Educational Studies in Mathematics (2024) 116:67–89
https://doi.org/10.1007/s10649-024-10306-8

Conceptions of span in linear algebra: from textbook examples to student responses

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Accepted: 25 January 2024 / Published online: 4 March 2024

Abstract

Using Balacheff's (2013) model of conceptions, we inferred potential conceptions in three examples presented in the spanning sets section of an interactive linear algebra textbook. An analysis of student responses to two similar reading questions revealed additional strategies that students used to decide whether a vector was in the spanning set of a given set of vectors. An analysis of the correctness of the application of these strategies provides a more nuanced understanding of student responses that might be more useful for instructors than simply classifying the responses as right or wrong. These findings add to our knowledge of the textbook's presentation of span and student understanding of span. We discuss implications for research and practice.

Keywords Linear algebra · Conceptions · Spans and spanning sets · Reading questions

The notion of span is central in linear algebra. From a mathematical point, the span of a finite set of vectors is simply and elegantly derived from basic mathematical operations to make a vector space, and determining whether a vector belongs to the span of a set is a central question in linear algebra. However, there has been ample and extensive research indicating that this notion is difficult for students to learn (Carlson, 1993; Harel, 1989; Hillel, 2000; Sierpinska, 2000). Much of the research on students' understanding of linear algebra ideas has been done in the context of individual interviews guided by specific theoretical approaches or by observing the implementation of theoretically designed curricula, both of which allow researchers to map the construction of notions by students. A smaller body of research has used students' responses to examination questions, to describe students' knowledge of procedures in solving linear algebra problems (Kontorovich, 2020). While quite important to the field, these findings have limitations: the samples are small and circumscribed to specific institutions and contexts. Moreover, when recommendations for practice are suggested, they are available once the research is concluded, months or even years after the participants provided information.

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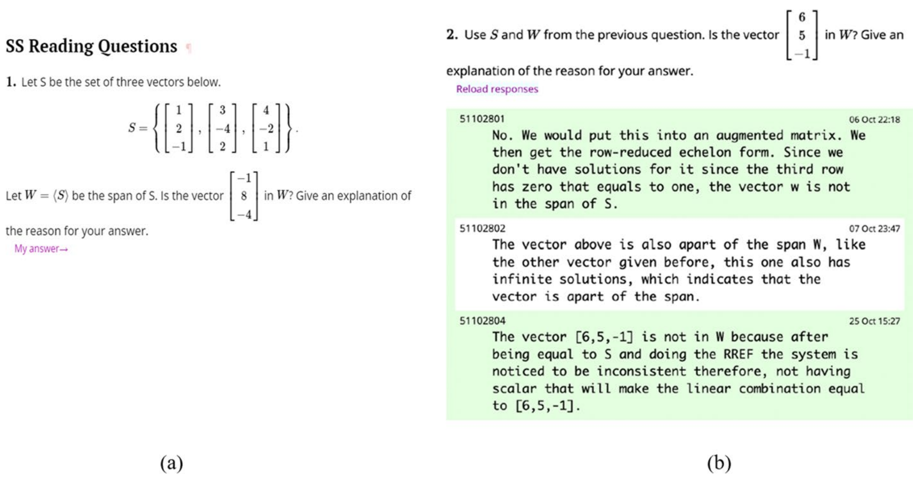


Fig. 1 **a** Reading question 1, student view. **b** Reading question 2, teacher view

In this paper, we explore student understanding of span in relationship with the textbook's presentation of the topic using student responses to two reading questions embedded in the textbook. As intended by its name, a *reading question* is a question that seeks to motivate students to read the textbook's content before attending a lesson where those specific contents will be discussed. Students are expected to provide responses to those questions ahead of the lesson. In the interactive textbooks we study, students can type their mathematical work and thoughts directly into the textbooks (Fig. 1a), and once they submit their answers, instructors can immediately view them (Fig. 1b).

Reading questions are designed to fulfill multiple purposes: to attune students to the new content, to uncover misunderstandings that can lead students to ask questions about the material, or to help students make connections to prior knowledge. These purposes are supported by research that documents that engaging with material prior to class can be beneficial for knowledge building (Graham et al., 2020). They are also included to support teaching when assigned before class, teachers can scan the student responses to ascertain their potential level of understanding of the textbook material and possibly adjust their teaching plans. Reading questions provide an important entry point into student understanding of ideas and can potentially allow investigation of learning progressions tied to particular textbooks.

In what follows, we present literature that describes our current knowledge of students' understanding of span, the theoretical approach to studying student responses, and the research questions guiding our investigation.

1 Literature review

We identified three main areas of research in the literature about spans[1]: students' ways of thinking about span, difficulties with the abstract nature and formal definition of span, and interventions to address those.

[1] Most research on student understanding of spans includes linear (in)dependence. See for example, Rasmussen and Wawro (2017) and Stewart et al. (2019) who provide a comprehensive review of these topics.

Drawing upon Tall and Vinner's (1981) construct of concept image, Plaxco and Wawro (2015) conducted individual interviews with five individual students enrolled in an inquiry-oriented instruction (IOI) course guided by the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1991) based on possible travels on a magic carpet (Magic Ride). The authors found a wide range of student concept images of span, which they organized into four main categories: travel ("everywhere you can get" to describe span), geometric (the area that the span of some vectors covers), vector algebraic ("every vector you can make with linear combinations of the columns"), and matrix algebraic (the properties of matrix related to span, such as dimension). These concept images are closely related to the models used in the curriculum to illustrate span (e.g., moving on a flat space using a hoverboard or a magic carpet parallel to the surface, each with their own moving constraints, see Wawro et al., 2012). Using the same curriculum and these concept images, Rasmussen et al. (2015) showed that students exhibited different trajectories through the four categories yet demonstrated individual progress over time. These investigations illustrate the connection between the representations used in the curriculum (means of transportation, e.g., a hoverboard or a magic carpet and vector representation in the Cartesian plane) and the images that students build about the span.

Student challenges in learning the notion of span have been extensively documented. Medina (2000) and Parker (2010) also used Tall and Vinner's (1981) concept image to show that students rarely stated a formal definition of span, struggled to interpret it, and preferred describing span using everyday words. Having a hard time connecting various related concepts, students relied on procedural rather than conceptual understanding to solve problems. Parker further illustrated how students' conceptual understanding of the definition of span varied more than their procedural understanding. Subsequent studies (e.g., Hannah et al., 2013) have confirmed students' struggles with understanding the formal definition and their inclination to use computational algorithms over theoretical methods.

To overcome these challenges, scholars have explored the implementation of various instructional interventions to aid students in transitioning to the abstract definition of span. Wawro et al. (2012), for example, employed the modeling task mentioned earlier, Magic Ride, to guide students in reinventing the notions of span and linear (in)dependence. They claimed that the modeling task eased students' transition from systems of equations to vector equations by leveraging their intuitive understanding. Cárcamo et al. (2016, 2017, 2018) devised modeling tasks about creating and using secure passwords that aimed to establish connections between span and the first-year students' experiences at a Spanish university. Although students had a hard time using mathematical notation and the procedure to find spanning sets, the instructional design helped them advance their understanding by transitioning "their informal mathematical knowledge to a more formal comprehension" (2016, p. 67).

Other intervention studies have been inconclusive. Hannah et al. (2013, 2016) investigated one instructor's students' understanding of span using visualization, talking and writing in the language of linear algebra, and emphasizing formal definitions. Using Tall's (2013) theory of three worlds (embodied/geometric, symbolic, and formal) and action, process, object, and schema (APOS; Dubinsky & McDonald, 2001), they showed that despite emphasizing visualization and formal language, more students preferred informal and visual ways of thinking about span (e.g., being able to get to any point in the space with the given vectors and covering a plane with two vectors) and more students performed better on algorithmic tasks compared to tasks that connected geometric, symbolic, and formal ways of thinking about span. Using the same frameworks, Stewart and Thomas (2007, 2009) examined the effects of tutoring with elements of embodied, symbolic, and formal concepts. Although most students still struggled with formal definitions, those who underwent the tutoring did better describing the concepts and were mostly able to link span to the idea of linear combination. Lastly, Bouhjar et al. (2021) investigated the impact of IOI on student reasoning about span and linear independence, using Tall and Vinner's (1981) concept image. The study showed that students exposed to IOI exhibited more varied and conceptually aligned concept images of span compared to their non-IOI counterparts. IOI students also engaged in reasoning about span with a higher frequency in terms of linear independence, dimensionality, and row reduction. Conversely, a significant proportion of non-IOI students approached span by treating vectors as geometric entities. This work highlights the importance of using diverse representations, modeling, and inquiry to support students' transition from less abstract to more formal definitions of span.

