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Spatial Stability for the Quasi-static Problem of Thermoelasticity

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Dedicated to C.O. Horgan on the occasion of his 60th birthday

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Abstract. This paper is concerned with the quasi-static problem of thermoelasticity. The classical system of equations of thermoelasticity is a coupling of an elliptic equation with a parabolic equation. It poses some new mathematical difficulties. Here we study the exponential spatial decay of solutions. An upper bound for the amplitude in terms of the boundary and initial conditions is obtained. The extension of the spatial stability results to thermoelasticity of type III is also treated.

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1. Introduction

Many studies on Saint-Venant's principle in elasticity and spatial decay estimates have been developed since the 1960's. Pioneering contributions were focused on elliptic and parabolic problems. In the 1990s much activity was concerned with hyperbolic problems. The history of this question is explained in [13] and up-dated by Horgan in [11, 12].

Here we are concerned with some thermoelastic theories. In [14] Lupoli obtained a spatial decay estimate for the static problem. We also recall the contribution by Bofill and Quintanilla [2]. The system of equations that governs this problem is a coupling of two elliptic equations. A different question is the dynamical one. In this case the couple is formed by a hyperbolic equation and a parabolic equation. In a recent contribution [16], the author proved that the spatial behavior of solutions of the dynamical problem of classical thermoelasticity is identical to that of the parabolic problem. Some extensions of these arguments to other thermoelastic theories have been proposed recently [17, 18]. But there are no contributions concerning quasi-static deformations. The quasi-static problem corresponds to the study of small thermoelastic deformations. The point of view is that the acceleration is so small that we can consider it to be zero. In the classical thermoelastic case the quasi-static problem corresponds to the coupling of an elliptic and a parabolic equation. This question is new in the context of spatial stability and poses new analytical difficulties. This kind of coupling has not been considered previously in the study of spatial decay estimates and so these issues for the quasi-static problems of thermoelasticity are worthy of investigation.

The major analytical difficulty is the combination of the terms $\beta_{ij}\theta$ in the mechanical equation and $\beta_{ij}\dot{u}_{i,j}$ in the heat equation (see (2.5), (2.6)). We overcome this difficulty by means of an idea that is similar to the one used when the logarithmic convexity method is applied to several thermoelastic theories [1, 15, 20]. The idea is to integrate the heat equation with respect to time.

In recent years an intense activity has been developed on alternative formulations of heat conduction in thermomechanics. Two recent surveys on this topic are the works of Chandrasekharaiah [3] and Hetnarski and Ignaczak [10]. In these references the authors recall several theories developed by different authors. In particular the theories proposed by Green and Naghdi [6–9] are considered. These theories are based on an equality involving the entropy rather than an inequality as is considered in classical thermoelasticity. Several results concerning these theories have been obtained recently [15, 19, 20]. Thus, it is natural to consider these alternative theories when we study thermoelasticity. It is worth recalling another theory proposed by Green and Lindsay [5]. We have tried to extend our arguments to other non classical thermoelastic theories. We have been able to extend our arguments to thermoelasticity of type III [7], but we cannot overcome the difficulties in the generalized thermoelasticity proposed by Green and Lindsay [5] nor in thermoelasticity without energy dissipation [8].

In this paper we study the quasi-static problem of thermoelastic deformations in the classical theory of thermoelasticity in two or three dimensions and in thermoelasticity of type III. In Section 2, we set down the classical thermomechanical problem. In Section 3 a spatial estimate for the solutions is obtained. An upper bound for the amplitude term is obtained in Section 4. In the last section we sketch the extension of the arguments to thermoelasticity of type III.

In this article we use the summation and differentiation conventions. Summation over repeated indices is assumed and we use subscripts preceded by a comma for partial differentiation with respect to the corresponding coordinate. Letters in boldface stand for vectors or tensors. A superposed dot denotes the derivative with respect to the time. Greek subscripts mean that they are restricted to the values two and three.

2. Preliminaries

We consider a thermoelastic beam that occupies the region $B = [0, \infty) \times D$ in the n-dimensional ($n = 2, 3$) Euclidean space, where D is a bounded domain in the $(n-1)$-dimensional space. We assume that B is sufficiently regular to apply the divergence theorem. In the two-dimensional case we consider a rectangular strip with axis along the x_1-direction, but in the three-dimensional case B means a cylinder with axis along the x_1-direction.

