bigg\}.
\end{aligned} \tag{18}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, where e_1, e_2, e_3 are determined by the unitary planes of reflections, the elasticity tensor has three independent components. These components are

$$\begin{aligned}
&c_{1111} = c_{2222} = c_{3333},\\
&c_{1122} = c_{1133} = c_{2233},\\
&c_{1212} = c_{1313} = c_{2323}.
\end{aligned} \tag{19}$$

8. ISOTROPIC. An elasticity tensor has isotropic symmetry if its symmetry group is $G_c = O(3)$. The corresponding group of rotations is $\widetilde{G}_c = SO(3)$, while the corresponding unitary group is $H_c = SU(2)$.

With respect to an arbitrary orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, the elasticity tensor has two independent components. These components are

$$\begin{aligned} &c_{1111} = c_{2222} = c_{3333} = 2c_{1212} + c_{1122}, \\ &c_{1122} = c_{1133} = c_{2233}, \\ &c_{1212} = c_{1313} = c_{2323}. \end{aligned} \tag{20}$$

In order to simplify the study of the invariance of an elasticity tensor we will investigate the invariance of related quantities in $\mathbb{C}^2$ under the action of $SU(2)$.

4. Maps between $\mathbb{C}^2$ and $\mathbb{R}^3$

We would like to construct a map ϕ: $\mathbb{C}^2 \to \mathbb{R}^3$ such that it intertwines the action of $SO(3)$ in $\mathbb{R}^3$ with the action of its double cover $SU(2)$ in $\mathbb{C}^2$. In other words, we want the following diagram to commute:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\phi} & \mathbb{R}^3 \\ \downarrow{\scriptstyle \tilde{g}_A} & & \downarrow{\scriptstyle g_{\psi(A)}} \\ \mathbb{C}^2 & \xrightarrow{\phi} & \mathbb{R}^3, \end{array}$$

where $\tilde{q}_A$ and $g_{\psi(A)}$ are linear isomorphisms of $\mathbb{C}^2$ and $\mathbb{R}^3$ induced by the action of A from $SU(2)$ and $\psi(A)$ from $SO(3)$ which are related by the double-covering projection ψ: $SU(2) \to SO(3)$. The fact that the diagram commutes means that

$$g_{\psi(A)}(\phi) = \phi(\tilde{g}_A).$$

To construct such a map, we consider two fixed points, $(0, R) \in \mathbb{C}^2$ and $(r^1, r^2, r^3) \in \mathbb{R}^3$, such that $(r^1)^2 + (r^2)^2 + (r^3)^2 = R^2$ and $R \in \mathbb{R}$, together with the orbits of $SU(2)$ and $SO(3)$ acting on these points by their natural actions in $\mathbb{C}^2$ and $\mathbb{R}^3$. We can map point $(z^1, z^2) \equiv (p^1 + \iota q^1, p^2 + \iota q^2)$ from the orbit of $SU(2)$ to point (ξ^1, ξ^2, ξ^3) from the orbit of $SO(3)$ by associating with (z^1, z^2) the unique element of $SU(2)$ that maps the fixed point $(0, R) \in \mathbb{C}^2$ to the point (z^1, z^2), and then using the corresponding element of $SO(3)$ to map the fixed point $(r^1, r^2, r^3) \in \mathbb{R}^3$ to the desired image (ξ^1, ξ^2, ξ^3) of the original point (z^1, z^2).

Since all the elements of $SU(2)$ are matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $|a|^2 + |b|^2 = 1$, the orbit of $SU(2)$ through $(0, R) \in \mathbb{C}^2$ is

$$\left\{(Rb, R\bar{a}) \in \mathbb{C}^2,\ |a|^2 + |b|^2 = 1\right\}.$$

The bar denotes complex conjugation. Hence, we can associate with any element (z^1, z^2) of the orbit the unique element

$$\begin{pmatrix} \frac{\bar{z}^2}{R} & \frac{z^1}{R} \\ -\frac{\bar{z}^1}{R} & \frac{z^2}{R} \end{pmatrix}$$

of $SU(2)$ that maps $(0, R)$ to (z^1, z^2).

This element of $SU(2)$ projects under the standard projection, e.g., Crampin and Pirani [8, p. 205], to the following element of $SO(3)$.

$$\frac{1}{R}\begin{pmatrix} \bar{z}^2 & z^1 \\ -\bar{z}^1 & z^2 \end{pmatrix}$$
$$\mapsto \frac{1}{R^2}\begin{pmatrix} \mathrm{Re}((z^2)^2-(z^1)^2) & \mathrm{Im}((z^2)^2-(z^1)^2) & 2\mathrm{Re}(z^1\bar{z}^2) \\ \mathrm{Re}(\iota((z^2)^2+(z^1)^2)) & \mathrm{Im}(\iota((z^2)^2+(z^1)^2)) & 2\mathrm{Im}(z^1\bar{z}^2) \\ -2\mathrm{Re}(z^1z^2) & -2\mathrm{Im}(z^1z^2) & |z^2|^2-|z^1|^2 \end{pmatrix}.$$

Hence, the fixed point $(r^1, r^2, r^3) \in \mathbb{R}^3$ maps to

$$\frac{1}{R^2}\begin{pmatrix} \mathrm{Re}((z^2)^2-(z^1)^2) & \mathrm{Im}((z^2)^2-(z^1)^2) & 2\mathrm{Re}(z^1\bar{z}^2) \\ \mathrm{Re}(\iota((z^2)^2+(z^1)^2)) & \mathrm{Im}(\iota((z^2)^2+(z^1)^2)) & 2\mathrm{Im}(z^1\bar{z}^2) \\ -2\mathrm{Re}(z^1z^2) & -2\mathrm{Im}(z^1z^2) & |z^2|^2-|z^1|^2 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix}.$$