Given that this research often aims to improve the teaching of linear algebra, often through the utilization of Realistic Mathematics Education models (RME; Gravemeijer, 1999), many studies examine student understanding by mainly relying on the notion of concept image—an individual's "cognitive structure associated with the concept" (Tall & Vinner, 1981, p. 153). While these studies offer insights into students' cognitive processes and conceptual development at the individual level, researchers recognize that this cognitive approach to understanding students' thinking has not produced much nuance regarding "when and whether the various conceptions occur across modes of thinking, Tall's (2013) worlds, or different metaphors or models" (Stewart et al., 2019, p. 1022). In other words, the current theoretical frameworks used to dissect students' understanding of span appear to be broadly linked to the representations of span, offering holistic descriptions without specific problem contexts. This lack of specificity makes it challenging for researchers to systematically study conceptions within a group of students.

In our study, we contribute to the literature by taking a different approach to studying student understanding of spans, through one type of problem, and during students' learning processes using Balacheff's cK¢ model of conceptions (Balacheff & Margolinas, 2005). Given that we are not interested in what students had learned by the time of data collection, but rather how they were thinking when learning about span, we integrate a characterization of "correctness" to our student conceptions, inspired by Harel's (1989, 2000) framework of correctness of students' answers. We provide an overview of our theoretical framework next.

2 Theoretical underpinnings

As they learn the material, students make statements, in writing or aloud, that suggest that they may be "mixing" contexts or definitions in ways that sound acceptable but are not quite right. Explanations for this phenomenon tend to rely on the assumption that what students do, say, or write maps directly to what they know (e.g., Sellers et al., 2021). These analyses are difficult to corroborate and are more reflective of the researcher's perspective. Balacheff's cK¢ model of conceptions (Balacheff & Gaudin, 2009; Balacheff & Margolinas, 2005) gives a process for generating explicit characterizations of conceptions grounded in the analysis of students' productions (what they write and what they say) and not on assumptions of what they are thinking.

In the French tradition, the word *knowing* (*connaissance*) is used as a noun to distinguish a learner's personal constructs of mathematical notions from *knowledge* (*savoir*), which refers to intellectual constructs recognized by the community. As learners encounter a mathematical task (a problem, an exercise, or a question) in which they must use their knowing of a specific mathematical notion, they are in a position in which their cognitive dimension interacts with those aspects of the task that bring into play the knowledge at stake (environment). The learners' actions, guided by their cognitive dimension, will generate feedback from the environment in repeated sequences (when multiple intermediate steps are needed). The sequence stops once the learners receive feedback that the task has been completed. The repeated exchanges emanate from the personal knowings learners have, and as such, they can reveal the coexistence of multiple knowings that may not be perceived as problematic or contradictory to the learner. Contradictory knowings can coexist at different times of a learner's history (e.g., "multiplication results in bigger numbers") or in cases in which mathematical situations they encounter repeatedly call for one knowing and not others (e.g., multiplication of natural numbers). The learner does not perceive the contradictions in new situations that demand the use of their knowings (e.g., multiplication by rational numbers in the interval (0,1)), even though they will appear contradictory to the observer—the teacher or the researcher is able to identify that the new contexts will require a different knowing. Balacheff uses "subject" (S) to refer to the cognitive dimension of the learner, and "milieu" (M) to refer to the aspects of the mathematical task that bring into play the knowledge at stake (Fig. 2). An arrow from the subject to the milieu is an action that the cognitive dimension of the learner is performing to address the task; an arrow from the milieu to the subject represents the feedback that the learner receives as a result of their action. The viability of the system of exchanges is regulated by conditions (constraints) needed to ensure that the system is such that the learner will engage with the task and that an equilibrium is reached (the task is solved). Constraints provide the subject with information for decision making. Thus, a conception is "a state of dynamic equilibrium of an action-feedback loop, between a subject and a milieu under proscriptive constraints of viability" (p. 189).

Learning in this context is defined as a process in which equilibrium between the learner and the milieu is reached after several action-feedback loops (a loss of equilibrium is perturbations that occur after an action is performed and a result is not exactly what is expected). Unresolved perturbations recognized by the subject can lead to learning or to repeated action-feedback loops. But in some cases, the subject does not identify the perturbation; when this occurs, the unnoticed perturbation is a symptom of a conception, the remnant of a "previous equilibrium of the subject/milieu system" (p. 190). To

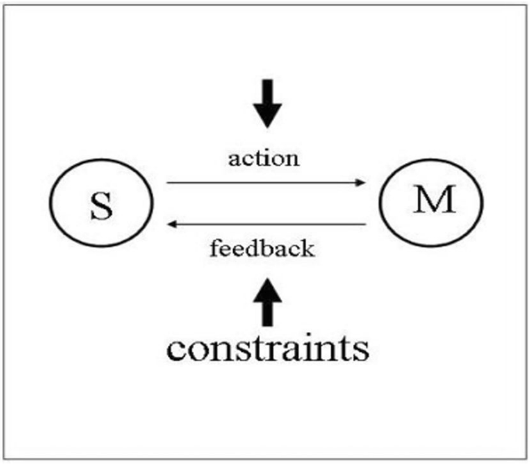


Fig. 2 Schematic for "conception" in the subject/milieu system (Balacheff & Gaudin, 2009, p. 189)

characterize a learner's knowings, Balacheff (2013) operationalizes a conception as a quadruplet: problems, operators, semiotic and representation systems, and control structures. Problems are the "class of the disequilibria the considered conception is able to recover from" (Balacheff & Gaudin, 2009, p. 190); these emerge from the need for mathematical notions to evolve. Operators are the tools for "actions on the milieu" needed to simulate what the learner does to tackle the problems and "to transform and manipulate linguistic, symbolic or graphical representations" (p. 190). Semiotic and representation systems are the "linguistic, graphical or symbolic means which support the interaction between the subject and the milieu" (p. 190). Lastly, control structures are the "components supporting the monitoring of the equilibrium of the [S – M] system" (p. 190) or the strategies (e.g., metacognitive processes, use of definitions, and checking existing answers) that allow learners to decide whether they have solved the problem and whether they have done so correctly.

This conceptualization of conceptions is practical because it allows researchers to define the domain of validity of a knowing as the collection of related conceptions exhibited by individuals. Because the collection corresponds to the expression of a learner's conceptions enacted by a situation (solving a problem), the definition allows the coexistence of more than one, possibly contradictory, conception in the subject. In addition, when there is a need for variations in the set of problems that learners must face (e.g., when the mathematical ideas are revisited in different contexts), the operators, representations, and metacognitive strategies needed to organize the work (control structures) can be used to describe the emergence of different conceptions of a mathematical notion. Moreover, as teachers care about whether responses are right or wrong answers, we believe that the cK¢ model provides a mechanism for characterizing correctness in terms of the students' process of justification. Thus, in this study, we seek to answer the following questions:

1. What conceptions of spanning sets can be inferred from the examples of spanning sets in an interactive undergraduate linear algebra textbook?
2. What control structures are evident in student responses to reading questions about spanning sets provided prior to the lesson on spanning sets?
3. What is the correctness of the students' responses using these control structures?

