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ORIGINAL ARTICLE

Strong solutions for Richards' equation with Cauchy conditions and constant pressure gradient

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Abstract
In this note, Richards' equation for two layered soils is considered in a two-dimensional spatial domain. It is endowed by pressure gradient and pressure condition at the top of domain, and no condition is posed at the bottom of domain. An existence and uniqueness result of strong solutions is obtained for such a problem assuming constant pressure gradient.

Keywords Richards' equation · Initial value problem · Strong solution · Layered soils · Unsaturated flow modeling

1 Introduction

The study of water movement into the vadose zone is a challenging problem, having many different applications, and needs to be faced by proper analytical and numerical methods, according to the target set and to the available measured data. Such a problem has significant importance in a environmental engineering context, for example for assessing the impact of vegetation on water balance variability (see for example [34]), or for designing infiltration trenches [29], or, classically, in an agronomic framework for modeling root uptake and growth (see for example [13, 40]), albeit the upscaling of such modeling at the field scale is still a though problem (see for example [25]).

Richards' equation is a well-established way for modeling such a movement into unsaturated soils and rocks: it underlies the hypothesis that domain is a porous medium, in which water movement is modeled in the void pores. This model arises from Darcy's equation for saturated soils, combined with a mass conservation law, assuming the air pressure constant (see for instance [33]). Infiltration dynamics is strongly dependent on the hydraulic properties of the soil in which the phenomenon occurs (see, for example, the highly heterogeneous behavior of Richards' equation solution in layered soils [8, 41]). It is worth stressing that an analytical solution of Richards' equation is very hard to obtain, and depends on the peculiar choice of hydraulic functions and boundary conditions (see for example [19]).

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Accordingly, such a flow equation is generally solved by numerical algorithms, based on finite difference and finite element methods or virtual elements methods [2, 3, 27, 35, 39]. An interesting and comprehensive survey about Richards' equation numerical issues can be found in [21]. In particular, as highlighted in [21], the highly nonlinear shape of empirical hydraulic functions requires tailored numerical schemes for solving the resulting nonlinear discrete problem: for instance, in [15] the mass balance property of Richards' equation in mixed form is demonstrated by using a modified Picard method; in [32] several comparisons are accomplished between Picard and Newton schemes, suggesting strategies for enhancing the performances of both schemes; finally, in [24], such linearization techniques are compared, together with L-scheme: the latter is analyzed by a theoretical point of view and its robustness is proved.

A separate chapter, albeit related to this, would be devoted by acknowledging data assimilation methods for incorporating dynamical observations into the model (see for example [9, 10, 12, 14]).

As for the state variable, Richards' equation can be solved with respect to the pressure head ψ or to the water content θ, assuming a given empirical relationship between the two (see for example [7] for a more extensive treatise).

A crucial issue when modeling such a process is setting the boundary conditions; in particular, if the water table is not known, or if it is below the spatial domain, then setting a realistic bottom boundary condition could be delicate. A motivation of our work comes from this difficulty: here just the knowledge of hydraulic features (state and vertical pressure gradient) at the ground surface is required. This paper arises from the need of providing a theoretical basis for the approach underlying the numerical papers [6–8].

The physical problem addressed in this paper concerns the infiltration phenomenon in layered soils: it is worth stressing that this is still an open problem in a modeling and numerical framework, see [6, 8, 19, 22, 37, 41].

Following [6], here we are going to consider the pressure form of Richards' equation in a bounded temporal domain $[0, T]$, for some $T > 0$, and in a two-dimensional spatial domain $[0, X] \times [0, Z]$, for some $X, Z > 0$, in which the z-direction is the vertical one, possibly unbounded:

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right)\right] + \frac{\partial}{\partial x}\left[K(\psi)\frac{\partial \psi}{\partial x}\right], \quad \text{if} \quad z \le \bar{z}, \tag{1a}$$

$$\hat{C}(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[\hat{K}(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right)\right] + \frac{\partial}{\partial x}\left[\hat{K}(\psi)\frac{\partial \psi}{\partial x}\right], \quad \text{if} \quad z > \bar{z}, \tag{1b}$$

being $\bar{z} \in (0, Z)$ the depth value where soil trespass occurs, where $K(\psi)$ and $\hat{K}(\psi)$ are the hydraulic conductivity functions and $C(\psi)$, $\hat{C}(\psi)$ are the specific moisture capacities of first and second soil, respectively: for sake of simplicity, we will assume the hydraulic conductivity equal in both the spatial directions; conversely, functions C and $\hat{C}$ represent a storage term and are defined as

$$C(\psi) := \frac{\partial \theta}{\partial \psi}, \quad \hat{C}(\psi) := \frac{\partial \hat{\theta}}{\partial \psi}. \tag{2}$$

The functions θ, $\hat{\theta}$ represent the volumetric water content defined by water retention curves, described by empirical functions of the relative head pressure, and which are constitutive characteristics of the media under consideration.

In (1) the hydraulic conductivity functions K, $\hat{K}$ are multiplied by a different factor because the diffusion in the vertical direction z takes into account also the gravity force.

Richards' equation needs to be associated with an initial condition, that is the pressure profile at time $t = 0$, i.e.

$$\psi(x, z, 0) = \psi^0(x, z), \qquad (x, z) \in [0, X] \times [0, Z]. \tag{3}$$

The knowledge of pressure state, or alternatively pressure flux at boundaries is generally needed; for example, within a Dirichlet framework, the following functions are assumed to be known:

$$\psi(0, z, t) = \psi_0(z, t), \quad \psi(X, z, t) = \psi_X(z, t), \quad (z, t) \in [0, Z] \times [0, T], \tag{4}$$

and, for the top and bottom conditions at the boundaries, respectively, one considers

$$\psi(x, 0, t) = \zeta_0(x, t), \qquad (x, t) \in [0, X] \times [0, T], \tag{5a}$$

$$\psi(x, Z, t) = \zeta_Z(x, t), \qquad (x, t) \in [0, X] \times [0, T], \tag{5b}$$

with ψ^0, ψ_0, ψ_X, ζ_0, ζ_Z smooth enough in their domain. Otherwise, as considered in [6], one could keep (3), (4) and (5a), while replacing (5b) by assigning the following vertical pressure gradient condition at the top of domain:

$$\left.\frac{\partial\psi(x, z, t)}{\partial z}\right|_{z=0} = \chi_0(x, t) \tag{6}$$

where $\chi_0(x, t)$ is a smooth function on $[0, X] \times [0, T]$. In [7] the choice of replacing condition (5b) by (6) is motivated by a practical point of view: typically the knowledge of hydraulic states at the bottom of the domain can be only supposed, since it can be difficult to have available realistic field measurements.

Existence and uniqueness of *strong solutions*, in the sense of [28], for Richards' equation with Dirichlet boundary conditions (3), (4), (5) is a well known problem, addressed and solved in [28].

When endowed with initial pressure gradient condition at the top of domain (6), together with (3), (4) and (5a), the existence and uniqueness problem of a solution to Richards' equation is extensively treated in literature, and researchers have tackled it from several points of view. In particular, in [16] authors prove that entropy solutions exist under general hypotheses; in [17] the problem of semi-classical solutions is analyzed; local classical solutions with $\chi_0(x, t) = 0$ is treated in [36]: in this context and in presence of capillary forces, local classical solutions are proven to exist as well (see for example [23]); finally, in [1, 30, 31] the proof of existence and uniqueness of weak solutions in a more general context have been considered, and in [4] (see also [5]) more general results of weak solutions defined on an arbitrary time interval are given.

In this note, we will consider the question of existence and uniqueness of *strong solutions* in the sense of [28], with Cauchy conditions (3), (4), (5a) and (6), under simplified hypothesis and in the case of layered soils; the general case is still open and will be addressed in another work.

However, despite the lacking of such general regularity results, a numerical technique based on a mixed MoL-TMoL has been proposed in [6] for handling such a problem in a 2D spatial domain with no restriction on constant flux, and numerical simulations are therein accomplished.