We can see that this map can be written in exactly the same way for any R and, hence, for any orbit. This way, we have found a map ϕ that intertwines the actions of $SU(2)$ and $SO(3)$. This map is

$$\phi\colon \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto \frac{1}{R^2}$$
$$\times \begin{pmatrix} \mathrm{Re}((z^2)^2-(z^1)^2) & \mathrm{Im}((z^2)^2-(z^1)^2) & 2\mathrm{Re}(z^1\bar{z}^2) \\ \mathrm{Re}(\iota((z^2)^2+(z^1)^2)) & \mathrm{Im}(\iota((z^2)^2+(z^1)^2)) & 2\mathrm{Im}(z^1\bar{z}^2) \\ -2\mathrm{Re}(z^1z^2) & -2\mathrm{Im}(z^1z^2) & |z^2|^2-|z^1|^2 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix}.$$

We note that this map is not linear and hence we cannot use it to pull back the elasticity tensor to form a tensor. However, if we use a function of ξ^1, ξ^2, ξ^3 that is associated with the elasticity tensor, then we can use this map to pull back this function to $\mathbb{C}^2$ and study the resulting function.

5. Polynomials Associated with Elasticity Tensor

A symmetry class of an elasticity tensor is determined by the invariance of associated polynomials under the action of the rotation group $SO(3)$. If we pull back these polynomials to $\mathbb{C}^2$, then their invariance under the action of $SU(2)$ will provide new characterizations for the standard symmetry classes of an elasticity tensor.

Consider a coordinate representation of an elasticity tensor, c_{ijkl}, with respect to some orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$. Then

$$P(\xi^1, \xi^2, \xi^3) = c_{ijkl}\xi^i\xi^j\xi^k\xi^l \tag{21}$$

is a homogeneous fourth-degree polynomial induced by the elasticity tensor. Also

$$\begin{aligned} P_{\mathcal{V}}(\xi^1, \xi^2, \xi^3) &= \mathcal{V}_{ij}\xi^i\xi^j, \\ P_{\mathcal{D}}(\xi^1, \xi^2, \xi^3) &= \mathcal{D}_{ij}\xi^i\xi^j, \end{aligned} \tag{22}$$

are two homogeneous second-degree polynomials induced by the Voigt tensor $\mathcal{V}$ and the dilatation tensor $\mathcal{D}$, respectively.

NOTE 5.1. Symmetry group $\widetilde{G}_c$ of an elasticity tensor coincides with the group of special orthogonal transformations that preserve the three polynomials. In other words,

$$\widetilde{G}_c = \big\{A \in SO(3),\ P(AX) = P(X),\ P_{\mathcal{V}}(AX) = P_{\mathcal{V}}(X),\\ P_{\mathcal{D}}(AX) = P(X),\ \forall X \in \mathbb{R}^3\big\}.$$

We want to pull back these three polynomials to $\mathbb{C}^2$ and study their invariance under the subgroups of $SU(2)$ that we studied in Section 3. In order to do this, we consider the family of maps discussed in the previous section. To narrow down this family to a single map, one can require an additional property of the map. Since we want to study polynomials, it is natural to consider a map that is homogeneous of some degree. For this purpose, we consider the following theorem.

THEOREM 5.1. *A map parameterized by* (r^1, r^2, r^3) *from the family of maps given by*

$$\phi\colon \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto \frac{1}{R^2} \\ \times \begin{pmatrix} \mathrm{Re}((z^2)^2 - (z^1)^2) & \mathrm{Im}((z^2)^2 - (z^1)^2) & 2\mathrm{Re}(z^1\bar{z}^2) \\ \mathrm{Re}(\iota((z^2)^2 + (z^1)^2)) & \mathrm{Im}(\iota((z^2)^2 + (z^1)^2)) & 2\mathrm{Im}(z^1\bar{z}^2) \\ -2\mathrm{Re}(z^1 z^2) & -2\mathrm{Im}(z^1 z^2) & |z^2|^2 - |z^1|^2 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix}$$

is homogeneous in (z^1, z^2) *if and only if* $(r^1, r^2, r^3) = a(-1, -\iota, 0)$ *for some complex* a.

NOTE 5.2. We are considering the complexification of the real space $\mathbb{R}^3$ in order to have polynomials in (z^1, z^2) that are complex homogeneous.

Hence, we consider the following map from Theorem 5.1 given by $a = R^2/2$.

$$\varphi\colon (z^1, z^2) \mapsto \big(\xi^1(z^1, z^2) = (z^1)^2 - (z^2)^2,\\ \xi^2(z^1, z^2) = -\iota\big((z^1)^2 + (z^2)^2\big),\ \xi^3(z^1, z^2) = 2z^1 z^2\big). \tag{23}$$

The pull backs of the three homogeneous polynomials P, $P_{\mathcal{V}}$ and $P_{\mathfrak{D}}$ by the homogeneous map φ of degree two are

$$P(z^1, z^2) = P\big((z^1)^2 - (z^2)^2,\ -\iota((z^1)^2 + (z^2)^2),\ 2z^1z^2\big),$$

which is an eighth-degree homogeneous polynomial with complex coefficients in z^1 and z^2,

$$P_{\mathcal{V}}(z_1, z^2) = P_{\mathcal{V}}\big((z^1)^2 - (z^2)^2,\ -\iota((z^1)^2 + (z^2)^2),\ 2z^1z^2\big)$$

and

$$P_{\mathfrak{D}}(z_1, z^2) = P_{\mathfrak{D}}\big((z^1)^2 - (z^2)^2,\ -\iota((z^1)^2 + (z^2)^2),\ 2z^1z^2\big),$$

which axe fourth-degree polynomials with complex coefficients in z^1 and z^2.

Since map φ preserves the action of $SO(3)$ and $SU(2)$, we can restate Note 5.1 as follows.

