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The Residence Time of Settling Particles in the Surface Mixed Layer

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Abstract. The transport from the upper mixed layer into the pycnocline of particles with negative buoyancy is considered. Assuming the hydrodynamic parameters to be time-independent, an adjoint model is resorted to that provides a general expression of the residence time in the mixed layer of the constituent under study. It is seen that the residence time decreases as the settling velocity increases or the diffusivity decreases. Furthermore, it is demonstrated that the residence time must be larger than z/w and smaller than h/w, where z, h and w denote the distance to the pycnocline, the thickness of the mixed layer and the sinking velocity. In the vicinity of the pycnocline, the residence time is not necessarily zero; its behaviour critically depends on the eddy diffusivity profile in this region. Closed-form solutions are obtained for constant and quadratic diffusivity profiles, which allows for an analysis of the sensitivity of the residence time to the Peclet number. Finally, an approximate value is suggested of the depth-averaged value of the residence time.

Key words: adjoint model, mixed layer, pycnocline, residence time, settling

1. Introduction

Wind stress or convective processes cause the development of a turbulent – or mixed – layer adjacent to the surface of estuaries, lakes or seas. In this region, vertical mixing is of a significant importance. The lower boundary of the mixed layer often is a region exhibiting strong stratification, the pycnocline; in the latter, turbulent mixing is negligible. Therefore, assuming horizontal homogeneity, a non-buoyant particle is likely to remain for a long time in the upper layer. However, particles such as phytoplankton

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cells – or aggregates of them – silts and marine snow generally have a density larger than that of water. As a result, they tend to migrate downward. In other words, the fate of sinking particles crucially depends on turbulent mixing and settling: the former tends to homogenize their concentration over the surface layer and the latter eventually allows the particles under consideration to leave the surface layer. To determine the relative importance of these phenomena, it is appropriate to determine the timescales over which they can significantly alter the concentration in the surface layer of a cloud of settling particles.

If H and K denote typical values of the thickness of the mixed layer and the vertical eddy diffusivity, the time needed to homogenize the concentration of a given constituent is of the order of $T_{\mathrm{m}} = H^2/K$. On the other hand, if W is a typical value of the downward migration velocity, the timescale characterising settling is $T_{\mathrm{s}} = H/W$, i.e. the time needed for all the sinking material to enter the pycnocline assuming turbulent diffusion to be negligible. The ratio of the mixing to the settling timescales is a dimensionless parameter known as the Peclet number, $Pe = T_{\mathrm{m}}/T_{\mathrm{s}} = WH/K$ (e.g. [1–3]). If $Pe \gg 1$, settling is more efficient than mixing and the concentration of settling particles is likely to exhibit significant variations over the height of the upper layer. By contrast, if the mixing timescale is much smaller than that related to settling ($Pe \ll 1$), it is appropriate to assume that sinking particles are evenly distributed throughout the surface layer. Then, between the air-water interface and the pycnocline, the gradient of the concentration of settling particles may be neglected, a simplifying assumption that was resorted to in a number of mathematical developments [2–5]. Unfortunately, this simple approach is not always valid, for the Peclet number is not necessarily small.

The depth of the mixed layer may lie in the range $10^1 - 10^2$ m, while the order of magnitude of the vertical eddy diffusivity is $10^{-3} - 10^{-1}\,\mathrm{m^2\,s^{-1}}$ (e.g. [6–8]). Using Stokes' formula (e.g. [4, 9]), the settling velocity may be estimated to range from 1 to $100\,\mathrm{m\,day^{-1}}$, i.e. $W \approx 10^{-5} - 10^{-3}\,\mathrm{m\,s^{-1}}$ [3–5, 10]. Thus, though the Peclet number may be as small as 10^{-3}, its maximum is of order 10^2, i.e. a value much larger than unity. At this point, one might object that the eddy diffusivity and the thickness of the mixed layer are not independent of each other, so that it is not appropriate to estimate the upper bound of the Peclet number by taking the smallest value of K and the maximum of H. Therefore, the range of acceptable values of Pe actually is narrower than that suggested above. Nonetheless, values of the order of unity, or larger, cannot be ruled out a priori. Therefore, in some circumstances, models are needed that are capable of predicting explicitly vertical profiles of concentrations in the upper turbulent layer (e.g. [10, 11]).