We consider the quasi-static problem. Thus, the displacement $(u_i(\mathbf{x}, t))$ and the temperature $\theta(\mathbf{x}, t)$ satisfy the system:

$$\tau_{ij,j} = 0, \tag{2.1}$$

$$\rho\frac{\partial\eta}{\partial t} = q_{k,k}. \tag{2.2}$$

Here τ_{ij} is the stress tensor, ρ is the mass density which is assumed to be a strictly positive function of the material points, η is the entropy and q_k is the heat flux vector.

We assume that the material is linear. Thus, we have the constitutive equations:

$$\tau_{ij} = C_{ijrs}u_{s,r} - \beta_{ij}\theta, \qquad \rho\eta = \beta_{ij}u_{i,j} + c\theta, \qquad q_i = k_{ij}\theta_{,j}. \tag{2.3}$$

The constitutive functions C_{ijrs}, β_{ij}, c and k_{ij} depend on the material points. They are assumed to be bounded above in B. We assume that the elasticity tensor and the thermal conductivity tensor satisfy the symmetries

$$C_{ijrs} = C_{srji}, \qquad k_{ij} = k_{ji}. \tag{2.4}$$

If we substitute the constitutive equations (2.4) into the evolution equations (2.1), (2.2) we obtain the system

$$(C_{ijrs}u_{s,r} - \beta_{ij}\theta)_{,j} = 0, \tag{2.5}$$
$$c\dot{\theta} = -\beta_{ij}\dot{u}_{i,j} + (k_{ij}\theta_{,i})_{,j}. \tag{2.6}$$

To complete the problem, we need to impose some boundary conditions:

$$u_i = 0, \qquad \theta = 0, \qquad [0,\infty) \times \partial D \times [0,\infty), \tag{2.7}$$
$$u_i = u_i^0(x_2, x_3, t), \qquad \theta = \theta^0(x_2, x_3, t), \qquad \{0\} \times D \times [0,\infty), \tag{2.8}$$

and initial conditions:

$$u_i(\mathbf{x}, 0) = \theta(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in B. \tag{2.9}$$

We assume the following compatibility conditions:

$$u_i^0(x_2, x_3, t) = \theta^0(x_2, x_3, t) = 0 \quad \text{when } (x_2, x_3) \in \partial D. \tag{2.10}$$

In the next section we obtain a decay estimate for the solutions of the problem determined by (2.5)–(2.9). We assume that there exist a positive constant C_2 such that

$$C_2 C_{ijkl}u_{i,j}u_{l,k} \geqslant u_{i,j}u_{i,j}. \tag{2.11}$$

In this paper we also assume that the thermal conductivity tensor k_{ij} is positive definite with smallest eigenvalue uniformly bounded below by $\bar{k}$ and that the heat capacity $c(\mathbf{x})$ is uniformly bounded below by a positive constant constant $\hat{c}$.

It is worth noting that the previous conditions imply the existence of a positive constant C_1 such that

$$\tau_{i1}\tau_{i1} \leqslant C_1(C_{ijkl}u_{i,j}u_{l,k} + c\theta^2). \tag{2.12}$$

3. Spatial Stability

In this section we prove exponential stability of the solutions of the problem determined by (2.5)–(2.9). To this end, we calculate the integral with respect to time of equation (2.6). We have

$$c\theta = -\beta_{ij}u_{i,j} + \int_0^t (k_{ij}\theta_{,i}(s))_{,j}\,\mathrm{d}s. \tag{3.1}$$

