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Reifying actions into artifacts: process–object duality from an embodied perspective on mathematics learning

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Abstract
Grasping mathematical objects as related to processes is often considered critical for mathematics understanding. Yet, the ontology of mathematical objects remains under debate. In this paper, we theoretically oppose *internalist approaches* that claim mental entities as the endpoints of process–object transitions and *externalist approaches* that stress mathematical artifacts—such as physical manipulatives and formulas—as constituting mathematical objects. We search for a view on process–object duality that overcomes the dualism of mind and body. One such approach is commognition that describes mathematical objects as discursive entities. This paper expands the nature of mathematical objects beyond discourse and highlights the role of learners' interaction with the environment by adopting ecological onto-epistemology. We develop a functional dynamic systems perspective on process–object duality in mathematics learning emphasizing embodied actions and the re-invention of artifacts' affordances. As a main result, we reconsider process–object duality as a reification of repetitive actions into a cultural artifact that consists of two steps: (1) forming a new sensory-motor coordination that brings new perception to the fore and (2) crystallizing a new artifact in a mathematical environment that captures this new perception. An empirical example from research on embodied action-based design for trigonometry illustrates our theoretical ideas.

Keywords Cultural artifacts · Embodied cognition · Functional dynamic systems · Mathematics education · Process–object duality · Reification

1 Introduction

For a few decades of mathematics education research's expansive development, the dual process–object nature of mathematical concepts has been prominent in explaining mathematics learning (Dubinsky, 2002; Sfard, 1991, 1994; Sierpinska, 1994). Those ideas, as developed in the 1990s, remain popular. For example, according to the Scopus database, there are 40–50 citations per year of a classical paper on the dual process–object nature of mathematical concepts by Sfard (1991). Based on the ideas of Piaget and Vygotsky

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about internalizing external actions, researchers highlighted the transition from actions with physical manipulatives and spatially articulated inscriptions to the abstract mathematical objects of the next level. The ontological nature of these mathematical objects was consonant with the, at that time, dominant constructivist view on mathematics learning, or described with the available notions from cognitive science. Researchers aligned mathematical objects with diverse internal cognitive entities, such as a scheme (Piaget, 1970), metaphorical reference to body schemata (Lakoff & Johnson, 1980), propositional structures, and visual images (Paivio, 1990; Richardson, 1969).

Cognitivist and constructivist approaches have been criticized from multiple sides. The first line of critique comes from the inside of mathematics education research (Thompson & Sfard, 1994). The internal cognitive structures used to ontologically ground mathematical objects are consonant with the idea of mental representations. However, the concept of mental representation presumes the existence of objects in reality, like chairs, to be represented in the mind. Such a priori existence of objects in external reality available for sensual experience is problematic for mathematical objects.

Another line of critique embraces the first one with a more general skepticism toward the idea of mental representations, developed in radical embodied cognitive science (Chemero, 2009; Kiverstein & Rietveld, 2018). Researchers question the ontological existence and the explanatory strength of mental representations and offer alternative models of the human mind: cognition is considered to be built to *act* in reality as efficiently as possible, without the need to *represent* reality precisely as it is.

The third line of critique comes from sociocultural approaches that stress social interaction and material culture as critical for mathematics learning. Researchers build on Vygotsky and Leontiev to describe learning as the development of operations with cultural artifacts rather than developing internal structures (e.g., Radford, 2005, 2014).

In this paper, we aim to reconsider process–object relations and transitions without appealing to mental representations of external objects and thus develop a view that overcomes opposition of mind and body. We build on a radical embodied cognitive science that has a non-dualistic view on cognition and assumes that cognition arises to enable acting in the world instead of representing it (Varela et al., 1991; Chemero, 2009). This non-dualistic perspective highlights the role of the body and of interaction with the environment. Although radical embodied cognitive science was initially developed to address low-level cognitive processes, currently intensive attempts are being made to extend those ideas to higher-level processes, such as thinking and conceptual understanding (Abrahamson, 2021; Kiverstein & Rietveld, 2018; Sanches de Oliveira et al., 2021). Our paper contributes to this challenge by investigating the research question of *how we can reconsider processes, objects, and their relations in mathematics learning without appealing to mental representations and accounting for learners' embodied interaction with their environment?*

Expansion of radical embodied ideas to higher-level cognitive processes led us to highlight the role of material culture as constantly transforming the environment. As a result, we bridge radical embodied cognitive science with sociocultural ideas (see also Abrahamson & Trninic, 2015; Baggs & Chemero, 2020; Pagnotta, 2018). Sociocultural ideas have been already appropriated to mathematics education research in the commognition framework (Sfard, 2008) and in objectification theory (Radford, 2021), to name just a few. Yet, we suggest the synthesis with radical embodied cognition perspective allows us to reconcile the cultural-historical approach with a new type of cognitive science, thus contributing to overcoming the contradiction between cognitive/constructivist and sociocultural approaches (Lerman, 1996).

Answering our research question, we dedicate Section 2 to re-thinking previous dialectics of processes and objects in mathematics education research: we discern the approaches that are focused on cognitive structures (Dubinsky, 2002; Sfard, 1991) and the approaches that already have attended to the role of material culture in the form of mathematical notations and inscriptions (Gray & Tall, 1994; Kaput, 1991). We pay special attention to the commognitive vision of process–object relations that offers discursive ontology to avoid mind-body dualism (Sfard, 2008).

Next, in Section 3, we set up our own theoretical scene by introducing the core concepts of ecological onto-epistemology and suggest specific ecological and cultural-historical meaning for the notions of ideal and material. We also introduce the notion of a body–artifacts functional dynamic system that becomes a core construct for further theoretical elaboration. In Section 4, finally, we develop a *functional dynamic systems* approach to processes, objects, and their relations in mathematics learning. We focus on flexibility and yet invariance of sensory-motor interactions that stabilize and further reify into cultural artifacts. Thus, material and perceptual transformations become critical in students' re-invention of mathematical environments and practices during learning.

In Section 5, we provide a brief empirical illustration of our theoretical statement that builds on a study of learning trigonometry with embodied action-based design (see more in Shvarts & van Helden, 2023).