NOTE 5.3. Symmetry group H_c of an elasticity tensor coincides with the group of special unitary transformations that preserves the three complex polynomials. In other words,

$$H_c = \big\{U \in SU(2),\ P(UZ) = P(Z),\ P_{\mathcal{V}}(UZ) = P_{\mathcal{V}}(Z),$$
$$P_{\mathfrak{D}}(UZ) = P(Z),\ \forall Z \in \mathbb{C}^2\big\}.$$

Using these three polynomials, we consider three related polynomials

$$\begin{aligned} P^{(8)}(t) &= P\left(\frac{z^1}{z^2}, 1\right), \\ P^{(4)}_{\mathcal{V}}(t) &= P_{\mathcal{V}}\left(\frac{z^1}{z^2}, 1\right), \\ P^{(4)}_{\mathfrak{D}}(t) &= P_{\mathfrak{D}}\left(\frac{z^1}{z^2}, 1\right), \end{aligned} \tag{24}$$

of eighth-degree and fourth-degree in a single variable $t = z^1/z^2$. If a map takes (z^1, z^2) to $(\tilde{z}^1, \tilde{z}^2)$, then the invariance of any of polynomials $P(z^1, z^2)$ under this map translates to the following properties of corresponding polynomials in $t = z^1/z^2$:

$$(z^2)^q P^{(q)}\left(\frac{z^1}{z^2}\right) = \big(\tilde{z}^2\big) P^{(q)}\left(\frac{\tilde{z}^1}{\tilde{z}^2}\right), \tag{25}$$

where q is 4 or 8.

By a straightforward calculation, one can check that the eighth-degree polynomial, $P^{(8)}$ has the following form:

$$P^{(8)}(t) = a_8t^8 + a_7t^7 + a_6t^6 + a_5t^5 + a_4t^4 - \overline{a_5}t^3 + \overline{a_6}t^2 - \overline{a_7}t + \overline{a_8}, \tag{26}$$

where the bar denotes a complex conjugation. The coefficients of this polynomial are

$$
\begin{aligned}
a_8 &= c_{1111} + c_{2222} - 4c_{1212} - 2c_{1122} + 4\iota(-c_{1112} + c_{1222}),\\
a_7 &= 8\big[c_{1113} - c_{1322} - 2c_{1223} + \iota(-c_{1123} + c_{2223} - 2c_{1213})\big],\\
a_6 &= 4\big[-c_{1111} + c_{2222} - 4c_{2323} + 4c_{1313} - 2c_{2233} + 2c_{1133}\\
&\quad + 2\iota(c_{1112} + c_{1222} - 2c_{1233} - 4c_{1323})\big],\\
a_5 &= 8\big[-3c_{1113} - c_{1322} + 4c_{1333} - 2c_{1223}\\
&\quad + \iota(c_{1123} + 3c_{2223} - 4c_{2333} + 2c_{1213})\big],\\
a_4 &= 2\big(3c_{1111} + 3c_{2222} + 8c_{3333} - 16c_{2323} - 16c_{1313} + 4c_{1212}\\
&\quad + 2c_{1122} - 8c_{1133} - 8c_{2233}\big).
\end{aligned}
\tag{27}
$$

Polynomial $P^{(8)}$ with coefficients (27) is the same as the one derived by Backus [1] and Baerheim [2]. However, in these two papers, polynomial $P^{(8)}$ is not derived directly from elasticity tensor c, but from a fourth-rank, totally symmetric, harmonic tensor induced by c.

Fourth-degree polynomial $P_{\mathcal{V}}^{(4)}$ is given by

$$P_{\mathcal{V}}^{(4)}(t) = a_4^{\mathcal{V}} t^4 + a_3^{\mathcal{V}} t^3 + a_2^{\mathcal{V}} t^2 - \bar{a}_3^{\mathcal{V}} t + \bar{a}_4^{\mathcal{V}}. \tag{28}$$

The coefficients of this polynomial are

$$
\begin{aligned}
a_4^{\mathcal{V}} &= c_{1111} - c_{2222} + c_{1313} - c_{2323} - 2\iota(c_{1112} + c_{1222} + c_{1323}),\\
a_3^{\mathcal{V}} &= 4\big[c_{1113} + c_{1223} + c_{1333} - \iota(c_{1213} + c_{2223} + c_{2333})\big],\\
a_2^{\mathcal{V}} &= 2\big(-c_{1111} - c_{2222} + 2c_{3333} - 2c_{1212} + c_{1313} + c_{2323}\big).
\end{aligned}
\tag{29}
$$

Fourth-degree polynomial $P_{\mathcal{D}}^{(4)}$ is given by

$$P_{\mathcal{D}}^{(4)}(t) = a_4^{\mathcal{D}} t^4 + a_3^{\mathcal{D}} t^3 + a_2^{\mathcal{D}} t^2 - \bar{a}_3^{\mathcal{D}} t + \bar{a}_4^{\mathcal{D}}. \tag{30}$$

The coefficients of this polynomial are

$$
\begin{aligned}
a_4^{\mathcal{D}} &= c_{1111} - c_{2222} + c_{1133} - c_{2233} - 2\iota(c_{1112} + c_{1222} + c_{1233}),\\
a_3^{\mathcal{D}} &= 4\big[c_{1113} + c_{1322} + c_{1333} - \iota(c_{1123} + c_{2223} + c_{2333})\big],\\
a_2^{\mathcal{D}} &= 2\big(-c_{1111} - c_{2222} + 2c_{3333} - 2c_{1122} + c_{1133} + c_{2233}\big).
\end{aligned}
\tag{31}
$$

Two polynomials $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$ with coefficients given by (29) and (31) are different from the fourth-degree polynomials derived by Backus [1] and Baerheim [2]. The coefficients of the polynomials used in the present paper, given by expressions (29) and (31), are simpler.

6. Characterizations of Symmetry Classes

In Section 3, we have seen that to each of the eight symmetry classes of an elasticity tensor corresponds a subgroup H_c of $SU(2)$. In this section, we shall prove that these eight subgroups of $SU(2)$ completely determine all symmetry classes of elasticity tensors. In order to do this, we will study the invariance of polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$ under H_c and derive necessary and sufficient conditions for an elasticity tensor to belong to one of the eight symmetry classes.