If we denote

$$\psi(\mathbf{x}, t) = \int_0^t \theta(\mathbf{x}, s)\,\mathrm{d}s, \tag{3.2}$$

equation (3.1) becomes

$$c\theta = -\beta_{ij}u_{i,j} + (k_{ij}\psi_{,i})_{,j}. \tag{3.3}$$

We define the function

$$F(z, t) = \int_0^t \int_0^s \int_{D(z)} (\tau_{i1}u_i + k_{i1}\psi_{,i}\theta)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s, \tag{3.4}$$

where $D(z) = \{\mathbf{x} \in B, x_1 = z\}$. Using the divergence theorem we see that

$$\begin{aligned} F(z+h, t) - F(z, t) &= \int_0^t \int_0^s \int_{B(z+h,z)} (C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 + k_{ij}\psi_{,i}\theta_{,j})\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \int_0^t \int_0^s \int_{B(z+h,z)} (C_{ijrs}u_{i,j}u_{s,r} + c\theta^2)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \int_{B(z+h,z)} k_{ij}\psi_{,i}\psi_{,j}\,\mathrm{d}v\,\mathrm{d}s, \end{aligned} \tag{3.5}$$

where $B(z+h, z) = \{\mathbf{x} \in B, z+h > x_1 > z\}$. Dividing by h and taking the limit as h tends to zero we have

$$\begin{aligned} \frac{\partial F}{\partial z} &= \int_0^t \int_0^s \int_{D(z)} (C_{ijrs}u_{i,j}u_{s,r} + c\theta^2)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \int_{D(z)} k_{ij}\psi_{,i}\psi_{,j}\,\mathrm{d}a\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Our next step is to estimate the function F in terms of its spatial derivative. We have:

$$|F(z,t)| \leqslant \left|\int_0^t\int_0^s\int_{D(z)} \tau_{i1}u_i\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right| + \left|\int_0^t\int_0^s\int_{D(z)} k_{i1}\psi_{,i}\theta\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right|. \tag{3.7}$$

In order to bound the first integral, we have that

$$\begin{aligned}&\left|\int_0^t\int_0^s\int_{D(z)} \tau_{i1}u_i\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right|\\ &\quad\leqslant \left(\int_0^t\int_0^s\int_{D(z)} \tau_{i1}\tau_{i1}\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_{D(z)} u_iu_i\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\quad\leqslant \left(\int_0^t\int_0^s\int_{D(z)} C_1(C_{ijkl}u_{i,j}u_{l,k}+c\theta^2)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\quad\quad\times\left(\lambda_1^{-1}\int_0^t\int_0^s\int_{D(z)} C_2C_{ijkl}u_{i,j}u_{l,k}\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\quad\leqslant (\lambda_1^{-1}C_1C_2)^{1/2}\frac{\partial F}{\partial z}.\end{aligned} \tag{3.8}$$

Here λ_1 is the first eigenvalue of the problem

$$\Delta u + \lambda u = 0, \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{3.9}$$

To bound the second integral on the right-hand side of (3.7), we have

$$\begin{aligned}&\left|\int_0^t\int_0^s\int_{D(z)} k_{i1}\psi_{,i}\theta\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right|\\ &\quad\leqslant \left(\int_0^t\int_{D(z)} (t-s)\psi_{,i}\psi_{,i}\,\mathrm{d}a\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_{D(z)} k_{i1}k_{i1}\theta^2\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\quad\leqslant \left(t\int_0^t\int_{D(z)} \psi_{,i}\psi_{,i}\,\mathrm{d}a\,\mathrm{d}s\right)^{1/2}\left(\hat{k}\int_0^t\int_0^s\int_{D(z)} \theta^2\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}.\\ &\quad\leqslant \frac{3}{2}\left(\frac{\hat{k}t}{\hat{c}\bar{k}}\right)^{1/2}\frac{\partial F}{\partial z}.\end{aligned} \tag{3.10}$$

Here $\hat{k}$ is the maximum of $k_{i1}k_{i1}$.

From (3.6)–(3.8) and (3.10) we can compute two positive constants

$$M_1 = (\lambda_1^{-1}C_1C_2)^{1/2}, \qquad M_2 = \frac{3}{2}\left(\frac{\hat{k}}{\hat{c}\bar{k}}\right)^{1/2}$$

such that

$$|F(z,t)| \leqslant \left(M_1 + M_2t^{1/2}\right)\frac{\partial F}{\partial z}. \tag{3.11}$$

The inequality (3.11) is well known in the study of spatial decay estimates (see [4]). We can conclude that either there exists z_0 such that $F(z_0, t) > 0$ and

$$F(z,t) \geqslant F(z_0,t)\exp\left(\frac{1}{M_1 + M_2 t^{1/2}}(z - z_0)\right), \quad z \geqslant z_0, \tag{3.12}$$

or we obtain the spatial estimate

$$-F(z,t) \leqslant (-F(0,t))\exp\left(-\frac{1}{M_1 + M_2 t^{1/2}} z\right), \quad z \geqslant 0. \tag{3.13}$$