2 Existence and uniqueness results

The existence and uniqueness of strong solutions of (1) together with Cauchy initial conditions seems to be a difficult problem. Here we approach this question in a simpler situation and leave the proof of the discontinuous general case to a more theoretical work. The proof will be done into two main steps: first we prove the existence and uniqueness for (1a) when K is smooth with respect to $z \in (0,\infty)$; then we consider the discontinuous case for K, regularize around $\bar{z}$ and apply previous result to (1b). In particular, here we consider Richards' equation (1) with the initial and boundary conditions in (3), (4), (5a), and (6) with constant flux at the top of our domain. Let us define the spatial domain as

$$\Omega := (0, X) \times (0, +\infty),$$

and, letting $F_0 := \{0\} \times (0, +\infty) \times (0, T)$, $F_X := \{X\} \times (0, +\infty) \times (0, T)$, $G_0 := (0, X) \times \{0\} \times (0, T)$ and $H_0 := \Omega \times \{0\}$, the boundary is given by the union of the following sets:

$$S_T := F_0 \cup F_X \cup G_0, \quad \Gamma_T := S_T \cup H_0.$$

Let also $Q_T := \Omega \times (0, T)$. Further, let us recall that, for $k \geq 1$, Sobolev spaces $W_2^{k,1}(Q_T)$ are defined as

$$W_2^{k,1}(Q_T) := \left\{ \psi \in L^2(Q_T) \ : \ D^\alpha \psi \in L^2(Q_T) \text{ for } 0 \leq |\alpha| \leq k,\ \frac{\partial \psi}{\partial t} \in L^2(Q_T) \right\},$$

where D^α denotes the spatial derivatives of order α. In particular, for $k = 2$, the space $W_2^{2,1}(Q_T)$ is naturally endowed with the norm

$$\|\psi\|^2_{W_2^{2,1}(Q_T)} := \int_0^T \int_\Omega \left(\psi^2 + |\nabla \psi|^2 + |D^2 \psi|^2 + \left(\frac{\partial \psi}{\partial t} \right)^2 \right) \mathrm{d}\mathbf{x}\, \mathrm{d}t, \tag{7}$$

with $\mathbf{x} = (x, z) \in \Omega$ and $\psi \in W_2^{2,1}(Q_T)$. Further, we set

$$H^1(Q_T) := W_2^{1,1}(Q_T).$$

Finally, the space of Richards' equation strong solutions we will investigate is set as

$$\Sigma := H^1(Q_T) \cap L^{\frac{4}{3}}((0, T), W_2^2(\Omega)),$$

where (see [20])

$$L^{\frac{4}{3}}((0, T), W_2^2(\Omega)) := \left\{ \psi \ : \ \psi(t) \in W_2^2(Q_T)\, \forall t \in (0, T) \text{ and } \int_0^T \|\psi\|^{\frac{4}{3}}_{W_2^2(Q_T)}\, \mathrm{d}t < \infty \right\},$$

and it is a Banach space when endowed with the norm

$$\|\psi\|_{L^{\frac{4}{3}}((0,T), W_2^2(\Omega))} := \left(\int_0^T \|\psi\|^{\frac{4}{3}}_{W_2^2(Q_T)}\, \mathrm{d}t \right)^{\frac{3}{4}}, \quad \psi \in L^{\frac{4}{3}}((0, T), W_2^2(\Omega)).$$

Specifying that, throughout this section, *divergence* operator is defined as

$$\mathrm{div}(f) := \nabla \cdot f, \quad f : Q_T \to \mathbb{R}^2,$$

where $\nabla := \left[\frac{\partial}{\partial x}\ \frac{\partial}{\partial z}\right]$, and leveraging (2), we observe that (1a), up to letting z go to $+\infty$, can be equivalently rewritten as

$$\frac{\partial \theta}{\partial t}(\psi) - \mathrm{div}(K(\psi)\nabla(\psi - z)) = 0, \quad (x, z, t) \in Q_T, \tag{8}$$

which is the so-called *mixed form* of Richards' equation, and will suit better for the following result. An analogous result holds, straightforwardly, for (1b).