THEOREM 6.1. *An elasticity tensor has monoclinic symmetry if and only if the three polynomials* $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$ *satisfy the following conditions:*

$$P^{(2q)}(t) = \varepsilon^q P^{(2q)}\left(\frac{t}{\varepsilon}\right), \quad q \in \{2, 4\}, \tag{32}$$

where $\varepsilon^2 = 1$, *and* $P^{(4)}$ *refers to both polynomials* $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$.

Proof. An elasticity tensor has monoclinic symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ are invariant under the action of the group given by expression (7), which implies invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(z^1, z^2), \\ \pm(\iota z^1, -\iota z^2). \end{cases} \tag{33}$$

The invariance of $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ under maps (33) is equivalent to the transformation of $P^{(8)}(t)$, $P_{\mathcal{V}}^{(4)}(t)$ and $P_{\mathcal{D}}^{(4)}(t)$ given by (25), which results in condition (32), where

$$t = \frac{z^1}{z^2} \quad \mapsto \quad \pm t = \pm\frac{z^1}{z^2}. \tag{34}$$

Next, we shall prove that condition (32) is equivalent to the fact that an elasticity tensor has – with respect to an orthonormal basis – thirteen independent components given by expression (8). Condition (32) is equivalent to the fact that all three polynomials are even since all coefficients of odd order are zero. This means that an elasticity tensor has monoclinic symmetry if and only if

$$a_7 = a_5 = a_3^{\mathcal{V}} = a_3^{\mathcal{D}} = 0.$$

The vanishing of these four complex coefficients is equivalent – using conditions (27), (29) and (31) – to a system of eight equations with eight unknowns. The solution of this system is

$$c_{1113} = c_{1223} = c_{1333} = c_{1213} = c_{2223} = c_{2333} = c_{1322} = c_{1123} = 0. \tag{35}$$

Thus, an elasticity tensor has a monoclinic symmetry if and only if it has – with respect to an orthonormal basis – thirteen independent components given by expression (8). □

THEOREM 6.2. *An elasticity tensor has orthotropic symmetry if and only if the three polynomials* $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$ *satisfy the following conditions:*

$$\begin{aligned} P^{(2q)}(t) &= \varepsilon^q P^{(2q)}(t/\varepsilon),\ q \in \{2, 4\}, \\ P^{(2q)}(t) &= \varepsilon^q t^{2q} P^{(2q)}(-\varepsilon/t),\ q \in \{2, 4\}, \end{aligned} \tag{36}$$

where $\varepsilon^2 = 1$, *and* $P^{(4)}$ *refers to both polynomials* $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$.

Proof. An elasticity tensor has orthotropic symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ are invariant under the action of the group given by expression (9), which implies invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(z^1, z^2), \\ \pm(\iota z^2, \iota z^1), \\ \pm(z^2, -z^1), \\ \pm(\iota z^1, -\iota z^2). \end{cases} \tag{37}$$

The invariance of $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ under maps (37) is equivalent to the transformation of $P^{(8)}(t)$, $P_{\mathcal{V}}^{(4)}(t)$ and $P_{\mathcal{D}}^{(4)}(t)$ given by expression (25), which results in condition (36), where

$$t = \frac{z^1}{z^2} \;\mapsto\; \begin{cases} \pm t = \varepsilon t, \\ \pm\dfrac{1}{t} = -\dfrac{\varepsilon}{t}. \end{cases} \tag{38}$$

Next, we shall prove that conditions (36) are equivalent to the fact that an elasticity tensor has – with respect to an orthonormal basis – nine independent components given by expression (10). The first condition in expression (36) is the same as condition (32), which is equivalent to the fact that all three polynomials are even. As we have seen in the proof of Theorem 6.1, this implies conditions (35). The second condition in expression (36) is equivalent to the fact that the remaining nonzero coefficients of all three polynomials are real. This is equivalent to

$$\mathrm{Im}(a_8) = \mathrm{Im}(a_6) = \mathrm{Im}(a_4^{\mathcal{V}}) = \mathrm{Im}(a_4^{\mathcal{D}}) = 0.$$

These four equations combined with conditions (35) imply that

$$c_{1112} = c_{1222} = c_{1233} = c_{1323} = 0. \tag{39}$$

From conditions (35) and (39) we see that an elasticity tensor has orthotropic symmetry if and only if it has – with respect to an orthonormal basis – nine independent components, given by expression (10). □

THEOREM 6.3. *An elasticity tensor has trigonal symmetry if and only if the three polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$ satisfy the following conditions:*

$$\begin{aligned} P^{(2q)}(t) &= \omega^q P^{(2q)}\left(\frac{t}{\omega}\right), \qquad q \in \{2, 4\}, \\ P^{(2q)}(t) &= \omega^q t^{2q} P^{(2q)}\left(-\frac{\omega}{t}\right), \quad q \in \{2, 4\}, \end{aligned} \tag{40}$$

where $\omega^3 = 1$, and $P^{(4)}$ refers to both polynomials $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$.

Proof. An elasticity tensor has trigonal symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ are invariant under transformations given by expression (11), which means invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(z^1, z^2), \\ \pm(z^2, -z^1), \\ \pm(e^{\pm\iota\pi/3} z^2, -e^{\mp\iota\pi/3} z^1), \\ \pm(e^{\pm\iota\pi/3} z^1, e^{\mp\iota\pi/3} z^2). \end{cases} \tag{41}$$

The invariance of $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ under maps (41) is equivalent to the transformation of $P^{(8)}(t)$, $P_{\mathcal{V}}^{(4)}(t)$ and $P_{\mathcal{D}}^{(4)}(t)$ given by expression (25), which results in condition (40), where

$$t = \frac{z^1}{z^2} \mapsto \begin{cases} \omega t, \\ -\dfrac{\omega}{t}. \end{cases} \tag{42}$$