If the settling velocity increases, the rate at which particles sink into the pycnocline also increases. This is obvious. However, no elementary reasoning may be used in order to elucidate the role of turbulent diffusion. If the Peclet number is very small, it is well known that the particles flux leaving the upper mixed layer tends to be independent of the eddy diffusivity and is approximately equal to the product of the settling velocity and the depth-averaged particle concentration [2, 3, 5]. However, if the Peclet number is not small, it is not clear a priori whether mixing tends to increase or decrease the number of particles at the bottom of the mixed layer, i.e. the number of particles that are on the verge of leaving the upper layer by entering the pycnocline. It is believed that no general formula can be derived that would predict the flux of material sinking into the pycnocline, since the latter largely depends on the details of the initial conditions and the relevant hydrodynamic parameters. Fortunately, another measure exists of the rate at which particles leave the upper layer. It consists in estimating their residence time, i.e. the time they spend in the mixed layer. Doing so provides no details on the time evolution of the particle flux leaving the upper layer, but allows one to gain significant insight into the role of turbulent diffusion. In particular, an answer can be given to the following question: does the residence time increase or decrease if the eddy diffusivity increases? This issue will be addressed hereinafter without invoking any assumption as to the order of magnitude of the Peclet number and the theoretical results derived below will be of use for many questions of practical importance in oceanography, such as the understanding of how "sinking phytoplankton species manage to persist" (e.g. [12]).

The present article is organised as follows. First, the model for the settling and diffusion of sinking material is outlined (Section 2). Next, in Section 3, a method for estimating the residence time is established. Then, the $Pe \to 0$ and $Pe \to \infty$ limiting cases are investigated. In Section 4, it is seen that the adjoint model of Delhez et al. [13] leads to a general expression of the residence time. Some of its properties are derived and illustrated (Section 5) by considering various profiles of the eddy diffusivity. In Section 6, the behaviour of the residence time in the vicinity of the pycnocline is examined. Finally, concluding remarks are made as to the usefulness of the analytical results derived in the present study (Section 7).

2. Settling and Diffusion Model

Consider a passive – or inert – constituent settling in a horizontally homogeneous mixed layer adjacent to the surface of a water body. The lower boundary of the surface layer is a pycnocline, in which turbulent diffusion is assumed to be negligible. The sinking velocity, w, and the thickness of the mixed layer, h, are assumed to be constant. Let z denote the vertical coordinate, which is equal to 0 and h at the top of the pycnocline and the water–air interface, respectively, so that the domain of interest is defined by inequalities $0 \leq z \leq h$ (Figure 1). This way of defining the vertical coordinate is not standard, but is well suited to the problem being tackled.

At any time t, the concentration of the constituent under consideration, $C(t, z)$, satisfies the following advection–diffusion equation:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(wC + \kappa \frac{\partial C}{\partial z}\right), \tag{1}$$

where κ denotes the eddy diffusivity. For the purposes of the present study, it is convenient to consider that the latter does not depend on time, but further assuming that it is also independent of the vertical coordinate would be too strong an idealisation [6, 8]. Hence, the eddy diffusivity is assumed to be a non-negative function of the vertical coordinate, i.e. $\kappa(z) \geq 0$.

It is hypothesised that there is no flux of the constituent under study through the air–sea interface, implying that the following boundary condition must be applied:

$$\left[wC + \kappa \frac{\partial C}{\partial z}\right]_{z=h} = 0. \tag{2}$$

As the pycnocline is a barrier to turbulent diffusion, the turbulent diffusion flux must be prescribed to be zero at the bottom of the mixed layer:

$$\left[\kappa \frac{\partial C}{\partial z}\right]_{z=0} = 0. \tag{3}$$

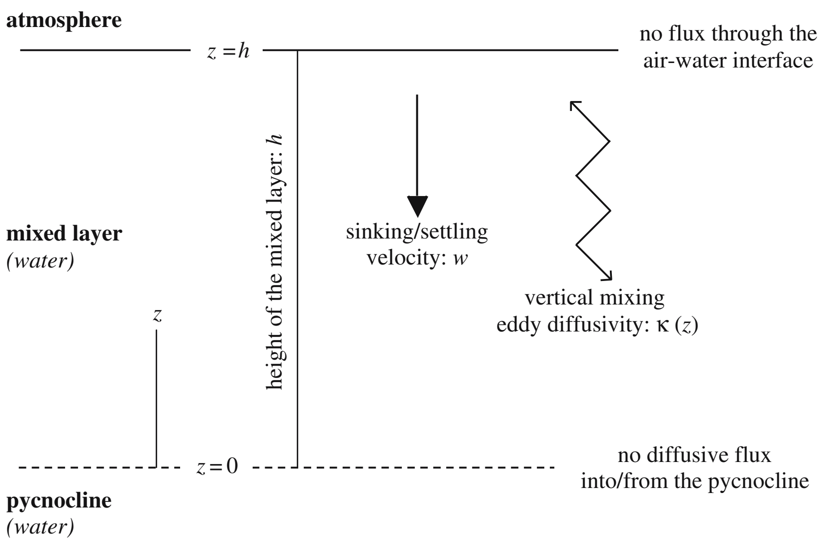

Figure 1. Sinking-diffusion model: illustration of its geometry, parameters and boundary conditions.

Upon specifying the initial profile of the concentration, $C(0,z)$, the latter may, in principle, be obtained at any point and any time $t > 0$. However, closed-form solutions are hard to derive for the present partial differential problem, especially when the eddy diffusivity is not constant in space. Fortunately, as will be seen, determining explicitly the time evolution of the concentration is not necessary to estimate the residence time.