Theorem 1 *Let us consider* (8) *with Cauchy initial conditions given by*

$$\begin{cases} \psi = \psi^D, & \text{on } F_0 \cup F_X \cup H_0, \\ \frac{\partial \psi}{\partial z} = 1, & \text{on } G_0, \end{cases} \tag{9}$$

where $\psi^D \in W_2^{2,1}(Q_T) \cap L^\infty(Q_T)$. *Then* (8)–(9) *admits a unique strong solution in* $\Sigma \cap L^\infty(Q_T)$. *Moreover, it holds that*

$$\|\psi\|_{L^\infty(Q_T)} \leq \|\psi^D\|_{L^\infty(Q_T)}. \tag{10}$$

Proof On the account of (9), let $\varphi \in H^1(Q_T)$ be an arbitrary test function with vanishing trace on $F_0 \cup F_X$, and let us seek for the weak formulation of (8). Thus, after integrating over $(0, T)$ and Ω, we get

$$\int_0^T \int_\Omega \frac{\partial \theta}{\partial t}(\psi)\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{dt} - \int_0^T \int_\Omega \mathrm{div}(K(\psi)\nabla(\psi - z))\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{dt} = 0,$$

from which, letting S be a suitable parameterization of S_T and $\mathbf{n}$ be the outer normal to S_T,

$$\int_0^T \int_\Omega \frac{\partial \theta}{\partial t}(\psi)\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{dt} - \int_{S_T} K(\psi)\nabla(\psi - z) \cdot \mathbf{n}\,\varphi \,\mathrm{d}S \\ + \int_0^T \int_\Omega (K(\psi)\nabla(\psi - z)) \cdot \nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{dt} = 0.$$

Now, on the account of selected test function set, we get that

$$\int_{F_0} K(\psi)\nabla(\psi - z) \cdot \mathbf{n}_{F_0}\varphi \,\mathrm{d}S_{F_0} = \int_{F_X} K(\psi)\nabla(\psi - z) \cdot \mathbf{n}_{F_X}\varphi \,\mathrm{d}S_{F_X} = 0,$$

and, since from (9) we have $\frac{\partial \psi}{\partial z} = 1$ on G_0, we obtain that

$$\int_{G_0} K(\psi)\nabla(\psi - z) \cdot \mathbf{n}_{G_0}\varphi \,\mathrm{d}S_{G_0} = \int_{G_0} K(\psi) \begin{bmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial z} - 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} \varphi \,\mathrm{d}S_{G_0} = 0,$$

concluding that

$$\int_{S_T} K(\psi)\nabla(\psi - z)\cdot \mathbf{n}\varphi \, \mathrm{d}S = 0.$$

Therefore, weak formulation of (8) comes out to be

$$\int_0^T \int_\Omega \frac{\partial\theta}{\partial t}(\psi)\varphi \, \mathrm{d}\mathbf{x}\, \mathrm{dt} + \int_0^T \int_\Omega \left(K(\psi)\frac{\partial(\psi - z)}{\partial z} \right) \frac{\partial\varphi}{\partial z} \, \mathrm{d}\mathbf{x}\, \mathrm{dt} = 0, \quad \forall\varphi \in H^1(Q_T),$$

with φ having vanishing trace on $F_0 \cup F_X$.

Now, we point out that the above weak formulation is the same as the one analyzed in [28], with the only difference that now spatial domain is unbounded: it is just an observation that results in [28] hold true in this case as well. Therefore, we infer existence and uniqueness of a strong solution for (8) with Cauchy conditions given by (9), and from Lemma 3.1 in [28] it follows that $\|\psi\|_{L^\infty(Q_T)} \le \|\psi^D\|_{L^\infty(Q_T)}$. □

Corollary 1 Let $c \neq 0$. Let us consider the partial differential equation

$$\frac{\partial\theta}{\partial t}(\psi) - \mathrm{div}(K(\psi)\nabla(\psi - cz)) = 0, \quad (x,z,t) \in Q_T, \tag{11}$$

with Cauchy initial boundary conditions given by

$$\begin{cases} \psi = \psi^D, & \text{on } F_0 \cup F_X \cup H_0, \\ \frac{\partial\psi}{\partial z} = c, & \text{on } G_0, \end{cases} \tag{12}$$

where $\psi^D \in W_2^{2,1}(Q_T) \cap L^\infty(Q_T)$.