We have proved:

THEOREM 3.1. *Let* (u_i, θ) *be a solution of the problem determined by the system* (2.5)*,* (2.6)*, the boundary conditions* (2.7)*,* (2.8) *and the initial conditions* (2.9)*. Then, either the asymptotic condition* (3.12) *holds or the following inequality*

$$\begin{aligned}
&\int_0^t \int_0^s \int_{B(\infty,z)} (C_{ijrs} u_{i,j} u_{s,r} + c\theta^2)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{B(\infty,z)} k_{ij}\psi_{,i}\psi_{,j}\,\mathrm{d}v\,\mathrm{d}s \\
&\quad \leqslant \left(\int_0^t \int_0^s \int_B (C_{ijrs} u_{i,j} u_{s,r} + c\theta^2)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \right.\\
&\qquad \left. + \frac{1}{2}\int_0^t \int_B k_{ij}\psi_{,i}\psi_{,j}\,\mathrm{d}v\,\mathrm{d}s\right)\exp\left(-\frac{z}{M_1 + M_2 t^{1/2}}\right),
\end{aligned} \tag{3.14}$$

is satisfied for $z \geqslant 0$.

4. The Amplitude Term

The aim of this section is to give an upper bound for the amplitude term multiplying the exponential on the right-hand side of (3.14), i.e. $-F(0,t)$ by means of the initial and boundary conditions. Let (v_i) and ϑ be two arbitrary functions that satisfy the boundary and initial conditions (2.7)–(2.9) and tends to zero as x_1 goes to infinity. We have

$$-F(0,t) = -\int_0^t \int_0^s \int_{D(0)} (\tau_{i1} v_i + k_{i1}\psi_{,i}\vartheta)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s. \tag{4.1}$$

Using the divergence theorem, we obtain

$$\begin{aligned}
-F(0,t) = \int_0^t \int_0^s \int_B (&C_{ijkl} u_{i,j} v_{l,k} - \beta_{ij}\theta v_{i,j} + c\theta\vartheta + \beta_{ij}\vartheta u_{i,j} \\
&+ k_{ij}\psi_{,j}\vartheta_{,i})\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned} \tag{4.2}$$

Repeated use of the Holder inequality gives

$$\begin{aligned}\int_0^t\int_0^s\int_B & C_{ijkl}u_{i,j}v_{l,k}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\\ &\leqslant \left(\int_0^t\int_0^s\int_B C_{ijkl}u_{i,j}u_{l,k}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B C_{ijkl}v_{i,j}v_{l,k}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \big(-F(0,t)\big)^{1/2}\left(\int_0^t\int_0^s\int_B C_{ijkl}v_{i,j}v_{l,k}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\int_0^t\int_0^s\int_B & c\theta\vartheta\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\\ &\leqslant \left(\int_0^t\int_0^s\int_B c\theta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B c\vartheta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \big(-F(0,t)\big)^{1/2}\left(\int_0^t\int_0^s\int_B c\vartheta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2},\end{aligned} \tag{4.4}$$

$$\begin{aligned}-\int_0^t\int_0^s\int_B & \beta_{ij}\theta v_{i,j}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\\ &\leqslant \left(\int_0^t\int_0^s\int_B \beta_{ij}\beta_{ij}\theta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B v_{i,j}v_{i,j}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \beta^{1/2}\big(-F(0,t)\big)^{1/2}\left(\int_0^t\int_0^s\int_B v_{i,j}v_{i,j}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\int_0^t\int_0^s\int_B & \beta_{ij}\vartheta u_{i,j}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\\ &\leqslant \left(\int_0^t\int_0^s\int_B \beta_{ij}\beta_{ij}\vartheta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B u_{i,j}u_{i,j}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant C_2^{1/2}\big(-F(0,t)\big)^{1/2}\left(\int_0^t\int_0^s\int_B \beta_{ij}\beta_{ij}\vartheta^2\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\int_0^t\int_0^s\int_B & k_{ij}\psi_{,j}\vartheta_{,i}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\\ &\leqslant \left(\int_0^t\int_0^s\int_B k_{ij}\psi_{,j}\psi_{,i}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B k_{ij}\vartheta_{,j}\vartheta_{,i}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \left(t\int_0^t\int_B k_{ij}\psi_{,j}\psi_{,i}\,\mathrm{d}v\,\mathrm{d}s\right)^{1/2}\left(\int_0^t\int_0^s\int_B k_{ij}\vartheta_{,j}\vartheta_{,i}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \big(-tF(0,t)\big)^{1/2}\left(2\int_0^t\int_0^s\int_B k_{ij}\vartheta_{,j}\vartheta_{,i}\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\right)^{1/2}.\end{aligned} \tag{4.7}$$