The first condition in expression (40) is equivalent to

$$\begin{aligned} &P^{(8)}(\omega t) = \omega P^{(8)}(t), \\ &P_{\mathcal{V}}^{(4)}(\omega t) = \omega^2 P_{\mathcal{V}}^{(4)}(t), \\ &P_{\mathcal{D}}^{(4)}(\omega t) = \omega^2 P_{\mathcal{D}}^{(4)}(t). \end{aligned} \tag{43}$$

We can check directly that the first condition in expression (43) results in

$$P^{(8)}(t) = a_7 t^7 + a_4 t^4 - \bar{a}_7 t = t\big(a_7 t^6 + a_4 t^3 - \bar{a}_7\big). \tag{44}$$

Similarly, we can see that the last two conditions in expression (43) result in

$$P^{(4)}(t) = a_2 t^2. \tag{45}$$

From expression (44) for $P^{(8)}$ and expression (45) for both $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$, we see that an elasticity tensor satisfies the first condition in expression (40) if and only if – with respect to an orthonormal basis – the following coefficients vanish:

$$a_8 = a_6 = a_5 = a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_4^{\mathcal{D}} = a_3^{\mathcal{D}} = 0. \tag{46}$$

If we consider the second condition of expression (40) with $\omega = e^{2\pi\iota/3}$ and $q = 4$, the only new equation we get is $\mathrm{Re}(a_7) = 0$.

Since all the seven coefficients in expression (46) are complex, their vanishing gives us fourteen independent equations. If we add $\mathrm{Re}(a_7) = 0$, we have fifteen equations. By solving these equations we see that an elasticity tensor has trigonal symmetry if and only if – with respect to an orthonormal basis – it has six independent components given by expression (12). □

THEOREM 6.4. *An elasticity tensor has tetragonal symmetry if and only if the three polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$ satisfy the following conditions:*

$$\begin{aligned} P^{(2q)}(t) &= \alpha^q P^{(2q)}\left(\frac{t}{\alpha}\right), \qquad q \in \{2,4\}, \\ P^{(2q)}(t) &= \alpha^q t^{2q} P^{(2q)}\left(-\frac{\alpha}{t}\right), \quad q \in \{2,4\}, \end{aligned} \tag{47}$$

where $\alpha^4 = 1$, and $P^{(4)}$ refers to both polynomials $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$.

Proof. An elasticity tensor has tetragonal symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ are invariant under transformations (13), which implies invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(z^1, z^2), \\ \pm(\mathrm{e}^{\pm\iota\pi/4} z^1, \mathrm{e}^{\mp\iota\pi/4} z^2), \\ \pm(\iota z^1, -\iota z^2), \\ \pm(\iota z^2, \iota^1), \\ \pm(z^2, -z^1), \\ \pm(\iota \mathrm{e}^{\pm\iota\pi/4} z^2, -\iota \mathrm{e}^{\mp\iota\pi/4} z^1). \end{cases} \tag{48}$$

The invariance of $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ under maps (48) is equivalent to polynomials $P^{(4)}(t)$, $P_{\mathcal{V}}^{(4)}(t)$ and $P_{\mathcal{D}}^{(4)}(t)$ satisfying conditions (47) under the following transformation:

$$t = \frac{z^1}{z^2} \mapsto \begin{cases} \alpha t, \\ -\dfrac{\alpha}{t}. \end{cases} \tag{49}$$

The first condition in expression (47) is equivalent to

$$\begin{aligned} P^{(8)}(\alpha t) &= P^{(8)}(t), \\ P_{\mathcal{V}}^{(4)}(\alpha t) &= \alpha^2 P_{\mathcal{V}}^{(4)}(t), \\ P_{\mathcal{D}}^{(4)}(\alpha t) &= \alpha^2 P_{\mathcal{D}}^{(4)}(t). \end{aligned} \tag{50}$$

One can check directly that the first condition in expression (50) is equivalent to

$$P^{(8)}(t) = a_8 t^8 + a_4 t^4 + \overline{a_8}. \tag{51}$$

Similarly, the last two conditions in expression (50) are equivalent to

$$P^{(4)}(t) = a_2 t^2. \tag{52}$$

The last condition in expression (47), where $\alpha = 1$ and $q = 4$, implies that $\mathrm{Im}(a_8) = 0$. From conditions (51) for $P^{(8)}$ and (52) for both $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$, we conclude that an elasticity tensor has tetragonal symmetry if and only if – with respect to an orthonormal basis – the following coefficients vanish:

$$\mathrm{Im}(a_8) = a_7 = a_6 = a_5 = a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_4^{\mathcal{D}} = a_3^{\mathcal{D}} = 0. \tag{53}$$

Since the above coefficients are complex, condition (53) gives us fifteen independent equations. By solving these equations, we see that an elasticity tensor has tetragonal symmetry if and only if it has – with respect to an orthonormal basis – six independent components given by expression (14). □

NOTE 6.1. For the trigonal and tetragonal symmetries, we have a common set of the following fourteen equations:

$$\operatorname{Im}(a_8) = \operatorname{Re}(a_7) = a_6 = a_5 = a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_4^{\mathcal{D}} = a_3^{\mathcal{D}} = 0.$$

Each symmetry case has one specific equation; namely, $\operatorname{Re}(a_8) = 0$ for trigonal, and $\operatorname{Im}(a_7) = 0$ for tetragonal.

THEOREM 6.5. *An elasticity tensor has transversely isotropic symmetry if and only if the three polynomials* $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$ *satisfy the following conditions:*

$$\begin{aligned} P^{(2q)}(t) &= z^q P^{(2q)}\left(\frac{t}{z}\right), \qquad q \in \{2, 4\}, \\ P^{(2q)}(t) &= z^q t^{2q} P^{(2q)}\left(-\frac{z}{t}\right), \quad q \in \{2, 4\}, \end{aligned} \tag{54}$$

where $z = \mathrm{e}^{\iota\theta}$ *is any complex number of modulus one, and* $P^{(4)}$ *refers to both polynomials* $P_{\mathcal{V}}^{(4)}$ *and* $P_{\mathcal{D}}^{(4)}$.