3 Methods

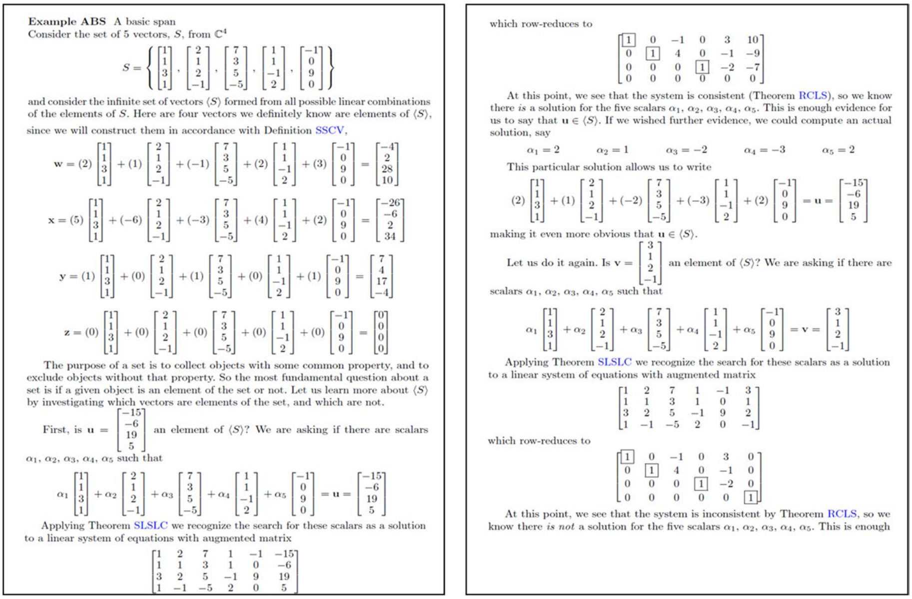
The data for this study come from a larger study that investigates the use of interactive textbooks in calculus, linear algebra, and abstract algebra courses by instructors and their students (Beezer et al., 2018). The linear algebra textbook, *A First Course of Linear Algebra* (Beezer, 2021), follows the definition-theorem-proof presentation style (Love & Pimm, 1996) and is designed as a bridge-to-proof course that uses mostly symbolic representations without visualizations. It is authored in PreTeXt (https://pretextbook.org/), a mark-up language that facilitates the publication of open source and open access textbooks and the inclusion of interactive features, such as computation or Sage cells, hyperlinks, automatic solution feedback systems, and short-answer questions (O'Halloran et al., 2018). PreTeXt textbooks can be reproduced in any output (e.g., HTML, PDF, ePub, and braille). Each section in the textbook has a set of three reading questions that students are supposed to answer directly in their textbooks before coming to class. Students' responses are collected in real-time making them immediately available to their instructors, who in turn can alter their lesson plan depending on those responses. The reading questions in the spanning sets (SS) section (Fig. 1) ask whether a vector is in the span of a set of given vectors. This section is divided into two subsections: span of set of vectors (SSV) and spanning sets of null spaces (SSNS). Our analysis focuses on the reading questions related to the first subsection, which starts with the definition of the span of a set of column vectors. Following the definition, the author gives three examples of the same problem using numerical and symbolic representations of vectors and matrices, without any visualizations. The subsection concludes with two Sage cells that guide students to generate the span of a finite set of vectors and use the notion of span to check the consistency of a linear system.

To investigate how the textbook content related to the reading questions in these sections, we analyzed the text and examples directly related to the reading questions. The study collected information from over 50 faculty, of which six used the interactive linear algebra textbook and assigned the reading questions to their students ($n=76$) as intended by the textbook's author, namely, to be answered before class. The student responses were collected during Fall 19, Fall 20, Spring 21, and Fall 21. The students were in six different universities (public, private; small, medium size) in the USA. The data were analyzed in three phases: (1) identifying conceptions in the textbook examples and the control structures in student responses to the reading questions using Balacheff's (2013) cK¢ model, (2) coding for the correctness of the use of the control structure in the responses, and (3) looking for patterns between the control structures in student responses and correctness to identify possible associations. These analyses map our three research questions. We describe each phase next.

3.1 Phase one: identifying conceptions and control structures

We analyzed three examples—Example ABS: A Basic Span; Example SCAA: Span of the Columns of Archetype A; and Example SCAB: Span of the Columns of Archetype B—that dealt with deciding whether a vector was in the span of a set or not. Figure 3 shows the first example in SS. The other two examples deal with two different linear systems (archetype A: three equations, three unknowns, with a singular coefficient matrix with dimension 1 null space; archetype B: three equations, three unknowns, with a nonsingular coefficient matrix, see http://linear.ups.edu/html/section-SS.html). Example ABS uses R^4; the other two examples use R^3.

The development of the codes followed a constant comparative method. After reading the texts (either in the textbook or in the student responses), we identified operators and control structures and assigned in vivo temporary codes to identify them (e.g., "find row reduction echelon form" was coded as "RREF"). We individually created a codebook as the reading of the texts went on. The four researchers held meetings to discuss the difficulties faced in identifying the components of the model of conceptions. The first two authors identified three different methods in the examples which suggested that multiple control structures could be used for the same problem (is the vector in the span?), which could lead to different conceptions. The last two authors, working with the student responses, noted that the operations were usually implicit. Consider for example: "Yes, this vector is in W, because the RREF [Row Reduced Echelon Form] is consistent" (RQ1, #46). In this response, there is no information on how the student found out that the RREF (or the system of equations that resulted in the RREF) is consistent because the steps taken are not



Example ABS A basic span
Consider the set of 5 vectors, S, from $\mathbb{C}^4$

$$S=\left\{\begin{bmatrix}1\\1\\3\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\\-1\end{bmatrix},\begin{bmatrix}7\\3\\5\\-5\end{bmatrix},\begin{bmatrix}1\\1\\-1\\2\end{bmatrix},\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}\right\}$$

and consider the infinite set of vectors $\langle S\rangle$ formed from all possible linear combinations of the elements of S. Here are four vectors we definitely know are elements of $\langle S\rangle$, since we will construct them in accordance with Definition SSCV,

$$\mathbf{w}=(2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(-1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(2)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(3)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}-4\\2\\28\\10\end{bmatrix}$$

$$\mathbf{x}=(5)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(-6)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(-3)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(4)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}-26\\-6\\2\\34\end{bmatrix}$$

$$\mathbf{y}=(1)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(1)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}7\\4\\17\\-4\end{bmatrix}$$

$$\mathbf{z}=(0)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(0)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(0)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

The purpose of a set is to collect objects with some common property, and to exclude objects without that property. So the most fundamental question about a set is if a given object is an element of the set or not. Let us learn more about $\langle S\rangle$ by investigating which vectors are elements of the set, and which are not.

First, is $\mathbf{u}=\begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$ an element of $\langle S\rangle$? We are asking if there are scalars $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ such that

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+\alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+\alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+\alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+\alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\mathbf{u}=\begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$$

Applying Theorem SLSLC we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

$$\begin{bmatrix}1&2&7&1&-1&-15\\1&1&3&1&0&-6\\3&2&5&-1&9&19\\1&-1&-5&2&0&5\end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix}\boxed{1}&0&-1&0&3&10\\0&\boxed{1}&4&0&-1&-9\\0&0&0&\boxed{1}&-2&-7\\0&0&0&0&0&0\end{bmatrix}$$

At this point, we see that the system is consistent (Theorem RCLS), so we know there *is* a solution for the five scalars $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$. This is enough evidence for us to say that $\mathbf{u}\in\langle S\rangle$. If we wished further evidence, we could compute an actual solution, say

$$\alpha_1=2\qquad\alpha_2=1\qquad\alpha_3=-2\qquad\alpha_4=-3\qquad\alpha_5=2$$

This particular solution allows us to write

$$(2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(-2)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(-3)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\mathbf{u}=\begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$$

making it even more obvious that $\mathbf{u}\in\langle S\rangle$.

Let us do it again. Is $\mathbf{v}=\begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$ an element of $\langle S\rangle$? We are asking if there are scalars $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ such that

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+\alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+\alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+\alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+\alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\mathbf{v}=\begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$$

Applying Theorem SLSLC we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

$$\begin{bmatrix}1&2&7&1&-1&3\\1&1&3&1&0&1\\3&2&5&-1&9&2\\1&-1&-5&2&0&-1\end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix}\boxed{1}&0&-1&0&3&0\\0&\boxed{1}&4&0&-1&0\\0&0&0&\boxed{1}&-2&0\\0&0&0&0&0&\boxed{1}\end{bmatrix}$$

At this point, we see that the system is inconsistent by Theorem RCLS, so we know there *is not* a solution for the five scalars $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$. This is enough

Fig. 3 Example ABS: a basic span in SS

given. However, the justification for the conclusion is explicit: a consistent RREF implies that the given vector is in the span. Because this happened in almost all the responses, we assumed that the semiotic system and the operators students used in their responses were similar and consistent with the textbook content and that the control structure must be the discriminating element of the conceptions. Therefore, we decided to focus on the control structures stated in the student responses. We also checked our use of the model with Balacheff, who deemed it "accurate and appropriate" (N. Balacheff, personal communication, January 6, 2023).