In (4.5) the parameter β can be obtained in terms of the tensor β_{ij} and c. From (4.1)–(4.7), we have that

$$-F(0,t) \leqslant \big(-F(0,t)\big)^{1/2}\big(I_1^{1/2} + I_2^{1/2} + \beta^{1/2} I_3^{1/2} + C_2^{1/2} I_4^{1/2} + (2t)^{1/2} I_5^{1/2}\big),$$

where

$$I_1 = \int_0^t \int_0^s \int_B C_{ijkl} v_{i,j} v_{l,k} \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.8}$$

$$I_2 = \int_0^t \int_0^s \int_B c\vartheta^2 \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.9}$$

$$I_3 = \int_0^t \int_0^s \int_B v_{i,j} v_{i,j} \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.10}$$

$$I_4 = \int_0^t \int_0^s \int_B \beta_{ij}\beta_{ij}\vartheta^2 \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.11}$$

$$I_5 = \int_0^t \int_0^s \int_B k_{ij}\vartheta_{,j}\vartheta_{,i} \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s. \tag{4.12}$$

Thus, we can compute two positive constants $N_1 = 5(1 + \beta + C_2)$, $N_2 = 10$ (for instance) such that

$$-F(0,t) \leqslant (N_2 t + N_1)(I_1 + I_2 + I_3 + I_4 + I_5). \tag{4.13}$$

There exist constants C^*, c^*, β^*, k^* such that the following inequalities

$$I_1 \leqslant C^* \int_0^t \int_0^s \int_B v_{i,j} v_{i,j} \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.14}$$

$$I_2 \leqslant c^* \int_0^t \int_0^s \int_B \vartheta^2 \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.15}$$

$$I_4 \leqslant \beta^* \int_0^t \int_0^s \int_B \vartheta^2 \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s, \tag{4.16}$$

$$I_5 \leqslant k^* \int_0^t \int_0^s \int_B \vartheta_{,i}\vartheta_{,i} \, \mathrm{d}v \, \mathrm{d}\tau \, \mathrm{d}s \tag{4.17}$$

are satisfied.

Now, we choose

$$v_i(\mathbf{x},t) = \exp(-\nu x_1) u_i^0(x_2, x_3, t),$$
$$\vartheta(\mathbf{x},t) = \exp(-\nu x_1)\theta^0(x_2, x_3, t), \tag{4.18}$$

where ν is an arbitrary positive constant and the functions $u_i^0(x_2, x_3, t)$ and $\theta^0(x_2, x_3, t)$ are defined in (2.8). Thus, the functions v_i and ϑ satisfy the required conditions. We easily derive that

$$I_1 + I_3 \leqslant \frac{(1+C^*)}{2\nu} \int_0^t \int_0^s \int_{D(0)} \big(u^0_{i,\alpha} u^0_{i,\alpha} + \nu^2 u^0_i u^0_i\big)\, \mathrm{d}a\, \mathrm{d}\tau\, \mathrm{d}s, \tag{4.19}$$

$$I_2 + I_4 \leqslant \frac{(c^*+\beta^*)}{2\nu} \int_0^t \int_0^s \int_{D(0)} (\theta^0)^2\, \mathrm{d}a\, \mathrm{d}\tau\, \mathrm{d}s, \tag{4.20}$$

$$I_5 \leqslant \frac{k^*}{2\nu} \int_0^t \int_0^s \int_{D(0)} \big(\theta^0_{,\alpha}\theta^0_{,\alpha} + \nu^2(\theta^0)^2\big)\, \mathrm{d}a\, \mathrm{d}\tau\, \mathrm{d}s. \tag{4.21}$$

The estimates (4.19)–(4.21) and (4.13) give an upper bound for the amplitude term in terms of the boundary data.