2 Process–object dialectics in the research of mathematics learning

In the 1980s–1990s, many researchers highlighted a transition from performing mathematical procedures to representing and operating with those procedures as objects to be critically important for the development of mathematical thinking. This phenomenon has acquired many names: *reification* (Sfard, 1991), *encapsulation* (Dubinsky, 2002, original publication in 1991), *entification* (Kaput, 1991), and formation of a *procept* (Gray & Tall, 1994). Despite being often called synonymous, these theoretical constructs essentially diverge in their ontological view on reified objects. From an embodied perspective, this divergence is critical as it uncovers the tension of considering mathematical concepts as mental entities versus objects embedded in material culture. Later, a non-dualist commognition perspective aims to overcome this opposition by addressing the process–object transition as a transformation of discourse (Sfard, 2008).

2.1 From processes to objects as internal abstract entities

The first view on process–object relations considers a reified object as a mental entity that can be described through psychological notions. Dubinsky (2002)—the main author of a constructivist APOS theory inspired by Piaget—describes mathematical thinking as developing from external *actions* to internalized *processes*, which are further encapsulated into *objects*. Further, multiple mental objects and mental processes are organized as a "more or less coherent collection" (p. 101) into a *schema* that exists "inside a person" (p. 102).

In her early version of reification theory, Sfard (1991) suggests a similar trajectory for the formation of mathematical concepts by a learner. Based on the analysis of mathematical concepts' historical development, three stages are distinguished: interiorization, condensation, and reification. The same stages are expected to be found in the development of an individual learner. Interiorization consists of automatizing "a process performed on the already familiar objects"; once automatized, the process becomes condensed, thus "turning this process into an autonomous entity"; and "finally the ability to see this new entity as an integrated, object-like whole" determines the final stage of reification (p. 18). Semantic propositional networks are described as plausible theorization of objects: "The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category" (p. 20). Sfard also refers to dual coding theory (Paivio, 1971)—developed in cognitive science at that time—that states the existence of not only verbal, but also visual representations.

2.2 From processes to objects as external symbols

We may distinguish a second view on the nature of reified objects in another set of theories. Researchers describe *entification* (Kaput, 1991) and *procept* formation (Gray & Tall, 1994) as transitioning from a process to an object. Rather than mental entities, these authors highlight concrete symbolic expressions as the ontological ground (or at least physical vehicles) of mathematical objects: "Complex ideas or mental processes can be chunked and thus represented by physical notations which, in turn, can be reflected on or manipulated to generate new ideas" (Harel & Kaput, 1991, p. 87). The choice of symbol appears to be critical for making symbolic manipulations easy or hard, as it can carry some external cues for interpreting it or not. For instance, unlike a non-elaborated notation T, the expression t_{ij} immediately delivers the entity as belonging to some two-dimensional array, thus directing possible manipulations (see p. 92). So, a particular notation determines the "allocation of responsibility" (Kaput, 1991, p. 69) between the notation's features and a person who is going to use it according to the embedded purpose.

Gray and Tall (1994) introduce a "procept" as an "amalgam of concept and process represented by the same symbol" (p. 121) to point at the double functionality of a symbol that can be interpreted and acted upon as an object for further manipulations and as a condensed way of writing down a procedure. So, the process–object transition is essentially entangled with introducing a new notation. This notation makes manipulation with a mathematical concept directly available in a truly physical, not metaphorical (cf., Sfard, 1994), sense: "Three is an abstract concept, but through using it [symbol 3] in communication and acting upon it with the operations of arithmetic, it takes a role as real as any physical object" (Gray & Tall, 1994, p. 122).

The theories within this branch highlight slightly different mechanisms for making symbolic notations so powerful. For Gray and Tall, the "ambiguous use of symbolism is at the root of powerful mathematical thinking" (p. 125). A symbol can be presented in short-term memory in a very compact form that facilitates mental manipulations; it can also trigger a sequence of actions that it has symbolized and that require some time. While Tall and Gray treat symbols as both mental and written entities, Harel and Kaput are more radical in treating the materiality of notations as critical in object formation. They stress the need for continuous perceptual availability of the notations: "By providing continual perceptual experience, material notations help provide the basis for continuing conceptual presence" (Harel & Kaput, 1991, p. 89).

2.3 The commognitive approach: discursively turning processes into objects

From a radical embodied perspective, the theories oriented on mental entities and theories oriented on external notations are drastically different. The notions of *encapsulation* and the initial notion of *reification* rely on the idea that cognitive structures change. The notions of *entification* and *procept* are based on external symbolization. This deliberately sharpened opposition is reminiscent of a body-mind dualism, typical for traditional cognitive science, and confronted by the radical embodied cognitive perspective (Chemero, 2009). A discursive approach to mathematical objects, elaborated by Sfard (2008) and influenced by Vygotsky and Wittgenstein, aims to overcome this dualistic tension between mathematical objects as mental entities and as tangible notations.

Sfard (2008) highlights the discursive ontology of mathematical objects. They exist within communication in mathematical words, formulas, sketches, and other inscriptions that realize signs of communicating partners who act and react to each other's expressions. As signs point at each other, they form realization trees. Realization assumes making a sign perceptually accessible in inscriptions or physical items that are manipulated. Reification, as a transition from processes to objects, is theorized as "discursively turning processes into object" (p. 43) by means of introducing nouns into a discourse that has previously described actions and procedures.

As new discursive means are introduced, mathematics objects do not become detached from materiality: "mathematical objects are not any less material than the primary objects, except that rather than being a single tangible entity that predates the discourse, they are complex hierarchical systems of partially exchangeable symbolic artifacts" (Sfard, 2008, p. 172). Moreover, through learning, automatization, and embodiment of discursive rules, operations with new discursive means become natural and immediate, thus making symbols (e.g., vertical notation of fractions) our "second nature" (p. 158). So, as new notations shape discursive practices, perception and action change to comply with discursive rules; thus, "visual perception plays as fundamental a role in mathematics as in any other discourse, except that the manner in which the sense of sight is employed is more complex and less obvious" (p. 162).

While discursive ontology highlights a special discursive status of mathematical objects, in our view, it might not pay sufficient attention to the transformations of material environments and embodied processes that happen in acquiring and developing cultural practices. We see such transformations as, for example, building a new 3D construction or drawing a graph to be reaching beyond discourse. In this paper, we also take a non-dualistic onto-epistemological stance, yet we intend to describe mathematical objects as conceived in deep layers of organism–environment interactions within cultural practices.