3. Residence Time

The amount of the constituent under study present at time t in the domain of interest is

$$m(t) = \int_0^h C(t,z)\mathrm{d}z, \tag{4}$$

a variable that, for simplicity, will be termed “mass”. Integrating equation (1) over the mixed-layer, using the boundary conditions (2) and (3), it is readily seen that the mass decreases monotonically as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = -wC(t,0). \tag{5}$$

Time t is the residence time of the mass leaving the domain during the time interval $[t, t+\Delta t]$, with $\Delta t \to 0$. The amount of mass that has left during this period is $-[m(t+\Delta t) - m(t)]$. Then, the mean residence time of the material that was present in the mixed layer at time $t = 0$ is given by the mass-weighted average of the elemental residence times [14], i.e.

$$\Theta = -\frac{1}{m(0)} \int_{m(0)}^0 t \, \mathrm{d}m. \tag{6}$$

This expression may be transformed to (e.g. [13])

$$\Theta = \frac{1}{m(0)} \int_0^\infty m(t) \, \mathrm{d}t. \tag{7}$$

Without any loss of generality, it may be assumed that $m(0) = 1$. Then, substituting (4) into (7) yields

$$\Theta = \int_0^\infty \int_0^h C(t,z) \, \mathrm{d}z \, \mathrm{d}t. \tag{8}$$

If the Peclet number tends to zero, i.e. if mixing occurs much faster than sinking, then the initial profile of concentration is of negligible importance, since the concentration becomes almost homogeneous over the mixed layer in an arbitrarily short time – and remains approximately homogeneous as time progresses. This implies

$$m(t) \sim hC(t), \quad Pe \to 0. \tag{9}$$

Then, substituting the latter expression into the mass budget (5) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}C(t) \sim -\frac{w}{h}C(t), \quad Pe \to 0. \tag{10}$$

As a result, the concentration of the sinking material under consideration decreases exponentially

$$C(t) \sim \frac{\mathrm{e}^{-wt/h}}{h}, \quad Pe \to 0. \tag{11}$$

By combining (8) and (11), the residence time is readily obtained:

$$\Theta \sim \frac{h}{w}, \quad Pe \to 0. \tag{12}$$

So, as the Peclet number tends to zero, the residence time is asymptotically independent of the eddy diffusivity and the initial concentration profile [13–16].

If vertical diffusion is negligible ($Pe \to \infty$), then, as time progresses, the initial concentration profile is shifted downward at velocity w, so that the concentration satisfies

$$C(t,z) = \begin{cases} 0 & \text{for } h - wt < z, \\ C(0, z + wt) & \text{for } z \le h - wt. \end{cases} \tag{13}$$

As a result, the residence time is

$$\Theta = \int_0^{h/w} \int_0^{h-wt} C(0, z + wt)\mathrm{d}z\,\mathrm{d}t, \tag{14}$$

implying that, in contrast to the limit $Pe \to 0$, the residence time critically depends on the initial concentration $C(0,z)$.

Except when the Peclet number is very small, the residence time significantly depends on the initial concentration profile. However, it is readily seen that the residence time Θ of particles whose initial concentration is $C(0,z)$ may be expressed as

$$\Theta = \int_0^{h} C(0,\xi)\theta(\xi)\mathrm{d}\xi, \tag{15}$$

where $\theta(z_0)$ is the residence time of material that is initially concentrated at the distance z_0 to the pycnocline. In other words, $\theta(z_0)$ is the residence time ensuing from the initial concentration $C(0,z) = \delta(z - z_0)$, where $\delta(z - z_0)$ is a Dirac impulse located at z_0. From here on, as in [17] or [13], the mathematical developments will aim at determining $\theta(z_0)$, from which the general residence time Θ may be obtained, for any initial concentration profile. Although $\theta(z_0)$ is a timescale of a specific nature that may be regarded as a function of the position rather than the initial concentration, it will be termed hereinafter "residence time" for the sake of simplicity. In the limit $Pe \to 0, \theta$ and Θ tend to be equal to h/w, and, when sinking is dominant $(Pe \to \infty), \theta$ is asymptotic to an expression considerably simpler than (14), i.e. $\theta(z_0) \sim z_0/w$. Rewriting the latter as a function of coordinate z, the asymptotic results established so far may be summarized as follows:

$$\theta(z) \sim \begin{cases} h/w, & Pe \to 0, \\ z/w, & Pe \to \infty. \end{cases} \tag{16}$$

One cannot be satisfied with only the asymptotic limits for large and small values of the Peclet number. It is desirable to obtain $\theta(z)$ for any eddy viscosity profile and Peclet number. However, even for a constant viscosity and an initial condition as simple as a Dirac impulse, the concentration $C(t,z)$ cannot be obtained easily. Thus, determining first the concentration at any time and position and, then, the residence time from formula (8) is not the most convenient method. Nonetheless, as is shown in the Appendix, this "direct" or "forward" approach (Table I) to determining the residence time may be somewhat modified so as to yield a general expression of the residence time. The latter may also be obtained in an easier way by resorting to the adjoint model method suggested by Delhez et al. [13], which is part of the Constituent-oriented Age and Residence time Theory (CART) (e.g. [13, 18, 19]). A brief description of CART and some of its applications is available on the World Wide Web (www.climate.be/CART).