Proof. An elasticity tensor has transversely isotropic symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ are invariant under transformations given by expression (16), which implies invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(\mathrm{e}^{\iota\theta/2} z^1, \mathrm{e}^{-\iota\theta/2} z^2), \\ \pm(\mathrm{e}^{\iota\theta/2} z^2, -\mathrm{e}^{-\iota\theta/2} z^1). \end{cases} \tag{55}$$

The invariance of $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathcal{D}}(z^1, z^2)$ under maps (55) is equivalent to polynomials $P^{(4)}(t)$, $P_{\mathcal{V}}^{(4)}(t)$ and $P_{\mathcal{D}}^{(4)}(t)$ satisfying condition (54) under the following transformation:

$$t = \frac{z^1}{z^2} \mapsto \begin{cases} \mathrm{e}^{\iota\theta} t, \\ -\dfrac{\mathrm{e}^{\iota\theta}}{t}. \end{cases} \tag{56}$$

For $P_{\mathcal{V}}^{(4)}$ and $P_{\mathcal{D}}^{(4)}$, the first condition in expression (54) is the same as for the trigonal and tetragonal cases, namely, $P_{\mathcal{V}}^{(4)}(t) = a_2^{(\mathcal{V})} t^2$ and $P_{\mathcal{D}}^{(4)}(t) = a_2^{(\mathcal{D})} t^2$. This implies the following eight equations:

$$a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_4^{\mathcal{D}} = a_3^{\mathcal{D}} = 0. \tag{57}$$

For $P^{(8)}$, the first condition in expression (54) is equivalent to $P^{(8)}(t) = a_4 t^4$, which implies eight more equations, namely,

$$a_8 = a_7 = a_6 = a_5 = 0. \tag{58}$$

The second condition in expression (54) does not imply any new restriction. Using equations (57) and (58), we see that an elasticity tensor has transversely isotropic symmetry if and only if it has – with respect to an orthonormal basis – five independent components given by expression (17). □

NOTE 6.2. We see that the sixteen equations that define a transversely isotropic symmetry are the combined equations for the trigonal and tetragonal symmetries.

THEOREM 6.6. *An elasticity tensor has cubic symmetry if and only if the two polynomials $P_{\mathcal{V}}^{(4)}$ and $P_{\mathfrak{D}}^{(4)}$ vanish, while the polynomial $P^{(8)}$ satisfies the following conditions:*

$$\begin{aligned} P^{(2q)}(t) &= \alpha^q P^{(2q)}\left(\frac{t}{\alpha}\right), \\ P^{(2q)}(t) &= \alpha^q t^{2q} P^{(2q)}\left(-\frac{\alpha}{t}\right), \\ 2^q P^{(2q)}(t) &= (t-\alpha)^{2q} P^{(2q)}\left(\frac{\beta}{a}\frac{t+\alpha}{t-\alpha}\right), \end{aligned} \tag{59}$$

where $\alpha^4 = 1$ and $\beta^2 = 1$.

Proof. An elasticity tensor has cubic symmetry if and only if $2^q P^{(2q)}(t) = (P(z^1, z^2), P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathfrak{D}}(z^1, z^2)$ are invariant under transformation (18), which implies the invariance under the following maps:

$$(z^1, z^2) \mapsto \begin{cases} \pm(z^1, z^2), \\ \pm(\iota z^2, \iota z^1), \\ \pm(z^2, -z^1), \\ \pm(\iota z^1, -\iota z^2), \\ \pm(\mathrm{e}^{\pm\iota\pi/4} z^1, \mathrm{e}^{\mp\iota\pi/4} z^2), \\ \pm(\mathrm{e}^{\pm\iota\pi/4} z^2, -\mathrm{e}^{\mp\iota\pi/4} z^1), \\ \pm\left(\cos\left(\frac{\pi}{4}\right)z^1 \pm \iota\sin\left(\frac{\pi}{4}\right)z^2, \pm\iota\sin\left(\frac{\pi}{4}\right)z^1 + \cos\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\cos\left(\frac{\pi}{4}\right)z^1 \mp \sin\left(\frac{\pi}{4}\right)z^2, \pm\sin\left(\frac{\pi}{4}\right)z^1 + \cos\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\mp\sin\left(\frac{\pi}{4}\right)z^1 + \iota\cos\left(\frac{\pi}{4}\right)z^2, \iota\cos\left(\frac{\pi}{4}\right)z^1 \mp \sin\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\pm\iota\sin\left(\frac{\pi}{4}\right)z^1 + \cos\left(\frac{\pi}{4}\right)z^2, -\cos\left(\frac{\pi}{4}\right)z^1 \mp \iota\sin\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\iota\cos\left(\frac{\pi}{4}\right)z^1 \mp \sin\left(\frac{\pi}{4}\right)z^2, \pm\sin\left(\frac{\pi}{4}\right)z^1 - \iota\cos\left(\frac{\pi}{4}\right)z^2\right), \end{cases} \tag{60_1}$$

$$(z^1, z^2) \mapsto \begin{cases} \pm\left(\pm\iota \sin\left(\frac{\pi}{4}\right)z^1 + \iota\cos\left(\frac{\pi}{4}\right)z^2, \iota\cos\left(\frac{\pi}{4}\right)z^1 \mp \iota\sin\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\pm \sin\left(\frac{\pi}{4}\right)z^1 + \cos\left(\frac{\pi}{4}\right)z^2, -\cos\left(\frac{\pi}{4}\right)z^1 \pm \sin\left(\frac{\pi}{4}\right)z^2\right), \\ \pm\left(\iota\cos\left(\frac{\pi}{4}\right)z^1 \mp \iota\sin\left(\frac{\pi}{4}\right)z^2, \mp\iota\sin\left(\frac{\pi}{4}\right)z^1 - \iota\cos\left(\frac{\pi}{4}\right)z^2\right). \end{cases} \quad (60_2)$$