We refined the definitions of the components of the model and of the control structures, and independently, pairs of authors coded the responses for control structures. The list of control structures grew as we coded the student responses. Once the authors had read and coded the responses individually, we met to compare and resolve disagreements. Disagreements were due to ambiguity in interpretations, which led to clarification of the definitions of the control structures. Once the definition of each control structure was finalized, we reread the responses to make sure that the coding aligned with the new definitions. The final definitions of each control structure are presented in Table 1, along with example responses that illustrate each of them. We use italics through the text to refer to the control structures.

As we discuss later in the findings, some of the student responses exhibited multiple control structures. For example, we identified three control structures in the following response to RQ1 from student 52 (the codes are underlined inside brackets):

> The vector [-1,8,-4] is in W. First, I wrote out the vectors as: x1 [1,2,-1] + x2 [3,-4,2] + x3 [4,-2,1] = [-1,8,-4]. Next, I formed these vectors into a 3x4 augmented matrix. I

Table 1 Control structures identified in textbook examples (in italics) and student responses with example responses

Name of control structure	Definition	Example response (reading question, #student ID)
*Consistency** (CS1)	*Consistency of the system of equations* alludes to the consistency of the system of equations obtained from expressing the given vector as a linear combination of the vectors in the set	"Yes, this vector is in W. This is because the RREF is consistent" (SS-RQ1, #46)
*Number of solutions** (CS2)	*Number of solutions to the system* alludes to the number of solutions (none, one, or infinitely many) to the system of equations obtained from expressing the given vector as a linear combination of the vectors in the set	"Yes because by row reducing then there are infinitely many solutions allowing this vector" (SS-RQ2, #68)
*Scalars** (CS3)	*Scalars that work as a linear combination* provides scalars to express the given vector as a linear combination of vectors in the given spanning set	"The vector [[-1],[8],[-4]] = 2[[1],[2],[-1]]-1[[3],[-4],[2]] and hence it is in W." (SS-RQ1, #76)
Linear combination (CS4)	*Existence of linear combination* alludes to the existence of a linear combination using the vectors in the spanning set to express the given vector, without providing the scalars	"Yes, the vector is equivalent to a linear combination of the three vectors." (SS-RQ1, #55)
SAGE (CS5)	*SAGE* alludes to providing a Sage code to generate the span of the vectors and test whether the test vector is in the span or quote the result generated using the code	"The vector [6,5,-1] is not in W for the same reason as above. We can also check this by using the.span() function in sage, which returns \\ false\\ "."" (SS-RQ1, #66)
Pivots (CS6)	*Pivots* alludes to mentioning the number or existence of pivot(s) (column) to justify whether the given vector is in the span	"No, there is a pivot column in the last column meaning it is NOT in the span." (SS-RQ2, Number44)
Number of free variables (CS7)	*Number of Free Variables* alludes to mentioning the number or existence of free variables(s) to justify whether the given vector is in the span	"Yes, because when you do RREF you don't get values that equal to no solution (0 = 1). You do indeed get 2 free variables and 2 dependent variables when you do RREF." (SS-RQ1, #8)

Note. *Control structures found in the textbook examples. Seven responses in RQ1 and 9 responses in RQ2 were unclear or uninterpretable then put the augmented matrix into RREF. We see there are infinetly [sic] many solutions [*CS2: Number of Solutions*], and x3 is a free variable [*CS3: Number of Free Variables*]. The solution set is S = [2-x3 , -1-x3 , x3]. Since x3 is a free variable, I choice x3 to be 3. That than gives us: [3 , -4, -1] as the solution set when x3=3. We know the vector [-1,8,-4] is in W because we get: (-1)u1 + (-4)u2 + (3)u3 = W [*CS7: Scalars*].

Thus, although one control structure was enough to justify that the vector is in the span of the other three vectors, the student provided three control structures in their response.

3.2 Phase two: coding for correctness

In the second phase, we analyzed the action-feedback loop by assessing the correctness of the student responses. By *correctness*, we do not mean to be "right" or "wrong" or being free of error. Instead, we conceptualize correctness as a two-folded tool to capture the nuances of student thinking during various steps of answering the reading questions. We analyzed the correctness of each *criterion* and its link to the final conclusion, which we refer to as *link by conception*. We define *criterion* as the intermediate result needed to apply the control structure that signals the student whether the given vector is in span. The description of the correct criteria for each control structure for both reading questions is shown in Table 2.

Next, we looked at the correctness of the link between the criteria to the final conclusion, that is, whether the student interpreted the criterion correctly (by means of mathematics) to reach the final conclusion answering the reading question. In this stage, we only analyzed the correctness of responses with valid control structures. We modified Harel's (1989, 2000) coding of the correctness of students' answers (correct final answer, incorrect final answer, correct justification, and incorrect justification) because in his case, an

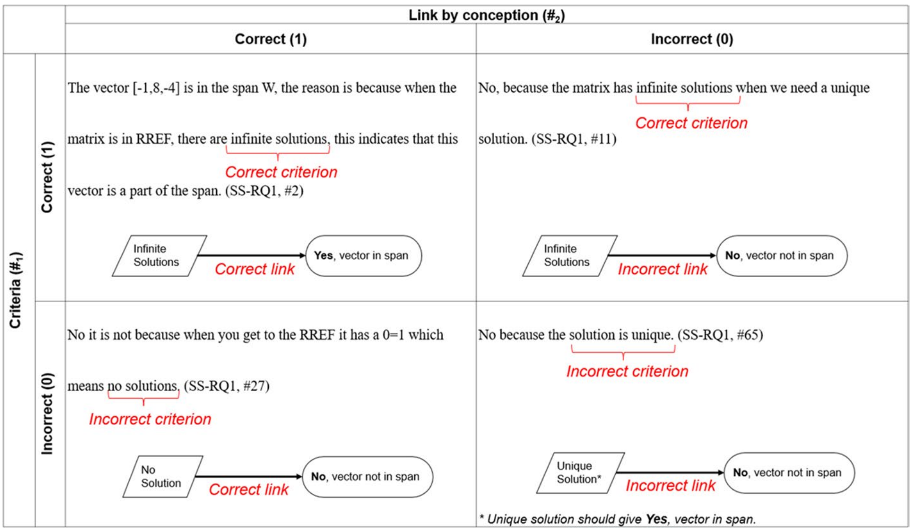


Fig. 4 Four example responses using *Number of Solutions* (CS2) in RQ1 with correct and incorrect links by conception and criteria

Table 2 Descriptions of correct criteria for each control structure used for each reading question

Control structure	Description of correct criteria	
	RQ1 (yes, vector in span)	RQ2 (no, vector not in span)
*Consistency** (CS1)	Assert that the system of equation is <u>consistent</u>	Assert that the system of equation is <u>inconsistent</u>
*Number of solutions** (CS2)	Assert that the system has (infinitely many) solutions	Assert that the system has <u>no solution</u>
*Scalars** (CS3)	Provide <u>correct scalars</u> that work as a linear combination, that imply the <u>existence of linear combination</u>	Assert that there are no scalars (that work as a linear combination)**
Linear combination (CS4)	Assert/imply that (scalars for) linear combination <u>exists</u>	Assert/imply that (scalars for) linear combination <u>do not exist</u>
SAGE (CS5)	Provide <u>correct SAGE</u> code that informs the student that vector is in span, OR Quote <u>“true”</u> obtained from the SAGE code	Provide <u>correct SAGE</u> code that informs the student that vector is not in span, OR Quote <u>“false”</u> obtained from the SAGE code
Pivots (CS6)	Assert the number of pivot columns is <u>less than the number of variables,</u> but it is equal to the number of equations, which implies that the system has <u>infinitely many solutions</u>	Assert the number of pivot columns is <u>less than the number of variables</u> which implies that the system <u>has no solution</u>
Number of free variables (CS7)	Assert that there is <u>one</u> free variable, OR Assert that free variable <u>exists</u>	Assert that nothing can be said about the number of free variables (because the system is inconsistent or the bottom row of the matrix is $0=1$)**

Note. The underlined text highlights the correct criterion for each control structure

*Control structures found in the textbook examples

**No responses were coded using this control structure incorrect justification could be obtained by correctly using a control structure. We created a binary code, C$\#_1\#_2$, where $\#_1$ represents the correctness of the criteria and $\#_2$ represents the correctness of the link by conception (1 = correct, 0 = incorrect). Thus, C10 is assigned when the response uses the correct criteria for the chosen control structure, but the link between the criteria and the final conclusion is incorrect. Figure 4 illustrates four responses using *Number of Solutions* (CS2) in RQ1 and that uses the criterion "has infinite solutions."