Table I. The equations of the direct/forward and adjoint problems, and the solution of the latter.

	Direct/forward problem	Adjoint problem
Unknown	Concentration: $C(t,z)$	Residence time: $\theta(z)$
Governing equation	$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(wC + \kappa \frac{\partial C}{\partial z}\right)$	$\frac{\mathrm{d}}{\mathrm{d}z}\left(w\theta - \kappa \frac{\mathrm{d}\theta}{\mathrm{d}z}\right) = 1$
Initial condition	$C(0,z) = \delta(z - z_0)$	Not applicable
Boundary conditions	$\left[wC + \kappa \frac{\partial C}{\partial z}\right]_{z=h} = 0$ $\left[\kappa \frac{\partial C}{\partial z}\right]_{z=0} = 0$	$\left[\kappa \frac{\mathrm{d}\theta}{\mathrm{d}z}\right]_{z=h} = 0$ $\left[w\theta - \kappa \frac{\mathrm{d}\theta}{\mathrm{d}z}\right]_{z=0} = 0$
Solution	$C(t,z) = ?$	$\theta(z) = \frac{z}{w} + \frac{1}{w}\int_z^h \exp\left[-w\int_z^\xi \frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi$

4. An Adjoint Model Approach

In accordance with the adjoint model approach of Delhez *et al.* [13], the residence time $\theta(z)$ satisfies the following differential problem (Table I):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(w\theta - \kappa\frac{\mathrm{d}\theta}{\mathrm{d}z}\right) = 1, \tag{17}$$

$$\left[\kappa\frac{\mathrm{d}\theta}{\mathrm{d}z}\right]_{z=h} = 0, \tag{18}$$

$$\left[w\theta - \kappa\frac{\mathrm{d}\theta}{\mathrm{d}z}\right]_{z=0} = 0. \tag{19}$$

Integrating relation (17) over the vertical coordinate is straightforward. Then, taking into account the boundary condition (19), the following first-order differential equation is obtained:

$$w\theta - \kappa\frac{\mathrm{d}\theta}{\mathrm{d}z} = z. \tag{20}$$

The solution of the latter that satisfies (18) is

$$\theta(z) = \frac{z}{w} + \frac{1}{w}\int_z^h \exp\left[-w\int_z^\xi \frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi. \tag{21}$$

It is immediately seen that at the top of the mixed layer, $z = h$, the residence time is equal to h/w, no matter the eddy diffusivity profile and magnitude!

If the settling velocity, eddy diffusivity and depth of the pycnocline are known, the residence time profile may be obtained by means of the above relation. In accordance with physical intuition, expression (21) indicates that the residence time decreases if the settling velocity increases. On the other hand, if the eddy diffusivity is increased, the residence time also increases, a property that could not have been inferred from elementary physical reasoning. In the non-diffusive limit ($Pe \to \infty$), the argument of the exponential tends to zero, so that the residence time tends to z/w. On the other hand, if vertical mixing is “infinitely” efficient ($Pe \to 0$), the exponential tends to unity, so that

$$\int_z^h \exp\left[-w\int_z^\xi \frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi \sim h - z, \qquad \frac{wh}{K} \to 0, \tag{22}$$

which implies that the residence time tends to the constant w/h. These two limiting case results are similar to those obtained by means of a different approach in the preceding Section. This is reassuring as to the relevance of general solution (21).

The argument of the exponential in (21) is negative, implying that

$$\int_z^h \exp\left[-w\int_z^\xi \frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi \le h - z. \tag{23}$$

As a consequence, the residence time is comprised between the non-diffusive limit, z/w, and the infinite-mixing limit, h/w,

$$\frac{z}{w} < \theta(z) < \frac{h}{w}, \tag{24}$$

implying that the average of the residence time over the height of the turbulent layer,

$$\bar{\theta} = \frac{1}{h}\int_0^h \theta(z)\mathrm{d}z \tag{25}$$

belongs to a rather narrow timescale interval (Figure 2),

$$\frac{1}{2}\frac{h}{w} < \bar{\theta} < \frac{h}{w}. \tag{26}$$

So, if the parameters of the processes under study are varied in such a way that the Peclet number increases from arbitrarily-large to arbitrarily-small values, the depth-averaged residence time increases by a factor of two only. Here, it is worth underscoring an important property of $\bar{\theta}$: comparing (25) with (15), it is immediately understood that the depth-averaged residence time is the residence time of particles which are uniformly distributed throughout the domain of interest at the initial time, i.e. particles whose initial concentration is $C(0, z) = 1$.