The invariance of $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathfrak{D}}(z^1, z^2)$ under maps (60) implies that the two polynomials vanish. The invariance of $P(z^1, z^2)$ under maps (60) is equivalent to the fact that polynomial $P^{(8)}(t)$ satisfies conditions (59). The two polynomials $P_{\mathcal{V}}^{(4)}$ and $P_{\mathfrak{D}}^{(4)}$ vanish if and only if

$$a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_2^{\mathcal{V}} = a_4^{\mathfrak{D}} = a_3^{\mathfrak{D}} = a_2^{\mathfrak{D}} = 0. \quad (61)$$

The first two conditions in expression (59) are the same as conditions (47). In view of expression (53), this is equivalent to

$$\mathrm{Im}(a_8) = a_7 = a_6 = a_5 = 0. \quad (62)$$

We can infer, from the last condition in expression (59) – for instance, by setting $t = 1$ and $\alpha = -1$ – that

$$a_4 = 14a_8. \quad (63)$$

The ten equations (61), the seven equations (62) and equation (63) bring the number of independent coefficients to three, as given by expression (19). □

THEOREM 6.7. *An elasticity tensor has isotropic symmetry if and only if the three polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathfrak{D}}^{(4)}$ vanish.*

Proof. An elasticity tensor has isotropic symmetry if and only if $P(z^1, z^2)$, $P_{\mathcal{V}}(z^1, z^2)$ and $P_{\mathfrak{D}}(z^1, z^2)$ are invariant under any special unitary transformation. This invariance implies that polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathfrak{D}}^{(4)}$ vanish.

The three polynomials $P^{(8)}$, $P_{\mathcal{V}}^{(4)}$ and $P_{\mathfrak{D}}^{(4)}$ vanish if and only if all coefficients vanish, namely,

$$a_8 = a_7 = a_6 = a_5 = a_4 = a_4^{\mathcal{V}} = a_3^{\mathcal{V}} = a_2^{\mathcal{V}} = a_4^{\mathfrak{D}} = a_3^{\mathfrak{D}} = a_2^{\mathfrak{D}} = 0.$$

These are nineteen equations that bring the number of independent components for an isotropic tensor to two, as given by expression (20). □

7. Summary

The elasticity tensor that possesses given symmetries must satisfy the following equations with the indicated parameters:

$P^{(2q)}(t) =$	$\varepsilon^q P^{(2q)}(t/\varepsilon)$	$\omega^q t^{2q} P^{(2q)}(-\omega/t)$	$2^{-q}(t-\alpha)^{2q} P^{(2q)}\left(\frac{\beta t+\alpha}{at-\alpha}\right)$
Monoclinic	$\varepsilon^2 = 1$		
Orthotropic	$\varepsilon^2 = 1$	$\omega^2 = 1$	
Trigonal	$\varepsilon^3 = 1$	$\omega^3 = 1$	
Tetragonal	$\varepsilon^4 = 1$	$\omega^4 = 1$	
Trans. isotropic	$\lvert\varepsilon\rvert = 1$	$\lvert\omega\rvert = 1$	
Cubic	$\varepsilon^4 = 1$	$\omega^4 = 1$	$\alpha^4 = 1, \beta^2 = 1$
Isotropic	$\lvert\varepsilon\rvert = 1$	$\lvert\omega\rvert = 1$	$\lvert\alpha\rvert = 1, \lvert\beta\rvert = 1$

The equations in the first column correspond to rotations about e_3. The equations in the second column correspond to reflections about the planes that contain e_3. The equations in the last column correspond to rotations about e_1 and e_2 by $\pi/2$.

If polynomial $P^{(2q)}(t)$ is of the form

$$P^{(2q)}(t) = a_0 + a_1 t + \cdots + a_{2q} t^{2q},$$

then the first equation translates to

$$\begin{aligned} a_0 &= \varepsilon^q a_0, \\ a_1 &= \varepsilon^{q-1} a_1, \\ &\vdots \\ a_n &= \varepsilon^{q-n} a_n \end{aligned}$$

and the second equation translates to

$$\begin{aligned} a_0 &= \omega^{-q} a_{2q}, \\ a_1 &= -\omega^{1-q} a_{2q-1}, \\ &\vdots \\ a_n &= (-1)^n \omega^{n-q} a_{2q-n}. \end{aligned}$$

From these expressions for the coefficients, we can see the following statements.*

For a monoclinic medium, only the first equation has to be satisfied with $\varepsilon^2 = 1$. Since q is an even number, we see that all odd coefficients have to be zero.

For an orthotropic medium, the first equation has to be satisfied with $\varepsilon^2 = 1$ and the second equation has to be satisfied with $\omega^2 = 1$. From the first equation we see that only the even coefficients are nonzero. From the second equation we see that $a_{2k} = a_{2q-2k}$.

* Note that the first equation does not yield any restriction on the coefficient a_n for which $n = q$, since in that case $\varepsilon^{n-q} = 1$.

For a trigonal medium, the first equation has to be satisfied with $\varepsilon^3 = 1$ and the second equation has to be satisfied with $\omega^3 = 1$. From the first equation we see that the only nonzero coefficients a_k are those for which $q - k$ is divisible by three. From the second equation we see that for these nonzero coefficients $a_k = (-1)^k \omega^{k-q} a_{2q-n}$.

For a tetragonal medium, the first equation has to be satisfied with $\varepsilon^4 = 1$ and the second equation has to be satisfied with $\omega^4 = 1$. We see that the only nonzero coefficients a_k are those for which $q - k$ is divisible by four. From the second equation we see that for these nonzero coefficients $a_k = (-1)^k \omega^{k-q} a_{2q-n}$.

For a transversely isotropic medium, the first equation has to be satisfied with $|\varepsilon| = 1$. We see that a nonzero coefficient is possible only if $n = q$, when $\varepsilon^{n-q} = 1$.