The first, second, and fourth authors analyzed the correctness of the criteria and their links to the final conclusion. Disagreements were resolved through consensus and resulted in refinements of the code definitions.

3.3 Phase three: connections between control structures and correctness

In the third phase, we examined the relationship between control structures and correctness. This analysis involved cross-tabulating two sets of data. First, we examined the control structures used and the frequency of each correctness category using UpSet plots (Conway et al., 2017; Lex et al., 2014) for each reading question, which allowed us to visualize the various combinations of control structures presented in student responses and their corresponding distribution across the correctness categories. Second, we performed a chi-squared test to determine if there was any association between the control structures observed in a response and the correctness of the response to investigate the connection between specific control structures and correctness. Finally, we used a chi-squared test to determine whether there was an association between the number of control structures in the responses and the correctness of the responses. The results of the chi-squared tests need to be taken with caution, as the observed frequency in one cell was less than five.

4 Findings

We present the findings in three sections, organized by the three analyses we performed: the analysis of the conceptions from the examples in the spanning sets (SS) section, the control structures present in the student responses, and the relationship between control structures present in the responses and the correctness of their use and conclusions.

4.1 Inferred conceptions from SS examples in the textbook

The three examples analyzed in the section addressed the problem of "whether a given vector is in the span of a set of vectors." All three examples in this section rely on similar symbolic representations R^4 in the first example, and R^3 in the other two. For each example, a set of vectors is given, followed by two vectors to be tested for inclusion in the span of the vector set. We identified three different control structures (CS for short, Fig. 5) and six operators (OP for short), used in two separate solution paths as shown in Fig. 5; by solution path, we mean a sequence of operators used for solving a problem with a specific control structure. For example, the solution path of using *Consistency* (CS1) in Fig. 5 is OP1 → OP2 → OP3 → OP4.1.

In all the solution paths illustrated in Fig. 5, the author starts by forming a system of equations expressing the given vector as a linear combination of the vectors in the set (OP1:

construct system). He then creates an augmented matrix for the system (OP2: *construct matrix*) and performs a row-reduced echelon transformation on the matrix (OP3: *row reduction*). He proceeds to inspect the resulting row-reduced echelon form (RREF) matrix either for consistency (OP4.1: *inspect consistency*) or for the number of solutions (OP4.2: *inspect number of solutions*) to conclude whether the given vector is in the span of S. Note that in the presentation, the author alludes to some operators rather than explicitly showing them. In Example SCAA and Example SCAB, the author states “Building the augmented matrix for the given system, and row-reducing, gives…” and then uses the results in the next steps. Such language suggests that a matrix was constructed (OP2: construct matrix), although it was not explicitly shown.

In justifying the decision about whether the vector is in the span or not, OP1→OP2→OP3→OP4.1 and OP1→OP2→OP3→OP4.2 are the main paths that imply two possible conceptions corresponding to two distinct control structures (*Consistency*, CS1 and *Number of Solutions*, CS2). Depending on the control structure, the criterion chosen to decide varies. Using CS1, the author asserts that we can conclude the vector is in the span if the system is consistent. (“… we see that the system is consistent … This is enough to say that $u \in \langle S \rangle$”, Beezer, 2021). Conversely, if the system is inconsistent, then the given vector is not in the span. Another alternative is to use CS2: the vector is in

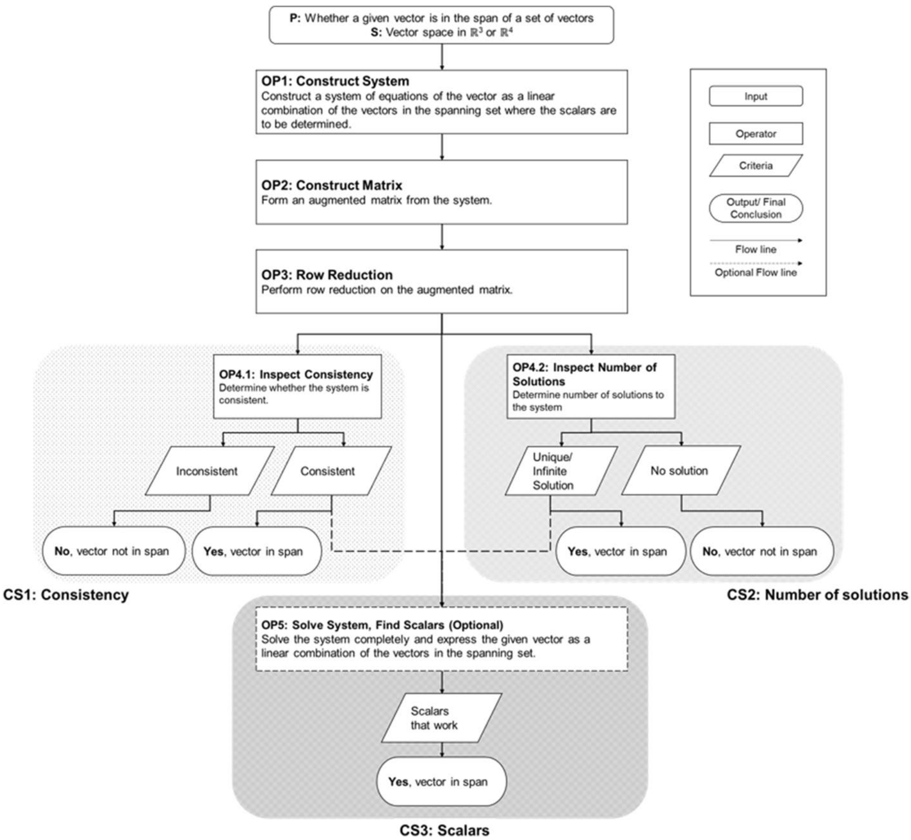


Fig. 5 Inferred conceptions from three examples in the SS section (dashed arrows indicate optional steps)

the span if the system has a unique solution ("This system has a unique solution …, so we are convinced that z really is in $\langle R \rangle$," Example SCAB) or infinitely many solutions ("This system has infinitely many solutions, but all we need is one solution vector, …, so we are convinced that w really is in $\langle S \rangle$", Example SCAA). Conversely, if the system has no solutions, then the given vector is not in the span.

The author also provides an optional operator that solves the system completely for the scalars in the linear combination (OP5: *solve system*, *find scalars*, represented using dashed lines in Fig. 5). This operator is depicted as a means of confirming the decision using either CS1 or CS2, but also implies the emergence of another control structure, *Scalars* (CS3). Using CS3, the author decides whether there are scalars that create a linear combination and if so, finds a set of such scalars ("we know there *is not* (emphasis in original) a solution for the five scalars ... If we wished for further evidence, we could compute an actual solution, say $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = -2$, $\alpha_4 = -3$, $\alpha_5 = 2$" Example ABS). In Example SCAA, after finding one such set of scalars that "work," the author explains that his set of scalars is in fact not unique, highlighting that *any* set of scalars that work is justifiable: "There is nothing magical about the scalars $\alpha_1 = 5$, $\alpha_2 = -3$, $\alpha_3 = 7$, they could have been chosen to be anything." Although one may find such scalars by guessing, the scalars in the three examples are found by solving the system.

4.2 Control structures in student responses to SS reading questions

Figure 6 shows the number of responses with each control structure for both reading questions and the distribution of the correctness of using each control structure. The three most observed control structures in RQ1 were *Consistency* (CS1), *Number of Solutions* (CS2), and *Scalars* (CS3). We had a similar set of common control structures in RQ2 except that we have *Linear Combination* (CS4) in place of CS3. We think this is because there are no scalars that work as a linear combination for RQ2; only one student used CS3 giving incorrect scalars. Other than *Pivots* (CS6), which was coded in 11 student RQ2 responses, the rest of the control structures (CS5–7) were seen five times or less across the responses to RQ1 and RQ2.

Given that the author provides *Scalars* (CS3) only as an optional control structure and he does not mention linear combination (CS4) in the examples, the relative high presence of CS3 and CS4 could be explained by assuming that those responses would rely on the textbook's definition of the span of a set given (Fig. 7) and not necessarily on the operator paths given by the textbook examples. Students could identify the scalars that work as a linear combination by using inspection or guessing.

Most responses had only one control structure (55 for RQ1 and 43 for RQ2), but some had two or more control structures (see Fig. 8). For example, out of 23 responses coded with *Consistency* (CS1) in RQ1, nine had other control structures (one with *Number of Solutions* (CS2), three with *Scalars* (CS3), two with *Linear Combination* (CS4), two with CS2 and CS4, and one with CS3 and *Number of Free Variables* (CS7)). Attending to specific combinations of control structures used in the RQ1 responses (Fig. 8a), we note that responses with CS7 always come in combination with one or more of the textbook's control structures, namely, CS1, CS2, and CS3.