3 Setting up a theoretical scene

As has been mentioned, one source of inspiration for revising reification is radical embodied cognitive science (Chemero, 2009; Varela et al., 1991). This perspective is rooted in ecological psychology, developed to explain perception (Gibson, 1986), and in coordination dynamics that address bodily movements (Kelso & Schöner, 1988; Thelen, 2000). The other source of inspiration is sociocultural approaches developed by Wenger (1998) and Leontiev (1978) following philosophical ideas about reification of Marx, Weber, and Lukacs. In this section, we introduce the core theoretical constructs that further ground our reconsideration of the transition from processes to objects in mathematics learning that avoids body-mind dualism and accounts for embodied interactions and material transformations.

3.1 Ecological ontology and cultural environments

We develop an approach to mathematics learning as a continuation of research on perception and movement; such a framework is meant to provide conceptualization of mathematics as a natural continuation of cultural and biological evolution (Abrahamson, 2021). Our theory is based on a monist ontological position that assumes motor behavior and thinking are of the same ontology. *Ecological ontology*[1], as it is elaborated by Turvey (2019), assigns to *a possibility of organism–environment interaction* the primary existence over an existence of a reality that is independent of any interaction. An *environment*, as a place for living beings, can be meaningfully described not in its physical parameters but in its relation/coupling to living beings' actions.

An action possibility—*an affordance*—is directly available for perception by an organism (Gibson, 1986) and within ecological ontology serves as a primary quality of the world (Turvey, 2019, p. 34). A cat sees surfaces as "jumpable," and a snake as "crawlable." To perceive an affordance, a person needs to have some capability for corresponding actions. This ability is embodied in a human in the form of *body potentialities*, namely possibilities for an action provided by the body, such as muscles and neurons (de Freitas, 2016; Shvarts et al., 2021). Just like an environment embeds forms of action in affordances, a body, taken ecologically, holds a potential for an action.

Expanding this approach to cultural practices and environments, we can talk about cultural affordances that only skillful members of society recognize (van Dijk & Rietveld, 2017). Similar to a surface looking "jumpable" to a cat, a cup of tea looks "drinkable'' to an adult, or an equation looks "solvable" to a mathematician. In a cultural environment, various affordances co-exist as a nested system within the same piece of environment (van Dijk & Rietveld, 2017): apart from serving the specific function of drinking tea, a cup has a less specific affordance as it can also be thrown at a threatening animal in case of danger.

3.2 Body–artifacts functional dynamic system

While affordances and body potentialities constitute an ecological landscape of potential actions, a *functional dynamic systems* (FDS) idea explains how they are actualized in organism–environment interactions. A motor action is constituted to solve *a motor problem*, namely reach some desirable outcome[2] (Bernstein, 1967). An organism is capable of forming FDSs; each of these systems is an emergent dynamic unit of body potentialities that are activated in fulfilling a functional request. For example, when walking, activation of muscles, proprioceptive receptors, and motor neurons are organized into FDS.

A critical aspect of an FDS is its emergent character and spontaneous adaptivity to the changes of the environment combined with stability in maintaining functional performance. This adaptive flexibility is explained by the principle of *synergy*, namely an

[1] Note that ecological ontology assumes taking an *onto-epistemological* rather than ontological stance. We offer a *possible* theoretical view that conceives students' learning as acquiring ways to act and constantly transform the environment, thus mixing ontology and epistemology; on a meta-level, this view does not attempt to create a precise description, but to support educator's interaction with the environment through designing it. See more on onto-epistemology in de Freitas and Sinclair (2018) and Stetsenko (2020).

[2] The question of where and how this desirable outcome exists within ecological ontology (without being a disembodied representation) is solved through the notion of multilevel intentionality and goes beyond the scope of this paper. See more details in Shvarts et al. (2021) and Shvarts & Abrahamson (2023).

emergent self-organization of muscular and neuronal elements into relatively stable coordination that allow fulfillment of higher-order aims without a direct guidance that would follow a pre-defined procedure (Kelso & Schöner, 1988). The notion of synergy is based on Haken's principle from theoretical physics: "faster individual microscopic elements in a system become 'enslaved' (entrained) to much slower varying macroscopic 'collective variables'" (Kelso & Engstrom, 2006, p. 113). This means that higher levels of an FDS provide only general guidance, thus fostering an invariant functionality of the target behavior (e.g., throw a ball into a basket) yet leaving freedom in shaping a concrete way to reach it. Simultaneously, the lower levels self-organize into coordinated independent units (e.g., maintain a vertical position, grab the ball) and thus flexibly and adaptively support target behavior reacting to small changes in the environment (e.g., the slope of the ground, size of the ball) (see the details on movement construction in Bernstein, 1967).

Expanding this approach to behavior in a cultural environment, we introduce the notion of *a body–artifacts functional dynamic system* (Shvarts et al., 2021). This notion is consonant with the ideas of a *fusion of a body and tool* and *human-with-media* exploited in mathematics education (Borba & Villarreal, 2005; Ferrara & Ferrari, 2020) and with the *extended cognition* ideas (Clark & Chalmers, 1998; Kirchhoff & Kiverstein, 2019). Yet its theoretical power lies in logically bridging coordination dynamics—the science of movement—with a cultural-historical approach—an approach that theorizes higher cognitive functions as mediated by artifacts. A *body–artifacts FDS* emerges in response to a task and dynamically couples a student's body potentialities and artifacts' affordances in perception-action loops, thus forming a body–artifacts system (Shvarts et al., 2021). For example, when eating, a person's body is extended by a spoon. When reading vector fields or coordinates of a point, students' bodies are extended by a Cartesian coordinate system that fosters eyes to move along the axes (Krichevets et al., 2014, Klein et al., 2018). So, in cultural practices, artifacts, including words, extend bodily capacities by orienting perception of and action in the environment in a mathematical way (Shvarts & Abrahamson, 2023). Discursive practice is understood as a subtype of organism–environment interaction; thus, ecological ontology embraces discursive ontology.

3.3 Reconsidering ideal and material ecologically

Traditional discourse in cognitive science and mathematics education is engaged with oppositions that are mostly considered as dualities excluding each other: mind–body, ideal–material, mental–tangible, and abstract–concrete. An idea of nested affordances—embedded in cultural environments and artifacts—allows us to overcome those dualities by reconsidering ideal and material from a sociocultural perspective.