5. Thermoelasticity of Type III

The aim of this section is to extend the arguments on spatial stability to thermoelasticity of type III. We recall that the equations which govern the quasi-static problem of thermoelasticity of type III are (see [7]):

$$(C_{ijrs}u_{s,r} - \beta_{ij}\theta)_{,j} = 0, \tag{5.1}$$

$$c\dot{\theta} = -\beta_{ij}\dot{u}_{i,j} + (k_{ij}\theta_{,i})_{,j} + (b_{ij}\alpha_{,i})_{,j}. \tag{5.2}$$

Here the variable α is defined by

$$\dot{\alpha} = \theta. \tag{5.3}$$

In view of equality (3.2) we note that

$$\alpha(\mathbf{x}, t) = \psi(\mathbf{x}, t) + \alpha(\mathbf{x}, 0).$$

From the point of view of Green and Naghdi α is regarded as representing some “mean” thermal displacement magnitude, and for brevity is usually referred to as the “thermal displacement”.

In this section, we assume conditions (2.11) and (2.12), and, in addition, we also assume that the tensor b_{ij} is positive definite. It is worth noting that this condition implies the existence of a positive constant γ such that the tensor $\gamma k_{ij} - 2b_{ij}$ is positive definite.

In order to define the problem, we need to impose initial and boundary conditions for the thermal displacement. We assume conditions (2.7)–(2.10) (but not $(2.8)_2$) and

$$u_i = 0, \qquad \alpha = 0, \quad [0,\infty) \times \partial D \times [0,\infty), \tag{5.4}$$

$$\alpha = \alpha^0(x_2, x_3, t), \qquad \{0\} \times D \times [0,\infty), \tag{5.5}$$

and initial conditions:

$$\alpha(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in B. \tag{5.6}$$

It is worth noting that in view of (5.6) we have that $\alpha(\mathbf{x}, t) = \psi(\mathbf{x}, t)$ and that we do not impose boundary conditions on the temperature because we impose the boundary conditions on α.

We assume compatibility conditions. Thus, we have that

$$\alpha^0(x_2, x_3, t) = 0 \quad \text{when } (x_2, x_3) \in \partial D. \tag{5.7}$$

Integrating Equation (5.2) with respect to time, we have

$$c\theta = -\beta_{ij} u_{i,j} + (k_{ij}\psi_{,i})_{,j} + \left(\int_0^t b_{ij}\psi_{,i}(s)\,\mathrm{d}s\right)_{,j}. \tag{5.8}$$

If we denote

$$\phi(\mathbf{x}, t) = \int_0^t \psi(\mathbf{x}, s)\,\mathrm{d}s, \tag{5.9}$$

Equation (5.8) becomes

$$c\theta = -\beta_{ij} u_{i,j} + (k_{ij}\psi_{,i})_{,j} + (b_{ij}\phi_{,i})_{,j}. \tag{5.10}$$

We define the function

$$F_\gamma(z, t) = \int_0^t \int_0^s \int_{D(z)} \exp(-\gamma\tau)\big(\tau_{i1}u_i + k_{i1}\psi_{,i}\theta + b_{i1}\phi_{,i}\theta\big)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s. \tag{5.11}$$

We have

$$\begin{aligned} &F_\gamma(z+h, t) - F_\gamma(z, t) \\ &\quad = \int_0^t \int_0^s \int_{B(z+h,z)} \exp(-\gamma\tau)\bigg(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 \\ &\qquad + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j} + \frac{\gamma^2}{2}b_{ij}\phi_{,i}\phi_{,j}\bigg)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \\ &\qquad + \int_0^t \int_{B(z+h,z)} \exp(-\gamma s)\left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}s \\ &\qquad + \int_{B(z+h,z)} \exp(-\gamma t)\frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}v. \end{aligned} \tag{5.12}$$