For a cubic medium, the first equation has to be satisfied with $\varepsilon^4 = 1$ and the second equation has to be satisfied with $\omega^4 = 1$. Also the third equation has to be satisfied with $\alpha^4 = 1$ and $\beta^2 = 1$. We see that the only nonzero coefficients a_k are those for which $q - k$ is divisible by four. From the second equation we see that for these nonzero coefficients $a_k = (-1)^k \omega^{k-q} a_{2q-n}$ and these coefficients must also satisfy the third equation, which can be easily checked by choosing, for instance, $t = 1$ and $\alpha = -1$.

8. Discussions

Using the summary of results in Section 7 and the proofs in Section 6, we can state all possible routes of increasing symmetries for an elasticity tensor, as shown in Figure 1. The two numbers that follow the names of given symmetries are the number of the independent components of the corresponding elasticity tensor and the number of the elements in the corresponding symmetry group.

The method described in this paper can also be used for another proof of the fact that an elasticity tensor cannot have discrete symmetry that is greater than

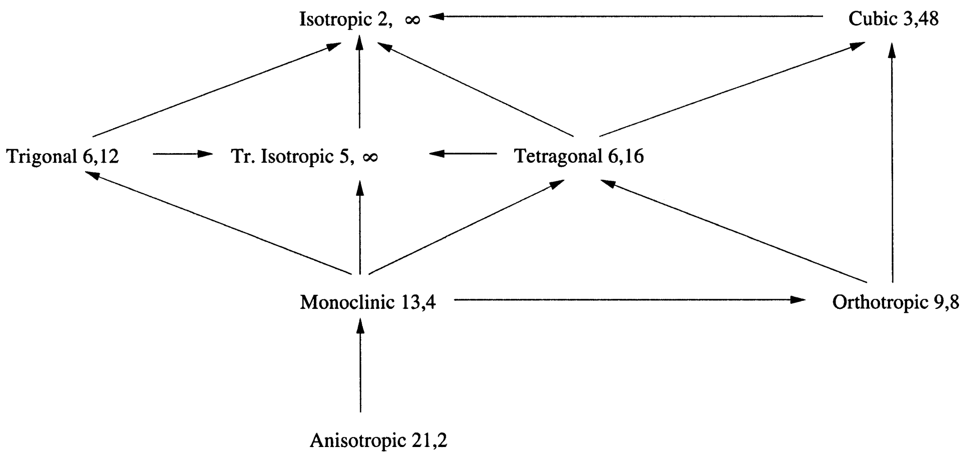

Figure 1.

four-fold. It is easily extendible for higher-rank tensors to obtain the result that was proved in 1945 by Hermann [11] in a different way.

The work presented in this paper is expressed in an arbitrary coordinate system, except in Section 6. The transformations in Section 6 are expressed in coordinates that are natural for the given symmetry class. This does not mean, however, that we would not obtain the same characterization using different coordinates. We would obtain expressions that characterize each class, but they would be more difficult to work with.

Acknowledgements

We would like to acknowledge fruitful comments by Michael Rochester and the anonymous reviewer.

This study has been done as part of the Geomechanics Project. Hence, the authors acknowledge EnCana Energy and Husky Energy for their support. Also, I. Bucataru acknowledges CNCSIS grant from the Romanian Ministry of Education for its partial support.

References

1. G. Backus, A geometrical picture of anisotropic elastic tensors. *Rev. Geophys. Space Phys.* **8**(3) (1970) 633–671.
2. R. Baerheim, Coordinate free representation of the hierarchically symmetric tensor of rank 4 in determination of symmetry. PhD Thesis. *Geologica Ultraiectina* **159** (1998).
3. A. Bóna, I. Bucataru, M.A. Slawinski, Material symmetries of elasticity tensors. *Quart. J. Mech. Appl. Math.* (in press).
4. L.P. Bos, P.C. Gibson, M. Kotchetov and M.A. Slawinski, Classes of anisotropic media: A tutorial. *Studio Geophysica Geodaetica* **48** (2004) 265–287.
5. P. Chadwick, M. Vianello and S.C. Cowin, A new proof that the number of linear elastic symmetries is eight. *J. Mech. Phys. Solids* **49** (2001) 2471–2492.
6. S.C. Cowin and M.M. Mehrabadi, On the identification of material symmetry for anisotropic elastic materials. *Quart. J. Mech. Appl. Math.* **40** (1987) 451–476.
7. S.C. Cowin and M.M. Mehrabadi, The structure of the linear anisotropic elastic symmetries. *Mech. Phys. Solids* **40**(7) (1992) 1459–1471.
8. M. Crampin and F.A.E. Pirani, *Applicable Differentiable Geometry*. Cambridge Univ. Press, Cambridge (1994).
9. S. Forte and M. Vianello, Symmetry classes for elasticity tensors. *J. Elasticity* **43**(2) (1996) 81–108.
10. S. Forte and M. Vianello, Symmetry classes and harmonic decomposition for photoelasticity tensors. *Internat. J. Engrg. Sci.* **35**(14) (1997) 1317–1326.
11. B. Herman, Some theorems of the theory of anisotropic media. *Comptes Rendus (Doklady) Acad. Sci. URSS* **48**(2) (1945) 89–92.
12. Y.Z Huo and G. del Piero, On the completeness of the crystallographic symmetries in the description of the symmetries of the elastic tensor. *J. Elasticity* **25** (1991) 203–246.
13. M.A. Slawinski, *Seismic Rays and Waves in Elastic Media*. Pergamon, New York (2003).
14. T.C.T. Ting, Generalized Cowin-Mehrabadi theorems and a direct proof that the number of linear elastic symmetries is eight. *Internat. J. Solids Struct.* **40** (2003) 7129–7142.

15. H. Xiao, On symmetries and anisotropies of classical and micropolar linear elasticities: A new method based upon a complex vector basis and some systematic results. *J. Elasticity* **49** (1998) 129–162.
16. W.N. Zou and Q.S. Zheng, Maxwell's multipole representation of traceless symmetric tensors and its application to function of high-order tensors. *Proc. Roy. Soc. London* **459** (2003) 527–538.