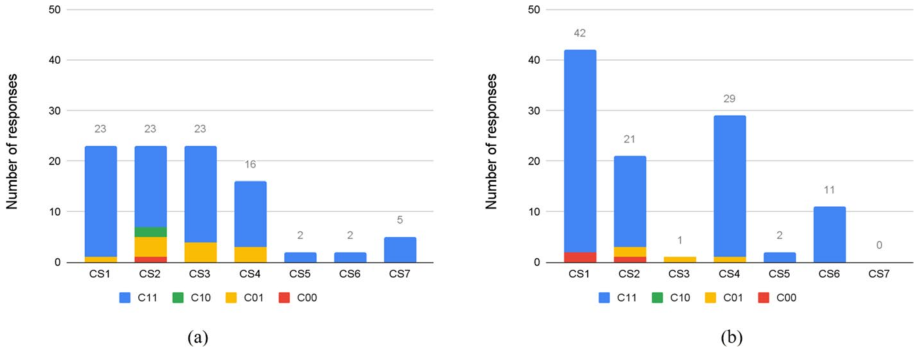


Fig. 6 Number of responses using each control structure and the correctness distribution when using each control structure for **a** RQ1 and **b** RQ2. Note. Control structures: CS1: consistency; CS2: number of solutions; CS3: scalars; CS4: linear combination; CS5: SAGE; CS6: pivots; CS7: Number of free variables. Correctness codes: C11: correct use of criteria for the chosen control structure and a correct link by conception; C10: correct criteria and incorrect link; C01: incorrect criteria and correct link; C00: incorrect criteria and incorrect link

Definition SSCV. Span of a Set of Column Vectors. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their ***span***, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S \rangle &= \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \,|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p\} \\ &= \left\{ \sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p \right\}.\end{aligned}$$

Fig. 7 Textbook's definition of the span of a set of column vectors located at the beginning of the SS section

4.3 Connections between control structures and correctness in student responses

Most of the interpretable responses (56 of 69 in RQ1 and 61 of 67 in RQ2) showed a correct use of criteria and a correct link (C11, see Table 3). The second common correctness category was the response with incorrect criteria but correct links by conception (C01). It is possible that some computational error in obtaining the criteria led students to incorrect conclusions (whether vector in span) despite having a correct understanding of the theorem (link by conception).

The chi-squared test of independence between the control structure observed in a response and the correctness of the response was significant ($\chi^2(200,6) = 14.08$; $p < 0.05$). Responses coded as using *Number of Solutions* (CS2) had, relative to other responses coded with other control structures, more incorrect criteria and links by conception ($\chi^2(200,6) = 4.15$; $p < 0.05$). In contrast, all 22 responses coded using *SAGE* (CS5), *Pivots* (CS6), and *Number of Free Variables* (CS7), which are not mentioned in this section of the textbook, showed correct criteria and had correct links by conception.

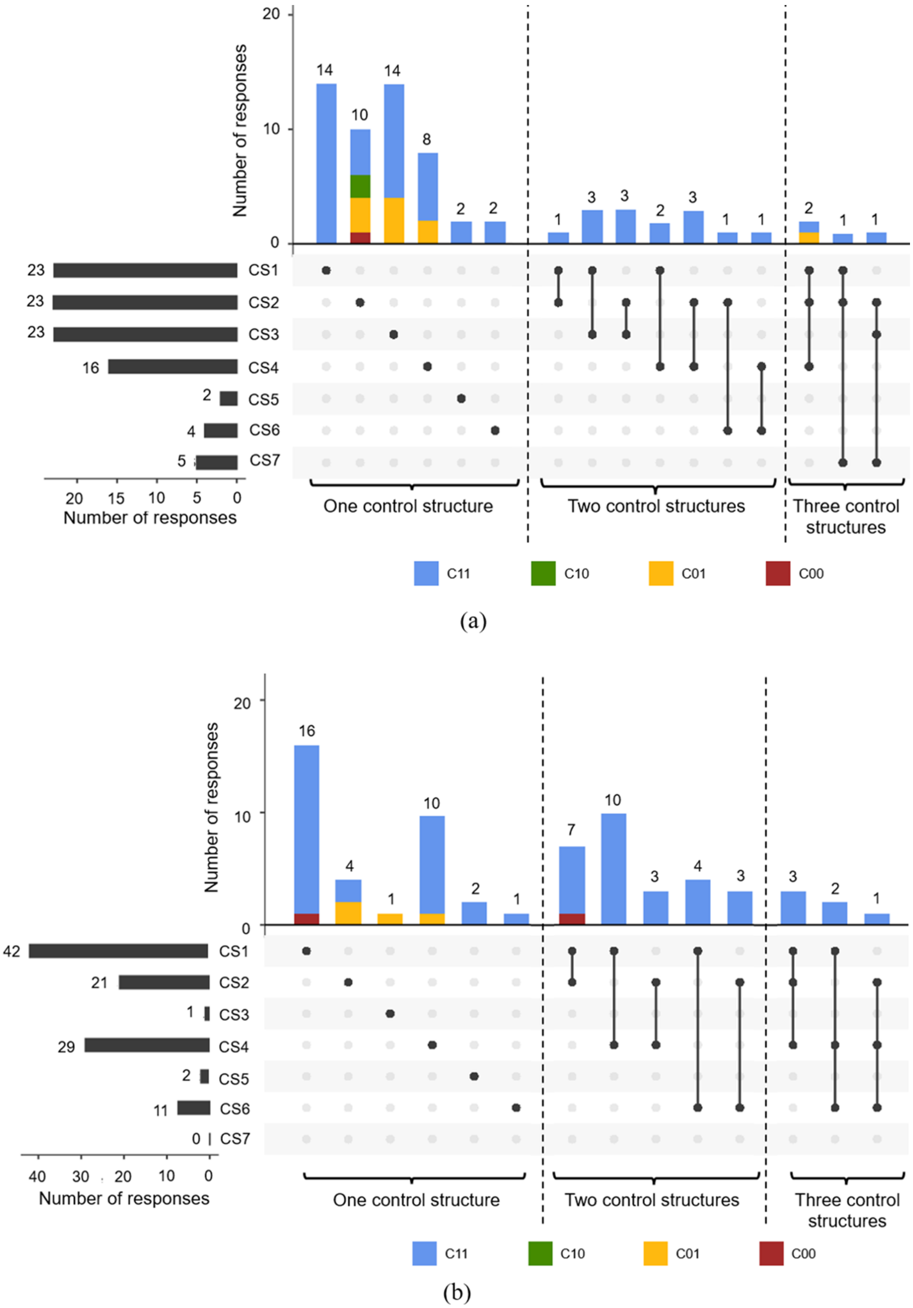


Fig. 8 Control structures in student responses and the correctness distribution when using one or more control structures in **a** RQ1 and **b** RQ2

Table 3 Number of interpretable responses in each correctness category for RQ1 and RQ2

Correctness category	RQ1	RQ2
C11: correct criteria and correct link	56	61
C10: correct criteria and incorrect link	2	0
C01: incorrect criteria and correct link	10	4
C00: incorrect criteria and incorrect link	1	2
Total	69	67

Table 4 Number of responses with one and multiple control structures and correctness in both reading questions combined $\chi^2(218,1) = 37.82$; $p < 0.001$

	C11: correct criteria and correct link	Other correctness categories	Total
One CS	65	17	82
Multiple CS	52	2	136
Total	117	19	218

The chi-squared test between the combined number of control structures present in the responses (one, two, or more) and the correctness of the use of the control structure and the links (two categories, C11 and C10 + C01 + C00) revealed a statistically significant association ($\chi^2(218,1) = 37.82$; $p < 0.001$; Table 4), which corroborates that responses with more than one control structure tended to have correct use of criteria and link by conception. Perhaps the various control structures work towards reaching a correct conclusion.