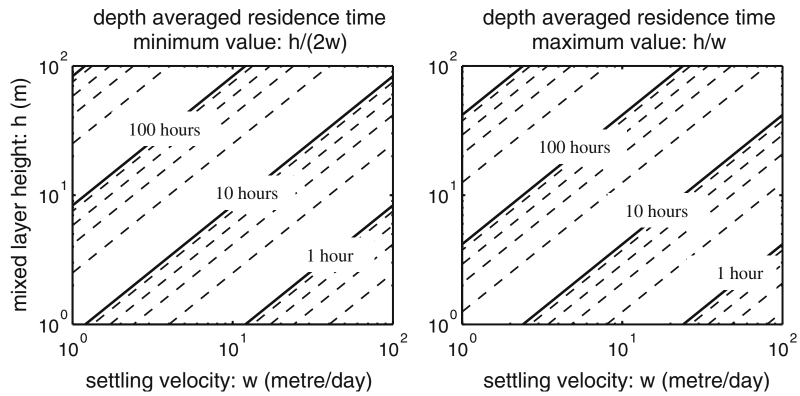

Figure 2. Isolines of the lower and upper bounds of $\overline{\theta}$, the average over the height of the mixed layer of the residence time. The minimum and maximum values are obtained from inequalities (26), and are equal to $h/(2w)$ and h/w, respectively.

5. Illustration

The properties of the residence time established above may be illustrated by considering a series of profiles of the eddy diffusivity. To do so, it is convenient to introduce dimensionless variables:

$$z' = \frac{z}{h}, \quad \kappa' = \frac{\kappa}{\overline{\kappa}}, \quad \theta' = \frac{\theta}{h/w}, \tag{27}$$

where $\overline{\kappa}$ is the average over the mixed layer of the eddy diffusivity, i.e.

$$\overline{\kappa} = \frac{1}{h}\int_0^h \kappa(z)\mathrm{d}z. \tag{28}$$

Using the latter, the Peclet number is naturally defined to be

$$Pe = \frac{wh}{\overline{\kappa}}. \tag{29}$$

From now on, unless otherwise stated, only dimensionless variables will be used, so that it is appropriate to drop the primes. Accordingly, the residence time formula (21) transforms to

$$\theta(z) = z + \int_z^1 \exp\left[-Pe\int_z^\xi \frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi, \tag{30}$$

and it is readily seen that $\partial\theta/\partial Pe < 0, z < \theta(z) < 1$, and $1/2 < \overline{\theta} < 1$.

If the eddy diffusivity is constant, $\kappa = 1$, the residence time is (Figures 3 and 4)

$$\theta(z) = z + \frac{1 - \mathrm{e}^{-Pe(1-z)}}{Pe}, \tag{31}$$

respectively. As mentioned earlier, selecting a constant eddy diffusivity is not entirely consistent with the main features of turbulent processes in the upper mixed layer. It is desirable that κ tend to zero as the bottom of the mixed layer is approached so as to take into account the inhibition of turbulence in the pycnocline. In addition, the maximum of the eddy diffusivity should not occur at the surface; in the vicinity of the latter, κ may well be rather small. A simple expression that satisfies these constraints is the following parabolic profile (Figure 3)

$$\kappa(z) = \lambda z(1 - \alpha z), \tag{32}$$

with $1/2 < \alpha \leq 1$. The constant α is likely to be very close to unity, say $0.95 \leq \alpha \leq 1$ (Burchard, personal communication, 2005), but smaller values cannot be ruled out *a priori*. On the other hand, the constant[1]

$$\lambda = \frac{6}{3 - 2\alpha} \tag{33}$$

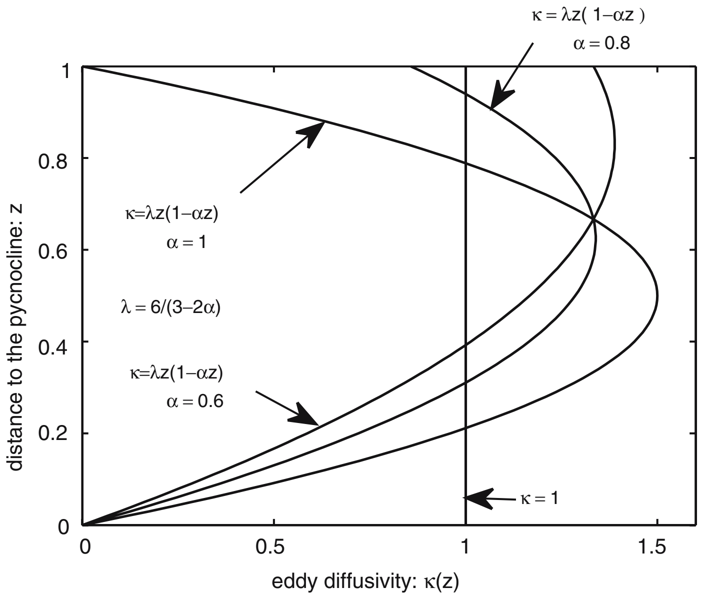

Figure 3. The eddy diffusivity profiles for which the residence time is estimated explicitly. All variables are dimensionless.