Then, (11) admits a unique strong solution in $\Sigma \cap L^\infty(Q_T)$ with initial conditions (12) and satisfying (10).

Proof Let $\psi(x,z,t)$ be the solution to (8) with Cauchy boundary conditions (9). Let then

$$\tilde{\psi}(x,z,t) := \psi(x, cz, t).$$

Easy computations show that $\tilde{\psi}$ solves (11) with Cauchy boundary conditions (12).

Now, if there exists $\tilde{\phi}$, different from $\tilde{\psi}$, solving the same problem, then $\phi(x,z,t) := \tilde{\phi}\left(x, \frac{z}{c}, t\right)$ would solve (8) with Cauchy boundary conditions (9), and so $\phi(x,z,t) = \psi(x,z,t)$. Therefore $\phi(x,cz,t) = \psi(x,cz,t)$, that is $\tilde{\phi} = \tilde{\psi}$, which is a contradiction. □

Next, we address the problem of two layered soils and prove, in this case, that existence and uniqueness of a strong solution, in the sense of Theorem 1, is still guaranteed.

We first need a technical result.

Lemma 1 *The space $\Sigma \cap L^\infty(Q_T)$ endowed with the norm*

$$\|\cdot\|_\Sigma := \max\left\{ \|\cdot\|_{H^1(Q_T)}, \|\cdot\|_{L^{\frac{4}{3}}((0,T),W_2^2(\Omega))} \right\}$$

is a Banach space.

Proof See, for example [38]. □

Theorem 2 *Let us consider the following partial differential equation*

$$\begin{cases} \frac{\partial\theta}{\partial t}(\psi) - \operatorname{div}(K(\psi)\nabla(\psi - z)) = 0, & (x,z,t) \in Q_T, \quad z \le \bar{z}, \\ \frac{\partial\theta}{\partial t}(\psi) - \operatorname{div}\big(\hat{K}(\psi)\nabla(\psi - z)\big) = 0, & (x,z,t) \in Q_T, \quad z > \bar{z}, \end{cases} \tag{13}$$

with Cauchy initial boundary conditions given by

$$\begin{cases} \psi = \psi^D, & \text{on } F_0 \cup F_X \cup H_0, \\ \frac{\partial\psi}{\partial z} = 1, & \text{on } G_0, \end{cases} \tag{14}$$

where $\psi^D \in W_2^{2,1}(Q_T) \cap L^\infty(Q_T)$.

Then (13) *admits a unique strong solution in* $\Sigma \cap L^\infty(Q_T)$ *with initial conditions* (14) *and satisfying* (10).

Proof Let $\overline{\varepsilon} > 0$ such that $\overline{z} - \overline{\varepsilon} > 0$ and let $\varepsilon \in (0, \overline{\varepsilon})$; let further K_ε be a sufficiently smooth function such that

$$K_\varepsilon(\psi) = K(\psi), \qquad \text{for } z < \bar{z} - \varepsilon,$$
$$K_\varepsilon(\psi) = \hat{K}(\psi), \qquad \text{for } z > \bar{z} + \varepsilon,$$

and $K_\varepsilon(\psi)$ smoothly interpolates $K(\psi)$, $\hat{K}(\psi)$ for $z \in [\bar{z} - \varepsilon, \bar{z} + \varepsilon]$.

Thus, considering the partial differential equation

$$\frac{\partial\theta}{\partial t}(\psi) - \operatorname{div}\big(K_\varepsilon(\psi)\nabla(\psi - z)\big) = 0, \quad (x,z,t) \in Q_T, \tag{15}$$

by Theorem 1 we deduce that there exists a unique strong solution $\psi_\varepsilon \in \Sigma \cap L^\infty(Q_T)$, and satisfies (10).