Expanding Marx's analysis of commodity to theorizing any cultural artifact, Ilyenkov explains: "All these objects are in their existence, in their 'determinate being' substantial, 'material', but in their essence, in their origin, they are 'ideal', because they 'embody' the collective thinking of people, the 'universal spirit' of mankind" (Ilyenkov, 2012, p. 171). So, cultural artifacts are ideal as "they contain, in coded form, the interactions of which they were previously a part and which they mediate in the present" (Cole, 2016, p. 250). At the same time, their materiality is also indispensable: "they [artifacts] are material because they exist only insofar as they are embodied in material artifacts,"—continues Cole. Artifacts, including words, are *sensible–supersensible* (Roth, 2020), which means that they can be involved in an activity based on their material form, ignorant of any cultural use, and based on their ideal form—in a culturally normative way. A spoon enables eating, but also hitting a fly if needed; a sine graph enables modeling oscillations, but also drawing sea waves in a computer program. Actual enactment is always an entanglement of those cultural and material practices, embedded into artifacts as potential forms of action.

Those elaborations allow us to re-define *material aspects of the environment* as affordances that hold potential for physical manipulations, available to a human as an animal with genetically pre-given body potentialities, such as the shape and size of our body. Those are affordances of supporting the body, hitting an object with a hand, stepping on, etc.; they can be described as size, hardness, and other material qualities. *Ideal aspects of the environment* afford cultural manipulations available to an educated human. Those are affordances of drawing with a pen, opening brackets in equations, and so on. Such cultural affordances come to be recognized when coupled with body potentialities developed in ontogenesis through learning, such as specific muscle strength and synaptic connections in the brain. Critical for our further analysis of reification within the body–artifacts FDSs approach, artifacts embody material and ideal aspects at the same time: these traditionally opposed aspects can be reconsidered as jointly shaping an artifact.

4 Theoretical proposal: process–object transition as actions reified in artifacts

4.1 A first step of reification: sensory-motor coordinations bringing new perceptual structures to the fore

We agree with the classical approaches to reification (Dubinsky, 2002; Sfard, 1991); that action automatization and fluency are the first move towards reifying a process into an object. Considering mathematics to be a cultural continuation of human biological abilities, we borrow the view on action automatization from the literature on motor action regulation and sports education.

Motor actions are constituted by assembling an FDS of neuronal and peripheral elements to fulfill a functional goal. In developing a new FDS, multiple levels of action regulation come to work together in a new way as the body establishes a new efficient sensory-motor coordination (Bernstein, 1967). A mechanism of synergy plays a crucial role in establishing automatization and fluency: low levels self-organize into a flexible, functional unit and substitute deliberate control of step-by-step enactment. Established coordinations stabilize and consolidate and stay as persistent motor skills, e.g., biking. Body potentialities, such as synapses of neurons, are gradually shaped by efficient activation and support future re-activation in performing functionally similar behavior (Sporns & Edelman, 1993). New neuronal paths support and reify established behavior, similar to how a new path in a forest emerges and facilitates further walking.

While mastering a new skill, the body exploits new affordances in the environment (Gibson, 1986). Once a skill is mastered and new body potentialities developed, those affordances become perceptually recognized by a skilled performer (van Dijk & Rietveld, 2017). We relate affordance recognition to the synergetic quality of functionally organized sensory-motor coordinations: while a motor synergy manifests for a subject as a fluent motor action, a sensorial synergy manifests as a new perceptual structure. Accordingly, sports education research evidences that regulation of complex actions can be best facilitated by attention to specific perceptual structures in the environment—attentional anchors (Hutto & Sánchez-García, 2015), such as figures imagined by jugglers.

From an embodied perspective, new coordinations enabled by sensory-motor synergies ground mathematical performance as well (Abrahamson, 2021). The studies that trace finger motions and eye movements uncover that new patterns of sensory-motor activity appear before participants can describe new perceptual structures that support their higher-level problem-solving strategies (Stephen et al., 2009; Tancredi et al., 2021). The emergence of mathematical objects is prepared by automatizing and stabilizing actions that enable new perception (Abrahamson & Sánchez-García, 2016).

Remarkably, action proficiency leads to different futures in sports and in mathematics learning. In sports, proficiency comes with automatization and leads to the disappearance of motor action regulation and attentional anchors from the consciousness. For example, while cycling, people normally do not notice dynamic synergies that regulate balance. In contrast, proficient stabilized actions in mathematics are preserved in learners' awareness and become new objects, as process–object dialectics highlights (Dubinsky, 2002; Gray & Tall, 1994; Harel & Kaput, 1991; Sfard, 1991). Learners can now distinguish new mathematically relevant perceptual structures in the environment—attentional anchors of action regulation that become new objects (Abrahamson, 2021; Abrahamson & Sánchez-García, 2016). Yet what makes the future of action proficiency in sport and mathematics so different?

4.2 A second step of reification: crystallization of artifacts reifying actions

We suggest that a new cultural artifact—a sketch, a symbol, or a definition—plays a critical role in preserving a stabilized action in the form of an object in the environment. From a sociocultural perspective, reification happens within cultural practice as established forms of action become fixed material objects: "human experience and practice are congealed into fixed forms and given status of object" (Wenger, 1998, p. 59). According to Wenger, the process of reification can be seen in "making, designing, representing, naming, encoding" and ends by producing names, poems, encyclopedias, formulas, recipes, and all other forms of material culture that preserve the experience of an efficient practice (an ideal form) by establishing a material form that facilitates this cultural practice.

The transformation of a stabilized action into an artifact in the history of mathematics is, again, similar to the gradual emergence of a path in a forest. In the evolution of culture, sharp stone spearhead-artifacts were co-shaped by stone materials and human bodies through iterative attempts and re-approaching within the hunting task (cf. FDS) (Malafouris, 2010). Similarly, the abacus gradually developed from a set of stones to the highly efficient modern Japanese soroban that has 4 and 1 stones in each row and perfectly matches a hand structure (Monaghan, 2016). Cartesian coordinates—initially represented as one axis and an arrow in a non-orthogonal direction—gradually become standardized as two orthogonal lines, thus adapting to the structure of an eye that moves more fluently horizontally and vertically (Krichevets et al., 2014). All these artifacts reify and preserve cultural actions in their material structure. Following Leontiev (1978), we call this process of shaping artifacts *crystallization*: as actions reify in artifacts, the artifacts are crystalized and facilitate further actions by their cultural affordances that naturally build upon their material structures.