Thus

$$\begin{aligned} \frac{\partial F_\gamma}{\partial z} = {} & \int_0^t \int_0^s \int_{D(z)} \exp(-\gamma\tau)\bigg(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 \\ & + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j} + \frac{\gamma^2}{2}b_{ij}\phi_{,i}\phi_{,j}\bigg)\,\mathrm{d}a\,\mathrm{d}\tau\,\mathrm{d}s \\ & + \int_0^t \int_{D(z)} \exp(-\gamma s)\left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}a\,\mathrm{d}s \\ & + \int_{D(z)} \exp(-\gamma t)\frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}a. \end{aligned} \tag{5.13}$$

Now, we can use a similar argument to the one used for classical thermoelasticity. We can compute two positive constants M_3, M_4 such that

$$|F_\gamma(t,z)| \leqslant \left(M_3 + M_4 t^{1/2}\right)\frac{\partial F_\gamma}{\partial z}. \tag{5.14}$$

We can conclude that either there exists a z_0 such that $F_\gamma(z_0,t) > 0$ and

$$F_\gamma(z,t) \geqslant F_\gamma(z_0,t)\exp\left(\frac{1}{M_3 + M_4 t^{1/2}}(z - z_0)\right), \quad z \geqslant z_0, \tag{5.15}$$

or we obtain the spatial estimate

$$-F_\gamma(z,t) \leqslant (-F_\gamma(0,t))\exp\left(-\frac{1}{M_3 + M_4 t^{1/2}}z\right), \quad z \geqslant 0. \tag{5.16}$$

Estimates (5.15) and (5.16) describe the spatial behavior of the solutions of the quasi-static problem of the thermoelasticity of type III. It is worth noting that (5.16) can be written as

$$\begin{aligned}
&\int_0^t \int_0^s \int_{B(\infty,z)} \exp(-\gamma\tau)\left(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j}\right. \\
&\quad \left. + \frac{\gamma^2}{2}b_{ij}\phi_{,i}\phi_{,j}\right) \mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_0^t \int_{B(\infty,z)} \exp(-\gamma s)\left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}s \\
&\quad + \int_{B(\infty,z)} \exp(-\gamma t)\frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}v \\
&\quad \leqslant \left(\int_0^t \int_0^s \int_B \exp(-\gamma\tau)\left(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j}\right.\right. \\
&\qquad \left. + \frac{\gamma^2}{2}b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad + \int_0^t \int_B \exp(-\gamma s)\left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}s \\
&\qquad \left. + \int_B \exp(-\gamma t)\frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}v\right)\exp\left(-\frac{1}{M_3 + M_4 t^{1/2}}z\right), \quad z \geqslant 0.
\end{aligned} \tag{5.17}$$

It follows that

$$\begin{aligned}
&\int_0^t \int_0^s \int_{B(\infty,z)} \left(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j}\right. \\
&\quad \left. + \frac{\gamma^2}{2}b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^t \int_{B(\infty,z)} \left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j}\right)\mathrm{d}v\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B(\infty,z)} \frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}v \\
&\leqslant \left(\int_0^t \int_0^s \int_B \left(C_{ijrs}u_{i,j}u_{s,r} + c\theta^2 + \left(\frac{\gamma}{2}k_{ij} - b_{ij}\right)\psi_{,i}\psi_{,j} \right.\right. \\
&\quad \left. + \frac{\gamma^2}{2} b_{ij}\phi_{,i}\phi_{,j} \right) \mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^t \int_B \left(\frac{1}{2}k_{ij}\psi_{,i}\psi_{,j} + \gamma b_{ij}\phi_{,i}\phi_{,j} \right) \mathrm{d}v\,\mathrm{d}s \\
&\quad \left. + \int_B \frac{b_{ij}}{2}\phi_{,i}\phi_{,j}\,\mathrm{d}v \right) \exp\left(\gamma t - \frac{1}{M_3 + M_4 t^{1/2}} z \right), \quad z \geqslant 0. \qquad (5.18)
\end{aligned}
$$

Estimate (5.18) is the counterpart of the estimate (3.14) obtained for classical thermoelasticity. An upper bound for the amplitude term $-F_\gamma(0, t)$ can be obtained as in Section 4, but we omit the details.

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