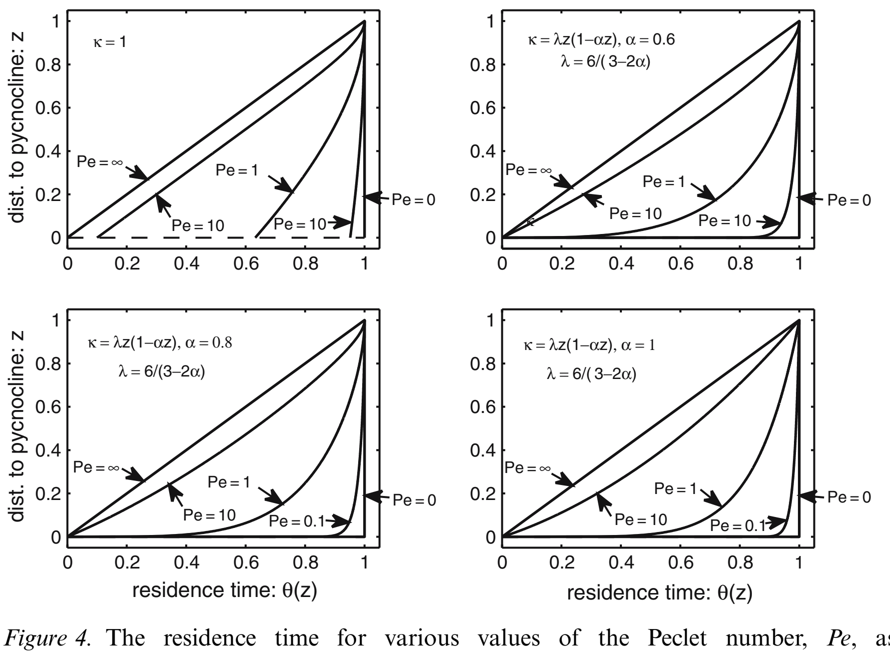

Figure 4. The residence time for various values of the Peclet number, *Pe*, as obtained by means of relation (30) for the eddy diffusivity profiles displayed in Figure 3. All variables are dimensionless.

is such that the depth-averaged eddy diffusivity is equal to unity, $\overline{\kappa}=1$. If

$$\mu=\frac{3-2\alpha}{6}Pe, \tag{34}$$

the residence time corresponding to eddy diffusivity (32) is (Figure 4)

$$\theta(z)=z+\frac{1}{\alpha}\left(\frac{\alpha z}{1-\alpha z}\right)^{\mu} B_{1-\alpha,1-\alpha z}(1+\mu,1-\mu), \tag{35}$$

where $B_{1-\alpha,1-\alpha z}(1+\mu,1-\mu)$ is a generalised incomplete beta function, i.e.

$$B_{1-\alpha,1-\alpha z}(1+\mu,1-\mu)=\int_{1-\alpha}^{1-\alpha z}\sigma^{\mu}(1-\sigma)^{-\mu}\,\mathrm{d}\sigma. \tag{36}$$

For $\kappa=1$, the average over the mixed-layer thickness of the residence time is (Figure 5)

$$\overline{\theta}=\frac{1}{2}+\frac{1}{Pe}-\frac{1-\mathrm{e}^{-Pe}}{Pe^2}. \tag{37}$$

Unfortunately, it seems that no closed-form expression of $\overline{\theta}$ can be obtained for quadratic eddy diffusivity profiles. Nonetheless, this variable can be estimated numerically. The depth-averaged residence times corresponding to all the eddy diffusivity profiles displayed in Figure 3 are so close to each other that expression (37) may be regarded as an excellent approximation

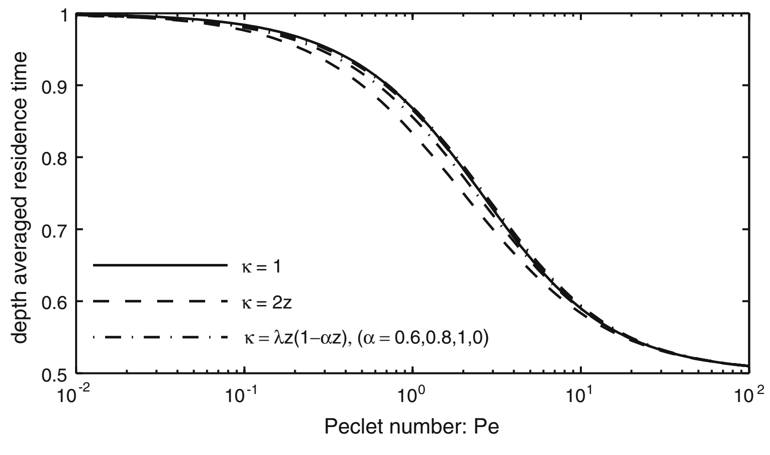

Figure 5. The average over the depth of the mixed layer of the residence time for various eddy diffusivity profiles. All variables are dimensionless.