A gradual reification of stabilized actions into cultural artifacts—such as verbal names, notations, sketches, and formulas—can also be traced in educational practice. While learning, gestures gradually transform from an action to an artifact: iconic gestures directly catch spatial characteristics of repetitive actions (Nathan & Alibali, 2011); further metaphorical gestures gradually substitute the iconic ones (Maffia & Sabena, 2020) and become descriptions stored in notations or sketches. Another mechanism that enables capturing actions into cultural artifacts is the synchronization of multimodal inscriptions within semiotic bundles (Arzarello, 2006): actions and gestures are tightly synchronized with verbal utterances and formulas, forming a unity, a bundle. Formulas allow one to capture and reify multimodal dynamics of interaction with other mathematical artifacts and later to re-activate these dynamics when a formula is brought into the environment again (Shvarts & Abrahamson, 2023). The effectiveness of a multimodal revoicing strategy—when a teacher enhances one modality of a student's gestures/utterances while repeating another modality (Flood, 2018)—evidences that gradual transition from actions to verbal artifacts supports understanding.

Altogether, when learning with understanding, artifacts crystallize for the students as preserving their embodied experiences. Artifacts become objects of students' ecological environments available for further mathematical operations. This transformation of dynamic actions into (relatively) static artifacts is, we submit, what researchers describe as process–object duality. Unlike in sports, body–artifacts FDSs in mathematics *transform their environment and create new artifacts that preserve emergent enactment and facilitate future actions*.

The presence of material and ideal aspects (as possibilities for physical and cultural actions) in the nested affordances of an artifact is critical for operating with an artifact, including its creative use in future practices. A chair's materiality provides an affordance to support; a chair's ideality provides an affordance to sit at a table. Cartesian coordinates' materiality provides an affordance for moving an eye vertically and horizontally; Cartesian coordinates' ideality provides an affordance to read the coordinates of a point. The solid material form of an artifact delivers "continual perceptual experience" (Harel & Kaput, 1991, p. 89) and availability for further manipulations. At the same time, ideal forms (cultural affordances) preserve mathematical meaning. We suggest that the peculiarity of mathematical practice over other cultural practices (cf. cooking) is its relative independence from the available artifacts in the environment: unlike chairs and spoons, mathematical artifacts can be *re-created at any moment* from body potentialities of a thinker (e.g., by drawing on paper or sand or by speaking words, out loud or not), thus enabling mathematical reasoning.

5 Empirical illustration: reification of an action with a unit circle and crystallization of a sine graph

To clarify the suggested approach, we provide a brief empirical illustration from a large design research (Bakker, 2018) that explores how technological environments can support students' understanding of a sine graph (Alberto et al., 2019; Shvarts & Alberto, 2021; Shvarts & van Helden, 2023). We work within the embodied action-based design genre (Abrahamson, 2014; Abrahamson et al., 2011; Alberto et al., 2021) that creates a specific lens making visible how students' understanding emerges from enactment and revealing the role of cultural artifacts.

In a designed digital environment, continuous color feedback supported students in discovering relations between a unit circle and a sine graph through coordinating hand movements along the unit circle and on the Cartesian plane. After solving sensory-motor tasks of coordinating arcs and sine values with the distances along the axes, students would reflect on their coordinating movement and further solve mathematical tasks related to the coordinated distances. In a post-test, the students estimated sine values and drew a graph of $y = \sin(x)$. Students worked in the digital environment on their own. Later, the interviews with the students were conducted aiming to investigate their understanding in depth. The interviews started from recalling the embodied activities based on the screenshots and continued by collaboratively exploring solutions of the post-test tasks. We refer the reader to Shvarts & van Helden (2023) for more details.

Lukas, a 10th-grade student (15 years old), was chosen as an example based on his proficiency in mathematics and vivid gesticulation: those two factors made the reification process observable in a short period of time. Before the study, Lukas had only seen a sine wave, which he drew in the pre-test as a general wave without a particular period or amplitude.

5.1 Analysis

Introducing the method of their Vygotskian semiotic approach, Radford and Sabena (2015) state: "methods are rooted in theoretical principles that convey worldviews" (p. 159). We cannot agree more. Introducing a shift towards ecological onto-epistemology, we frame the analysis in line with the introduced theoretical principles thus focusing on (1) actions and affordances in the environment that make those actions possible and (2) artifacts that mediate these actions as students notice them in the environment and incorporate into body–artifacts FDSs. The unit of our analysis is the reification of actions and crystallization of artifacts. Thus, we trace how the action dynamics transform to become a static artifact in the environment based on the student's drawings and multimodal discourse as the student unpacks his understanding.

Analyzing the verbal part of the discourse, we used the classifications elaborated by Sfard, who described reification as "substituting talk about actions with talk about objects" (Sfard, 2008, p. 44), which can be traced as transitioning from describing a process of doing using verbs to describing stable entities using nouns. We coded verbal utterances as referring to *the student's action* (e.g., "I move…," marked in blue in the transcripts) and as referring to *artifacts and their properties* (e.g., "It is a regular curve," marked in red). We also noticed that the student often expressed *actions of an artifact*; we considered them as an intermediate code (e.g., "Graph goes down…," marked in violet).

Similarly, we coded different types of gestures that allowed us to distinguish whether the student expresses his actions or describes artifacts available in the environment. *Pointing* gestures (McNeill, 1992; Nathan & Alibali, 2011) evidenced a separate artifact in the student's environment that he could point at. *Representational* gestures were subdivided in *iconic* gestures—in which the student delivered a crystallized artifact quickly, in one stroke—and *enacting* gestures (Masson Carro et al., 2015; McNeill, 1992) characterized by slowly gesturing on the table, closely observing his own gestures, as if *enacting* again his previous action. *Pointing* and *iconic* gestures evidenced accomplished reification (marked in red) and *enacting* gestures evidenced non-reified action (marked in blue).

Accumulating the analysis of the drawings and multimodal discourse into our theoretical categories, we distinguished *previously incorporated artifacts into the body–artifacts FDS*, *new actions* (*stabilized* performance, enabled by fluent *sensory-motor coordinations* and *new perceptual structures*), and *newly crystallized artifacts*.

5.2 Results

Our empirical results aim to illustrate the two steps of reification so that theoretical construction can be better grasped. Episode 1 presents step 1: a new sensory-motor coordination brings to the fore a new perceptual structure; episode 2 presents step 2: crystallization of a new artifact that reifies previous actions.

5.2.1 Episode 1: a new sensory-motor coordination brings forth new perceptual structure

Looking at a screenshot of an embodied activity (Fig. 1), Lukas recalled how he was coordinating an arc on a unit circle and a segment on the *x*-axis (see Table 1, R stands for researcher and L stands for Lukas).