5 Discussion

When dealing with the problem of deciding "whether a given vector is in a span of a given spanning set," Beezer's (2021) textbook presents three solution methods, each representing a distinct conceptual approach as defined by Balacheff: (1) the consistency of the system of the equation obtained by expressing the given vector as a linear combination of the spanning vectors, (2) the number of solutions to that system, and (3) the scalars found by solving the system completely. We observed a wide variety of control structures in student responses to the two reading questions that address the same problem, with some students using multiple control structures to justify their answers. Moreover, when comparing individual students' responses across the two reading questions, we discovered that they often fall into the different categories of control structures. Although we observed some variation in the level of correctness when using different control structures, most of the student responses offered solutions with correct criteria and linked by conception to both reading questions. We discuss these findings of our research questions next.

5.1 Conceptions inferred from the textbook examples

We identified three distinct potential conceptions given the different control structures in the relevant textbook examples. While the textbook does not name the different strategies, the justifications used to arrive at the final conclusions clearly differ. Each example uses a separate control structure. The textbook does not have a rationale for using a control structure, instead, the examples illustrate different approaches depending on whether the vector is in the span or not, offering multiple ways in which the problem can be treated. In terms of the model of conceptions, such work codifies different conceptions of the notion of spanning sets. Because in linear algebra there are multiple equivalences (Payton, 2019; Wawro, 2014), it is likely that students will struggle to recognize the equivalence of the various conceptions. It might be useful to alert students that although the ways of deciding may seem different, they will soon learn that all are basically the same.

In the analyzed examples and throughout the textbook, the author deliberately uses symbolic representations (matrices, vectors, and equations) instead of geometric or real-world representations. This choice is tailored to the targeted audience and the course's objectives as the textbook aims to provide a formal introduction to linear algebra concepts to second- and third-year students in bridge-to-proof courses. Given the author's heavy use of symbolic representations, it is no surprise that students predominantly used symbolic control structures in their responses.

5.2 Control structures evident in student responses

Our analysis of the control structures used in student responses suggests that students hold a wider variety of conceptions relative to those in the textbook. Perhaps students rely on other resources on the definition of span (e.g., Desmos, YouTube videos, other textbooks, and classmates), instead of reading the accompanying examples and using them to solve the problem. This could explain the use of two of these control structures (*Scalars* (CS3) and *Linear Combinations* (CS4)) because the textbook defines span in terms of linear combinations. Alternatively, students may rely on other sources to answer the questions; this could explain the use of *Pivots* (CS6) and *Number of Free Variables* (CS7), which are not present in the textbook content up to this point. Our prior work regarding students' schemes of use of the reading questions shows that students do attempt to answer the reading questions without reading the preceding text and that they use multiple resources as they work with the reading questions (Castro et al., 2022; Quiroz et al., 2022).

We found responses with multiple control structures. It is not clear that students are aware that one control structure suffices, whether using multiple control structures is a strategy to increase their chances of getting the "correct" answer if they wanted to check their answer or show that they knew more than one way to justify their answers, or if they realized that they were using multiple control structures. It is possible that the decision on which control structure to use does not always happen right after students identify the problem, but rather as they engage with the action-feedback loop, meaning as they carry out the operators until certain feedback hints at a choice of control structure.

Recognizing the equivalence between the control structures may be an epistemological obstacle. *Number of Free Variables* (CS7), for example, was always used in combination with other control structures; perhaps students see this control structure as additional support for their justification. Because the textbook does not yet state the equivalence between these control structures, we presume that it will be the instructor's responsibility to demonstrate how and why they are equivalent to one another. Nevertheless, recognizing their equivalence is an important milestone in the learning of linear algebra because it allows students to perceive how key concepts relate to one another. This suggests a longitudinal study of students' responses to reading questions across sections to track the evolution of their conceptions of span. Such study would provide valuable insights into how students' understanding, specifically their use of control structures, develops over time during the course, informing targeted interventions and instructional strategies to enhance their learning experience.

We also observed different combinations of control structures in the student responses for the two reading questions when comparing control structures used by the same students across the two questions. Why do students choose not to replicate their own work from the previous question when it is essentially the same problem? This illustrates how the nature of a problem and the operators needed to solve it contribute to choosing control structures. For instance, in the second reading question, more students mentioned *Pivots* (CS6) compared to the first, as they may have received feedback indicating that there are no pivots, thus realizing that they can stop. This reaffirms how the cK¢ model works; the system gives students feedback, prompting students to choose control structures that may signal that they have solved the problem.

5.3 Correctness of student responses using control structures

We found that most of the responses had a solution with both correct criteria and link by conception. We do not claim, however, that students who wrote those answers fully understand the material they have read; we are aware that inferring understanding is problematic based on these responses alone. However, the variations in the correctness observed among the responses categorized into different control structures, especially those mentioned in the textbook examples, suggest that the difficulty in understanding and employing them varies. For example, responses that used *Number of Solutions* (CS2) had relatively more incorrect criteria and links by conception than those that used *Consistency* (CS1) and *Scalars* (CS3). This suggests that relying on the number of solutions as a control structure for this type of problem might be more complex. Additionally, it raises the possibility that a misinterpretation concerning the case of a unique solution could be linked to the presence of incorrect criteria. Given the limited evidence, we propose that this connection warrants further investigation.

Even though a conception is not right or wrong, the proposed fine-grained analysis of the correctness of the answer based on the control structure observed by looking at the correctness of criteria and the correctness of link by conception can provide a useful tool to map a progression of conceptions. We think that this analysis is more informative than solely assessing the correctness of the final conclusion. As reading questions are designed to be answered prior to a lesson, they allow instructors to delve into the intricacies of the students' thinking and adjust teaching to students' needs. For instance, if errors in criteria occur more frequently than errors in links, an instructor could revisit the role of the operators (e.g., row reduction) and what is behind them. If link errors are more prevalent, instructors may expand on their mathematical justification.

5.4 Limitations

There are two limitations in the study. First, as the students typed their responses in a textbook textbox, it was not possible to ask follow-up questions to clarify how they made decisions about specific control structures and their connections to local conclusions. This made it difficult to infer all the elements in the conceptions (e.g., operators used and alternative representations) used to arrive at typed answers. However, we were careful to only use the information provided to make the classification and avoid inferring students' intentions. For that reason, and because the operations and the representations could be assumed to be similar, the control structures can serve as a reliable proxy for identifying different conceptions that students have about deciding whether a vector is in the span of a set or not.

Second, our coding for correctness is not comprehensive for students who used multiple control structures but had different correctness based on each control structure. For example, in response to RQ2, student #70 wrote: "No, because the matrix is inconsistent and has infinitely many solutions for α_1 and α_2." This response has two control structures, *Consistency* (CS1), and *Number of Solutions* (CS2). Both the criterion and link for CS1 are correct (inconsistent → not in span), but both the criterion and the link for CS2 are incorrect (infinite solutions → not in span, should be no solution → not in span; infinite solutions should imply vector is in span). We coded this response as C00 (incorrect criteria and incorrect link by conception) to account for the errors when using CS2. This case illustrates a difficulty in the process of coding for correctness that might be significant as a larger number of responses are gathered. In our sample, there was only one instance of this issue.

6 Implications

Balacheff's model (Balacheff & Margolinas, 2005) enabled us to describe potential conceptions from textbook examples and student responses related to the problem of deciding set belonging. While only describing a particular component of students' conceptions due to the nature of the data (typed responses), the analysis shows the potential of extending the use of this model to other data sources (exams and interviews), which can shape curricular and instructional interventions. We note that this model has been successfully used to analyze textbook content (Mesa, 2004, 2010; Mesa & Goldstein, 2016).

That some students answered the two reading questions without using the textbook examples highlights a tension when designing reading questions. The author may have deliberately selected simple vectors for the reading questions so that practicing the replication of the process illustrated in the example would be straightforward. However, doing so may have curtailed the need for students to read the text, as some of them solved the problem without the information in the examples. This points to an area of further research regarding the design of these reading questions.

Finally, we believe that these analyses can facilitate learning and teaching by providing students with immediate feedback and instructors with potential conceptions, which would allow them to plan accordingly for their lessons. Given the advances in natural language processing (NLP), large language models (LLM) with mathematical interpretability, such as MathBERT (Shen et al., 2021), have the potential to automate this process within interactive textbooks. We are planning to leverage Runestone (https://landing.runestone.academy/), an interactive learning analytics platform, to streamline the collection of student and instructor data, thereby supporting our overarching vision of supporting both students and teachers as they learn and teach mathematics with interactive textbooks.

Acknowledgements Thanks to the Undergraduate Research Opportunity Program and to the Research on Teaching in Undergraduate Settings lab at the University of Michigan.