of them all (Figure 5). On the other hand, for the linear eddy diffusivity profile $\kappa(z) = 2z$, the residence time and its depth-average are

$$\theta(z) = \begin{cases} z + \dfrac{z^{Pe/2} - z}{1 - Pe/2} & \text{if } Pe \neq 2, \\ z - z \log z & \text{if } Pe = 2 \end{cases} \tag{38}$$

and

$$\overline{\theta} = \frac{1}{2} + \frac{1}{2 + Pe}, \tag{39}$$

respectively. As is seen in Figure 5, the latter expression is also very close to (37). Needless to say, neither constant nor linear diffusivity profiles are physically acceptable and, yet, they seem to lead to approximate values of $\overline{\theta}$ that might be fairly reliable. Then, one may conjecture that the average over the thickness of the mixed layer of the residence time is almost independent of the diffusivity profile and is essentially a function of the Peclet number, the simplest form of which might be (39). The profound reason thereof is not known. It is worth pointing out, however, that the properties established above indicate that $\overline{\theta}$ is independent of the eddy diffusivity profile in the limits $Pe \to 0, \infty$, which probably diminishes the number of degrees of freedom of the behaviour of the function $\overline{\theta}(Pe)$ between these limits.

6. The Neighbourhood of the Pycnocline

The residence time associated with the quadratic eddy diffusivity (32) tends to zero as the distance to the pycnocline decreases. By contrast, the residence time associated with a constant eddy diffusivity exhibits a different behaviour in the vicinity on the lower boundary of the domain of interest. For instance, at $z = 0$, the residence time (31) is $(1 - e^{-Pe})/Pe$, a value larger than zero. To gain some insight into the cause of this discrepancy, it is useful to first examine the timescales characterising settling and mixing near the bottom of the mixed layer. If the eddy diffusivity is asymptotic to z^a $(a \geq 0)$ in the limit $z \to 0$, then the time needed to mix the constituent under consideration over a height of order z may be estimated as

$$\tau_m \sim \frac{z^2}{\kappa(z)/Pe} \sim \frac{z^2}{z^a/Pe} \sim Pe z^{2-a}, \quad z \to 0. \tag{40}$$

The time needed to travel a distance z with a unit velocity is a relevant settling timescale:

$$\tau_s \sim z, \quad z \to 0. \tag{41}$$

The ratio τ_s/τ_m, which may be regarded as a local Peclet number, is then

$$\frac{\tau_s}{\tau_m} \sim \frac{z^{a-1}}{Pe}, \quad z \to 0. \tag{42}$$

If $a > 1$, the ratio τ_s/τ_m tends to zero as $z \to 0$, implying that the particles sink into the pycnocline so fast that turbulent mixing cannot have any significant influence. Therefore, the residence time must tend to zero as the bottom of the domain is approached. On the other hand, if $0 \le a < 1$, turbulent diffusion is more efficient than settling so that particles are likely to disperse before they can all leave the domain by entering the pycnocline. As a consequence, the residence time should tend to a finite, non-zero value in the limit $z \to 0$. There is no clear answer, however, if $a = 1$.

It is possible to develop a mathematically rigorous – though less physically appealing – approach to the behaviour of the residence time in the limit $z \to 0$ that is valid for any $a \ge 0$ and is consistent with the above reasoning based on the comparison of local timescales. If $\eta(z)$ is the indefinite integral of $-1/\kappa(z)$, then expression (30) may be transformed to

$$\theta(z) = z + e^{-Pe\eta(z)} \int_z^1 e^{Pe\eta(\xi)} d\xi. \tag{43}$$

It is again assumed that $\kappa(z) \sim z^a$, so that $\eta(z) \sim z^{1-a}/(a-1)$ provided $a \ne 1$, and $\eta(z) \sim -\log z$ if $a = 1$. This immediately leads to the value of the residence time at the lower boundary of the domain:

$$\lim_{z\to 0} \theta(z) = \begin{cases} 0 & \text{if } a \ge 1, \\ > 0 & \text{if } a < 1. \end{cases} \tag{44}$$

So, the residence time in the vicinity of the pycnocline crucially depends on how “fast” the diffusivity tends to zero.

7. Conclusion

Using the adjoint model approach of Delhez *et al.* [13], an expression was derived from which the residence time at any point of the mixed layer may be obtained. This formula is valid for steady-state hydrodynamics. If the latter hypothesis does not hold true, the theory of Delhez *et al.* [13] may again be resorted to. However, such a generalisation, though technically straightforward, is likely to provide neither analytical solutions nor clear conclusions, especially if particles that sank into the pycnocline can be re-entrained back into the mixed layer as the latter deepens.