In this episode, we can see the first step of reification, namely a new action and new forms of perception (see Table 2 for a summary). As Lukas recalls the embodied activity, he reports a new perception of the equality of an arc and a segment as he developed new sensory-motor coordination (“I saw that…,” Table 1: 1.2). For Lukas, new perceptual structures are interconnected with bodily actions with previously acquired artifacts—namely, a unit circle and *x*-axis. The answer to the researcher’s direct request of showing the acquired action (Table 1: 1.3) evidences stabilization and fluency in this newly established coordination (Table 1: 1.4): although, initially, he needed “a little trial and error,” now Lukas reproduced the action very smoothly.

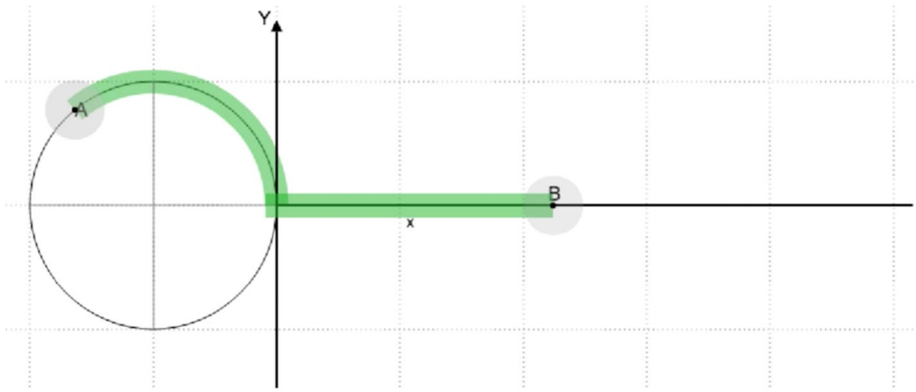


Fig. 1 A screenshot from an embodied activity of coordinating an arc of a unit circle with an *x*-coordinate

Table 1 Transcript of episode 1

(1.1) **R:** What were you doing here?

(1.2) **L:** This one was kind of tricky, but after a while, I saw that ((*Gesturing a segment on an imaginary x-axis two times*)) … mmm… the length of the arc on the circle should … has to be ((*A gesture of an arc on the imaginary unit circle*))… the same as the length on the line ((*Multiple gestures along the imaginary x-axis*))

(1.3) **R:** Cool, and can you show me how you were moving two points together?

(1.4) **L:** Like that, but with a little trial and error ((*Slowly showing a movement that coordinates distances on a unit circle and x-axis*))

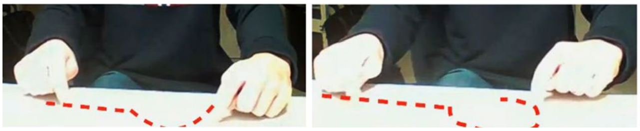


Table 2 First step of reifying action with unit circle: new sensory-motor coordination brings a new perception to the fore

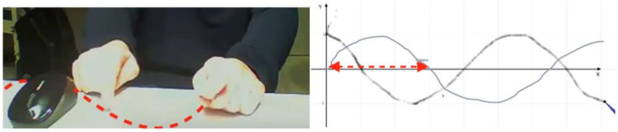
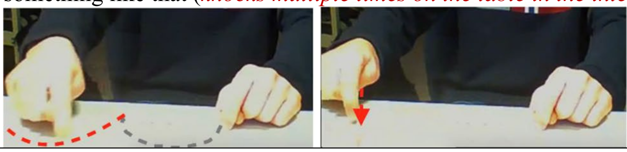
Previously incorporated artifacts within FDS	New sensory-motor coordination	New perception	Evidence of stabilization
(Unit) circle, *x*-axis	One hand runs along a (unit) circle and the other hand runs on a line (*x*-axis) with the same speed	Equal length of an arc and a segment of the line (*x*-axis)	Lukas reports that he needed trial and error during the activity, yet he is able to gesture the action smoothly

5.2.2 Episode 2: from action to a new artifact

The second episode presents an extract from the interview about a sketch of a sine graph drawn in a post-test task (Fig. 2). As we will see, this sketch is based on the new perception described above (coordination between the length of a unit circle arc and a segment of the *x*-axis; episode 1), and on new perception that vertically aligns a sine value in a unit circle and *y*-axis (not presented here). The researcher invited Lukas to explain his graph: “Can you explain to me how you did it?” Lukas immediately says that the graph was not drawn correctly, as it should start from zero and instantly redraws it. Then, he explains (Table 3).

In this episode, the student brings forth two new artifacts—a sine graph (a “regular curve”) and the *x*-intercept—and explains how they emerged for him (see Table 4 for a summary). The nouns and quick-iconic and pointing gestures in his discourse evidence that some artifacts were already available for him (the “line” [*x*-axis]; circle, sine on the circle; Table 3: 2.3, 2.5). He uses those artifacts to build an environment where new actions and new artifacts can be introduced through verbs and enacting gestures. For example, he highlights the feeling of “arc going smoothly,” which is grounded in the actions, as

Table 3 Transcript of episode 2

(2.1) **L:**	It's mostly… I kinda understand a feel of… how the arc goes really smoothly (*gestures the sine curve with the finger two times: just starts and then continues with more confidence*) I also looked at… (*gestures with the mouse along the x-axis*). 
(2.2) **R:**	Could you, please, remove the mouse?
(2.3) **L:**	Really, it is a very regular curve (*slowly gestures the curve holding the finger of his other hand at the origin, where a unit circle would start*), but also it [computer system] has stripes over there on the line (*points at a few places on the screen along x-axis*). So, I know, it [the line] had to cross the *x*-axis at about 3.14 (*smiles, gestures the first half period of the graph from the finger of the left hand with special attention to the moment of crossing*). 
(2.4) **R:**	Aha, how do you know it?
(2.5) **L:**	Well… to get back to zero in the sine … on the circle, you had to do half a pi, which is half a circle, so… and…(*Multiple times gestures half an arc, forward and backward; gradually, the gestures on the wave of the graph become shorter and become gestures of the unit circle, thus identifying movement on the arc with the movement on the circle*) … no, not half a pi, but pi, because full circle is two pi (*quickly gestures around the circle*), I know that over here it will be zero again (*Gestures half a circle again, arriving at the point of pi*). 
(2.6) **L:**	So, in my curve (*talks slowly, synchronous with gesturing out the curve until the imaginary point of crossing the x-axis*) it should be zero again at 3.14, or something like that (*knocks multiple times on the table in the intercept point*). 

gestures evidence (Table 3: 2.1). Further, he uses nouns to point to this newly emerged artifact ("regular curve," Table 3: 2.3, 2.6), which is also drawn as a solid form on the screen (Fig. 2). In the same way, the action of crossing the *x*-axis while synchronously traveling half a circle (Table 3: 2.5) becomes a stable feature of the curve "it should be zero again at 3.14," supported by a pointing (knocking) gesture at this newly crystallized artifact (that mathematically would be called *x*-intercept, see Table 3: 2.6).