Let now $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon)$, with $\varepsilon_2 < \varepsilon_1$. Let us define, for any $\varepsilon \in (0, \overline{\varepsilon})$,

$$R_\varepsilon :\, = (0, X) \times (\bar{z} - \varepsilon, \bar{z} + \varepsilon) \times (0, T).$$

In particular, then, both R_{ε_1} and R_{ε_2} are finite measure and $R_{\varepsilon_2} \subseteq R_{\varepsilon_1}$. Also, on the account of Theorem 1, we have that

$$\psi_{\varepsilon_1}\big|_{Q_T \setminus R_{\varepsilon_1}} = \psi_{\varepsilon_2}\big|_{Q_T \setminus R_{\varepsilon_1}}.$$

Since $H^1(Q_T) \subseteq L^6(Q_T)$, with compact embedding, and $W_2^2(\Omega)$ is continuously embedded in the set of $\frac{1}{2}$-Hölder continuous functions, we now observe that

$$\begin{aligned} \big||\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\big\|_\Sigma &= \Big\|\psi_{\varepsilon_1}\big|_{Q_T \setminus R_{\varepsilon_1}} - \psi_{\varepsilon_2}\big|_{Q_T \setminus R_{\varepsilon_1}}\Big\|_\Sigma + \Big\|\psi_{\varepsilon_1}\big|_{R_{\varepsilon_1}} - \psi_{\varepsilon_2}\big|_{R_{\varepsilon_1}}\Big\|_\Sigma \\ &= \Big\|\psi_{\varepsilon_1}\big|_{R_{\varepsilon_1}} - \psi_{\varepsilon_2}\big|_{R_{\varepsilon_1}}\Big\|_\Sigma \le c\big(X, T, \|\psi^D\|_{L^\infty(Q_T)}\big)\varepsilon_1, \end{aligned}$$

where $c\big(X, T, \|\psi^D\|_{L^\infty(Q_T)}\big) > 0$ is a constant depending on X, T and $\|\psi^D\|_{L^\infty(Q_T)}$. Last inequality implies that the sequence $\{\psi_\varepsilon\}_{\varepsilon\in(0,\overline{\varepsilon})}$ is Cauchy in Σ. Thus, from Lemma 1, there exists $\psi \in \Sigma$ such that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi$$

strongly with respect to $\|\cdot\|_\Sigma$. It is straightforward to see that ψ solves (13) with Cauchy conditions (14). Finally, let us note that $\left\|\psi_\varepsilon\right\|_{L^\infty(Q_T)} \le \left\|\psi^D\right\|_{L^\infty(Q_T)}$ for all $\varepsilon \in (0, \overline{\varepsilon})$ implies, by uniqueness of limit, that $\psi_\varepsilon \overset{*}{\rightharpoonup} \psi$ in L^∞ as $\varepsilon \to 0$. Resorting to weakly-$*$ lower semi-continuity of $\|\cdot\|_{L^\infty(Q_T)}$ provides (10).

Uniqueness easily comes from observing that any other solution ϕ of (13) would provide a solution ϕ_ε of (15) such that, for $z \notin [\bar{z}-\varepsilon, \bar{z}+\varepsilon]$, $\phi = \phi_\varepsilon$. By uniqueness of solution to (15), we deduce that $\phi_\varepsilon = \psi_\varepsilon$, and therefore, with respect to above norm,

$$\phi = \lim_{\varepsilon\to 0} \phi_\varepsilon = \lim_{\varepsilon\to 0} \psi_\varepsilon = \psi,$$

which proves the claim. □

3 Conclusions and future works

In this note we have proved that Richards' equation for two layered soils, with given initial head pressure and constant initial pressure gradient at the top of domain, is a well posed problem providing a unique strong solution in the sense of [28]: this work represents a theoretical basis for the approach underlying our numerical papers [6, 8]. We have used a regularization technique and well known results originally proposed for Richards' equation with Dirichlet boundary conditions; for arbitrary Cauchy initial conditions, the problem needs to be faced differently, and will be studied in a forthcoming work.

Furthermore, we think that the approach of treating a conservation law as an initial value problem could be of interest also for applications different from the infiltration into the unsaturated zone: for example, for modeling the non-equilibrium solute transport in porous media by a mobile-immobile model (see for instance [26]), or in a broad class of problems modeled by transport equations, e.g. [11], or [18], for cases in which the location of boundary between two distinct geomaterials is uncertain.

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