A detailed sensitivity analysis was carried out, showing that the residence time decreases as the Peclet number increases. At the top of the mixed layer, the residence time was seen to be independent of the eddy diffusivity, whereas the profile of the latter is of a critical importance in the neighbourhood of the pycnocline. Finally, it is conjectured that the average of the residence time over the thickness of the mixed layer, $\overline{\theta}$, is largely independent of the space dependency of the eddy diffusivity and is given to a good degree of accuracy by

$$\overline{\theta} \approx \frac{h}{w} \frac{\overline{\kappa} + wh/4}{\overline{\kappa} + wh/2}, \tag{45}$$

an expression in which only dimensional quantitie are employed, making it ready for use once the relevant physical parameters are known.[2] It must be stressed that relation (45) is the dimensional counterpart of expression (39). In other words, (39) and (45) are equivalent, but dimensionless variables are used in the former while the latter resorts to dimensional quantities.

The present theory may be applied to particles that tend to float out of the bottom mixed layer. No significant modifications are needed, except that the vertical coordinate would then measure the distance to the top of mixed layer and increase downward. In addition, the vertical velocity w would denote the floating velocity rather than the settling one.

As is well known, Lagrangian transport models must include a term taking into account the space variations of the eddy diffusivity (e.g. [20, 21]). Such models are not easy to validate for want of relevant analytical solutions. However, an indirect validation may be possible: the residence time of Lagrangian particles would be computed and compared with that ensuing from the theory developed herein, for the latter is applicable to any type of eddy diffusivity profile.

Appendix

Upon assuming that $C(0, z) = \delta(z - z_0)$, the residence time $\theta(z_0)$ may, in principle, be obtained from integral (8). However, the latter requires the knowledge of the concentration $C(t, z)$ at any time t and position z. This is generally impossible. This difficulty may be circumvented by first inverting the time integration and space integration in expression (8), allowing the latter to be transformed to

$$\theta(z_0) = \int_0^h \int_0^\infty C(t, z) \mathrm{d}z \, \mathrm{d}t \tag{A1}$$

Upon introducing the variable

$$\tilde{C}(z) = \int_0^\infty C(t, z) \, \mathrm{d}t, \tag{A2}$$

relation (A1) may be rewritten as follows:

$$\theta(z_0)=\int_0^h \tilde{C}(z)\,\mathrm{d}z. \tag{A3}$$

No particle of the constituent under study remains in the domain of interest in the limit $t\to\infty$, i.e. $C(\infty,z)=0$. Taking this end value into account, bearing in mind the initial value $C(0,z)=\delta(z-z_0)$, Equations (A1)–(A3) are integrated in time, yielding the following ordinary differential problem:

$$-\delta(z-z_0)=\frac{\mathrm{d}}{\mathrm{d}z}\left(w\tilde{C}+\kappa\frac{\mathrm{d}\tilde{C}}{\mathrm{d}z}\right), \tag{A4}$$

$$\left[w\tilde{C}+\kappa\frac{\mathrm{d}\tilde{C}}{\mathrm{d}z}\right]_{z=h}=0, \tag{A5}$$

$$\left[\kappa\frac{\mathrm{d}\tilde{C}}{\mathrm{d}z}\right]_{z=0}=0. \tag{A6}$$

Lengthy calculations are necessary to derive the solution to (A4)–(A6). Substituting the latter into (A3) eventually leads to

$$\theta(z_0)=\frac{z_0}{w}+\frac{1}{w}\int_{z_0}^h \exp\left[-w\int_{z_0}^{\xi}\frac{\mathrm{d}\zeta}{\kappa(\zeta)}\right]\mathrm{d}\xi, \tag{A7}$$

which may be rewritten as a function of z that is equivalent to the general expression of the residence time provided in Section 4, i.e. formula (21).

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Notes

1. It is appropriate to bear in mind that dimensionless variables are being dealt with: the constant λ is necessary for the average over the height of the mixed layer of the – dimensionless – eddy diffusivity to be equal to unity, as is required by (27).

2. In general, the eddy diffusivity cannot be estimated directly from in situ measurements. An estimate that is hopefully acceptable and consistent with (32), assuming $\alpha = 1$, is as follows: $\kappa(z) = \gamma u_* z(1 - z/h)$, where $\gamma \approx 0.4$ and u_* are the von Karman constant and the surface friction velocity, i.e. the square root of the ratio of the wind stress norm to the water density. Then, the depth-averaged eddy diffusivity reads $\overline{\kappa} = \gamma u_* h/6$, a value that can be substituted into (45), yielding

$$\bar{\theta} \approx \frac{h}{w}\frac{2\gamma u_* + 3w}{2\gamma u_* + 6w}.$$

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