The student's gestures evidence the combination of *material* and *ideal* ways to approach a unit circle and other artifacts. On the one hand, the coupling of the student's body with the unit circle's material affordances enables the student to infer the length that the point travels on x-axis to reach zero again. On the other hand, the student aligns the length of

Table 4 Second step of reifying action with a unit circle: the crystallization of a sine graph

Previously incorporated artifacts within FDS	New sensory-motor coordination	New perception	Crystallized artifacts
(Unit) circle; *x*-axis;	The movement on the Cartesian plane in coordination with the movement on the (unit) circle	Smoothness and regularity of the curve	A regular curve (sine graph)
An arc length and a sine value on a (unit) circle; *x*-axis, a regular curve (sine graph)	Stopping at half a circle and one wave	Correspondence between zero *y*-coordinate on the circle and on the graph; correspondence between half a period of a graph half a circle (π)	A point of crossing curve and *x*-axis at π or 3.14 (*x*-intercept and half period of the sine graph)

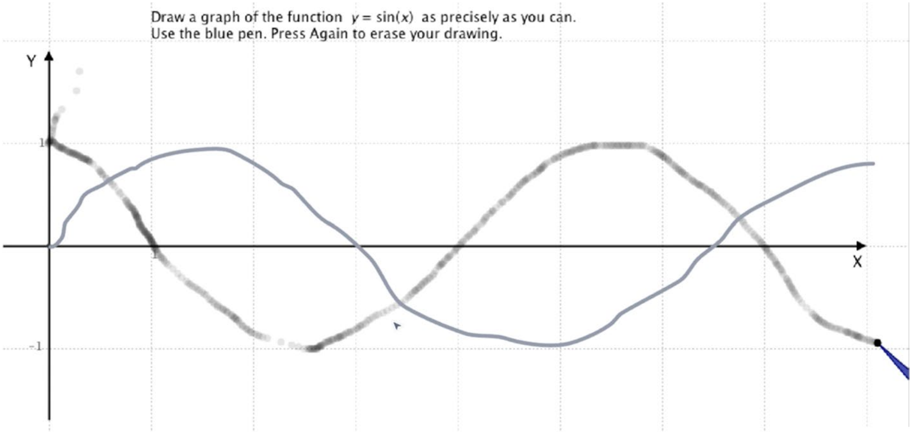


Fig. 2 Student's drawing of the sine graph in a post-test (thick line) and its correction at the beginning of the interview (thin line)

the arc with the *x*-coordinate, thus exhibiting a newly acquired cultural form of action—an ideal aspect of the artifact. Notice that in the situation at hand, the previously acquired unit circle is not initially present in the environment, but is re-created by the student through gesturing.

Overall, the artifacts of a unit circle and *x*-axis partake in the performance of the body–artifacts FDS, becoming a material|ideal instrument of the student's further thinking and being re-created once needed. Enactment of this *body–unit-circle–x-axis system* is further reified into a new artifact—a sine graph—which is perceived in a cultural way. Having been crystallized within previous actions, a sine graph is immediately born within an FDS of a student that is further extended.

5.3 Summarizing description

According to the suggested theory, *the first step* in reification is enabled by the natural ability of a sensory-motor system to self-organize into dynamic synergies (Kelso & Schöner, 1988): those FDSs enable coordinated action in solving motor problems of a particular function despite the variability of conditions (Bernstein, 1967). Such FDSs are responsible for walking, biking, or any other sensory-motor actions, including such actions as finding coordinates of a point on a Cartesian plane accomplished by moving the eyes (Krichevets et al., 2014). In our empirical example, new coordination was developed in a distance-learning embodied activity (episode 1): the student learned to maintain the length of a unit circle arc equal to the *x*-coordinate of a sine graph. As we have seen in the analysis of the interview, the development of this sensory-motor coordination was accompanied by new perception—distinguishing two lengths and their equality—that grounded the next step of reification.

Theoretically, in *the second step*, a new material artifact is produced and transforms a learner's environment. New artifacts preserve and facilitate previous forms of persistent actions: artifacts emerge within body–artifacts FDSs and stabilize some of the system's functionality in their affordances. It can be a sketch or a formula that helps to see or calculate some relations. In our empirical example, a sine graph and a point of its crossing with the *x*-axis are crystallized (episode 2) as they reify the actions with a unit circle and a new perception of equality between the arc length and *x*-axis segment. As we see, newly emerged artifacts are directly embedded into the students' environment (a sine graph is already exposed on a screen); previously acquired artifacts (e.g., a unit circle) can also be re-created in the environment when needed based on body potentialities through drawing or gesturing.

Acting with the artifacts, the student combines the newly acquired cultural (ideal) forms of action (aligning arcs with an axis, moving on the waves) and already available for his body (material) ways of acting (sensing the equal speeds, tapping, making a smooth movement). Mathematical artifacts provide affordances for material and ideal aspects of practice.

6 Discussion and conclusions

In this paper, we focused on relations between processes and objects in learning mathematics. We theoretically discerned two branches of theories in mathematics education research, which provide different ontological grounds for mathematical objects. The first branch of theories (Dubinsky, 2002; Sfard, 1991) considers objects as internal mental entities embedded in students' cognitive systems. The second branch of theories highlights the importance of introducing external mathematical notations in transition to objects (Gray & Tall, 1994; Harel & Kaput, 1991). The first branch of theories still attracts considerable attention in the field despite the critique of the cognitivism that underpins them. An alternative non-dualist view is known as commognitive approach (Sfard, 2008), which considers process–object transition as a transformation of discourse, thus developing a sociocultural view that grants discourse with primary ontology. Rapidly developing theories of embodied cognition call for reconsidering cognition as formed in interaction with and extended into an environment (e.g., Varela et al., 1991; Chemero, 2009; Kiverstein & Rietveld, 2018; Sanches de Oliveira et al., 2021). Coordinating radical embodied ideas with cultural-historical approaches, we strive to overcome the critique of internalist approaches, synthesize both branches, and offer a non-dualist perspective that expands beyond discourse.