Funding This work was supported by the National Science Foundation under Awards IUSE 1624634, 1821509, 1625223, 1626455, and 1821329.

Declarations

Disclaimer Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Competing interests The authors declare no competing interests.

References

Balacheff, N. (2013). cK¢, a model to reason on learners' conceptions. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2–15). University of Illinois at Chicago.

Balacheff, N., & Gaudin, N. (2009). Modeling students' conceptions: The case of function. *Research in Collegiate Mathematics Education, 16*, 183–211. https://doi.org/10.1090/cbmath/016/08

Balacheff, N., & Margolinas, C. (2005). Modele de connaissances pour le calcul de situations didactiques [Model of conceptions in didactic situations]. In *Balises pour la didactique des mathématiques* (pp. 1–32). La Pensée Sauvage.

Beezer, R. (2021). *A first course in linear algebra*. http://linear.ups.edu/

Beezer, R., Judson, T., Farmer, D., Morrison, K., Mesa, V., & Lynds, S. (2018). *Undergraduate teaching and learning in mathematics with open software and textbooks (UTMOST)* (National Science Foundation, DUE 1821706,1821329,1821509,1821114) [Grant].

Bouhjar, K., Andrews-Larson, C., & Haider, M. Q. (2021). An analytical comparison of students' reasoning in the context of inquiry-oriented instruction: The case of span and linear independence. *The Journal of Mathematical Behavior, 64*, 100908. https://doi.org/10.1016/j.jmathb.2021.100908

Cárcamo, A., Fortuny, J., & Fuentealba, C. (2018). The emergent models in linear algebra: An example with spanning set and span. *Teaching Mathematics and Its Applications: An International Journal of the IMA, 37*(4), 202–217. https://doi.org/10.1093/teamat/hrx015

Cárcamo, A., Fortuny, J., & Gómez, V. (2017). Mathematical modelling and the learning trajectory: Tools to support the teaching of linear algebra. *International Journal of Mathematical Education in Science and Technology, 48*(3), 338–352. https://doi.org/10.1080/0020739X.2016.1241436

Cárcamo, A., Gómez, V., & Fortuny, J. (2016). Mathematical modelling in engineering: A proposal to introduce linear algebra concepts. *JOTSE: Journal of Technology and Science Education, 6*(1), 62–70. https://doi.org/10.3926/jotse.177

Carlson, D. (1993). Teaching linear algebra: Must the fog always roll in? *The College Mathematics Journal, 24*(1), 29–40. https://doi.org/10.1080/07468342.1993.11973503

Castro, E., Mali, A., & Mesa, V. (2022). University students' engagement with digital mathematics textbooks: A case of linear algebra. *International Journal of Education in Mathematics, Science and Technology*. 1–23. https://doi.org/10.1080/0020739X.2022.2147104

Conway, J. R., Lex, A., & Gehlenborg, N. (2017). UpSetR: An R package for the visualization of intersecting sets and their properties. *Bioinformatics, 33*(18), 2938–2940. https://doi.org/10.1093/bioinformatics/btx364

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275–282). Springer.

Freudenthal, H. (1991). *Revisiting mathematics education*. Kluwer.

Graham, S., Kiuhara, S. A., & MacKay, M. (2020). The effects of writing on learning in science, social studies, and mathematics: A meta-analysis. *Review of Educational Research, 90*(2), 179–226. https://doi.org/10.3102/0034654320914744

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177.

Hannah, J., Stewart, S., & Thomas, M. (2013). Emphasizing language and visualization in teaching linear algebra. *International Journal of Mathematical Education in Science and Technology, 44*(4), 475–489. https://doi.org/10.1080/0020739X.2012.756545

Hannah, J., Stewart, S., & Thomas, M. (2016). Developing conceptual understanding and definitional clarity in linear algebra through the three worlds of mathematical thinking. *Teaching Mathematics and Its Applications: An International Journal of the IMA, 35*(4), 216–235. https://doi.org/10.1093/teamat/hrw001

Harel, G. (1989). Applying the principle of multiple embodiments in teaching linear algebra: Aspects of familiarity and mode of representation. *School Science and Mathematics, 89*, 49–57.

Harel, G. (2000). Three principles of learning and teaching mathematics: Particular reference to linear algebra-old and new observations. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 177–189). Kluwer Academic Publishers.

Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191–207). Kluwer.

Kontorovich, I. (2020). Theorems or procedures? Exploring undergraduates' methods to solve routine problems in linear algebra. *Mathematics Education Research Journal, 32*(4), 589–605. https://doi.org/10.1007/s13394-019-00272-3

Lex, A., Gehlenborg, N., Strobelt, H., Vuillemot, R., & Pfister, H. (2014). UpSet: Visualization of intersecting sets. *IEEE Transactions on Visualization and Computer Graphics, 20*(12), 1983–1992. https://doi.org/10.1109/TVCG.2014.2346248

Love, E., & Pimm, D. (1996). 'This is so': A text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Vol. 1, pp. 371–409). Kluwer.

Medina, E. (2000). *Student understanding of span, linear independence, and basis in an elementary linear algebra class* [Unpublished doctoral dissertation]. University of Northern Colorado.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics, 56*, 255–286. https://doi.org/10.1023/B:EDUC.0000040409.63571.56

Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *Research in Collegiate Mathematics Education, 16*, 235–265.

Mesa, V., & Goldstein, B. (2016). Conceptions of angles, trigonometric functions, and inverse trigonometric functions in college textbooks. *International Journal of Research in Undergraduate Mathematics Education, 3*, 338–354. https://doi.org/10.1007/s40753-016-0042-1

O'Halloran, K. L., Beezer, R. A., & Farmer, D. W. (2018). A new generation of mathematics textbook research and development. *ZDM-Mathematics Education, 50*(5), 863–879. https://doi.org/10.1007/s11858-018-0959-8

Parker, C. F. (2010). *How intuition and language use relate to students' understanding of span and linear independence in an elementary linear algebra class* [Unpublished doctoral dissertation]. University of Northern Colorado.

Payton, S. (2019). Fostering mathematical connections in introductory linear algebra through adapted inquiry. *ZDM-Mathematics Education, 51*(7), 1239–1252. https://doi.org/10.1007/s11858-019-01029-9

Plaxco, D., & Wawro, M. (2015). Analyzing student understanding in linear algebra through mathematical activity. *The Journal of Mathematical Behavior, 38*, 87–100. https://doi.org/10.1016/j.jmathb.2015.03.002

Quiroz, C., Gerami, S., & Mesa, V. (2022). Student utilization schemes of questioning devices in undergraduate mathematics dynamic textbooks. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the twelfth Congress of European Research in Mathematics Education (CERME 12)* (pp. 4030–4037). ERME / Free University of Bozen-Bolzano.

Rasmussen, C., & Wawro, M. (2017). Post-calculus research in undergraduate mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 551–581). National Council of Teachers of Mathematics.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Education Studies in Mathematic*s, *88*(2), 259–281. https://doi.org/10.1007/s10649-014-9583-x

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *The Journal of Mathematical Behavior, 61*, 100802. https://doi.org/10.1016/j.jmathb.2020.100802

Shen, J. T., Yamashita, M., Prihar, E., Heffernan, N., Wu, X., Graff, B., & Lee, D. (2021). Mathbert: A pre-trained language model for general nlp tasks in mathematics education. *arXiv preprint arXiv:2106.07340*. https://doi.org/10.48550/arXiv.2106.07340

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). Kluwer.

Stewart, S., Andrews-Larson, C., & Zandieh, M. (2019). Linear algebra teaching and learning: Themes from recent research and evolving research priorities. *ZDM-Mathematics Education, 51*, 1017–1030. https://doi.org/10.1007/s11858-019-01104-1

Stewart, S., & Thomas, M. O. (2007). Embodied, symbolic and formal thinking in linear algebra. *International Journal of Mathematical Education in Science and Technology, 38*(7), 927–937. https://doi.org/10.1080/00207390701573335

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951–961. https://doi.org/10.1080/00207390903200984

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Wawro, M. (2014). Student reasoning about the invertible matrix theorem in linear algebra. *ZDM-Mathematics Education, 46*, 389–406. https://doi.org/10.1007/s11858-014-0579-x

Wawro, M., Rasmussen, C. L., Zandieh, M. J., Sweeney, G. F., & Larson, C. (2012). An inquiry-oriented approach to span and linear independence: The case of the magic carpet ride sequence. *Primus, 22*(8), 577–599. https://doi.org/10.1080/10511970.2012.667516