We offer a process–object dialectics within a functional dynamic systems (FDS) perspective that explains process–object relations without appealing to mental representations and that accounts for embodied interactions and transformations of mathematical environments. We suggest distinguishing two steps in reification processes, namely, (1) fluency of a new action by forming sensory-motor coordination through dynamic synergy that bring new perception to the fore and (2) crystallization of an artifact as reifying actions. This way, reification includes establishing a new synergetic unity of sensory-motor enactment within cultural practice, which is further supported by new crystallized artifacts in the environment. This two-step structure of reification allows for an integration of the two aforementioned branches within ecological ontology: Self-organization of sensory-motor actions into a synergy—which enables the perception of new affordances—leads to the changes in a student's body (consonant to internalist approaches). Reification of actions into artifacts leads to the changes in an external environment (consonant to externalist approaches). We propose that a unique quality of cultural artifacts over biological environments is a human ability to re-create them in external physical space based on previously established body potentialities. Seemingly, also, a bird creates a nest based on a pre-programmed genetic potentiality of its body. Moreover, some species can adapt objects for their functional use, for example, to bend a wire. However, humans are apparently quite unique in expanding through generations their much more complex and flexible abilities of (re)producing artifacts (Heersmink, 2022). The particularity of mathematical practice over other practices with artifacts (e.g., chairs and spoons) lies in the possibility to re-produce those artifacts in the environment easily once they have been crystallized. This flexibility, fluency, and at the same time, precision in re-creating the needed environment, we submit, is exactly what makes mathematical thinking possible by flexibly instrumentalizing it (see Drijvers, 2019; Shvarts et al., 2021).

Along with the commognitive approach, our view is non-dualistic as it does not separate body and mind, material and ideal. While for the commognitive approach, discursive ontology becomes the source of resolving dualism, we offer ecological onto-epistemology as serving a similar function. The theory presented in Sfard (2008) aims to describe mathematical activity as specifically human; mathematical discourse is an autopoietic system specific for human practice (p. 174). The conceptual aims of our approach are different, as we search for a framework that would allow us to conceptualize mathematics as a natural continuation of cultural and biological evolution (Abrahamson, 2021; Cole, 2016). We try to establish an ontological continuum from biological activity and evolution exhibited by animals to mathematical activity and learning, specific to educated humans. In both cases, an ecological environment provides action opportunities. Yet, creation and re-creation of artifacts presents the particularity of human mathematical activity. For the commognition theory, a process–object transition is a transformation of discourse (Sfard, 2008). For the functional dynamic systems approach, a process–object transition is a transformation of an environment through the introduction of a new cultural artifact (only some of the artifacts are discursive means). Taken as ecological entities, cultural artifacts provide students with new action possibilities, thus fostering new (embodied) actions.

Consonant with the cultural-historical tradition of formative experiments, embodied action-based design was used in this research as a method to make sensory-motor actions in process–object transition visible to us as researchers. A question of this theoretical proposal's applicability to other teaching/learning situations requires special attention.

Particularly, action-based embodied design was helpful in creating a situation where sensory-motor actions support the emergence of new perceptual structures for the students. Yet studies of students' gestures and manipulations (e.g., Radford, 2021) and eye-tracking studies (e.g., Shvarts, 2018) reveal that in other pedagogical settings sensory-motor processes are also critical in coming to perceive mathematical structures and, thus, artifacts' affordances. However, further analysis is needed to create better empirical ground for our theoretical view in more common teaching/learning situations. Such a study might require longitudinal work, as we expect that stabilization of new sensory-motor coordinations and further reification of them in artifacts usually take a long period of time. For example, fluency in operating with a covariance table is an action that is reified in algebraic notation for functions; mastering such fluency to further operate with algebraic notations happens over the course of months.

As an outcome of the design research, we provide new theoretical insights but also come to reconsider design heuristics that were previously suggested within action-based embodied design genre (Abrahamson, 2014; Abrahamson et al., 2020; Alberto et al., 2021). Particularly, we address the issue of postponing the introduction of symbolic artifacts, such as grids and numbers. According to our theoretical proposal, an artifact becomes an organic and fully understood extension in a student's body–artifacts FDS when it has been crystallized as a reification of students' actions. To achieve this understanding, technological environments need to provide opportunities not only for developing sensory-motor coordinations (step 1 in reification) but also for re-inventing mathematical artifacts (step 2 in reification). As Kaput states, "building symbols is a major part of what it means to be human" (1991, p. 151). We call for (technological) design solutions that would allow learners themselves to introduce and interlink notations as reifying their sensory-motor actions, thus re-inventing mathematics. Our approach does not support introducing material artifacts and symbols as co-present in the environment (Coles & Sinclair, 2019; Reinschlüssel et al., 2018) and designs developed on the basis of automatically interlinking multiple representations (criticized already in Yerushalmy, 1991). Instead, the re-invention of mathematical artifacts as a design heuristic resonates with realistic mathematics frameworks of progressive mathematization (Freudenthal, 1972) and emergent models (Gravemeijer, 1999), and with material-making and constructionist ideas that focus on students' abilities to create their own mathematical environments (e.g., Abrahamson & Chase, 2020; Ng & Ferrara, 2020).

We argue that symbolization with new inscriptions leads to transforming—not representing—students' reality and to extending their cognitive systems by new cultural artifacts. Theoretically embedding cultural artifacts into embodied cognitive systems, we contribute to overcoming the gap between sociocultural and cognitivist theories, and see cognition as indispensable from material|ideal culture. Focusing on the environment as developing for a learner in the education process, we invite educators to create environments that foster cultural forms of action and perception as well as further transformations of those environments by the students themselves through re-invention of the cultural artifacts.

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Data availability The paper uses a video recording of one participant; this video recording cannot be made public for privacy reasons.

Declarations

Consent to participate and ethical review The participant has given his voluntary consent to participate in the study and gave permission to use the data in pseudonymized form in publications. The study was approved by the Ethical Board, ERB Review Bèta S-2039.

Competing interests The authors declare no competing interests.

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