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# Extrema points: concept images, mis-in and mis-out examples

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## Abstract
This paper reports on students' conceptions of minima points. Written assignments and individual interviews uncovered salient, concept images, as well as erroneous *mis-out examples* that mistakenly regard examples as non-examples and *mis-in examples* that mistakenly grant non-examples the status of examples. We used Tall and Vinner's theoretical framework to analyze the students' errors that were rooted in mathematical and in real-life contexts.


**Keywords** Concept image · Extremum · Minimum · Mis-in examples · Mis-out examples · Representation

Topics associated with functions are quite prominent in secondary school and in academic mathematics curricula. Learners' understanding of concepts, such as function, tangent, derivative, and limit have gained considerable research attention (e.g., Bezuidenhout, 2001; Biza, 2021; Selden & Selden, 2001; Sierpińska, 1987; Tall, 1981; Ubuz, 2007; Vinner & Dreyfus, 1989). However, students' conceptions of extrema points have only scarcely been examined. Extrema points (maxima and minima) are function-related concepts that are widely discussed in high school mathematics classes, as well as in academic courses. According to the Israeli curriculum, students are generally introduced to functions in Grade 7 during algebra lessons, acquiring the vocabulary of extrema points and monotonicity. Then they discuss straight lines and absolute value functions, and afterward, through high school, in calculus lessons, extrema points are commonly examined by means of derivatives and serve, for instance, in the investigations of functions and optimization problems. Later, in academic calculus courses, the concepts of minimum point and maximum point play a significant role in other domains, such as science, computers, economics, and engineering. Because familiarity with students' conceptions of mathematical concepts has been regarded as beneficial for the formulation of instructional tools and generally in teaching (e.g., Borasi, 1996; Hill et al., 2008; Fischbein, 1987; Sierpińska, 1987; Tirosh, 2000), the awareness of

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students' grasp of this concept seems indispensable, and we found the related lack of information quite troubling.

In this paper, we investigate students' conceptions of extrema point and their reactions to examples and non-examples. We also offer some possible, theory-based sources of their ideas, when they do not arrive at the expected conclusions.

## 1 Theoretical background

We first briefly review the literature regarding the possible sources of the students' mathematical errors, then we attend to the question: What does research tell us about students' conceptions of extrema points?

### 1.1 What are the possible sources of students' mathematical errors?

The analysis of students' conceptions calls for the use of theoretical frameworks that provide psycho-mathematical tools and terminology to investigate, interpret, and communicate research observations (Mason, 2006). Our examination of students' reasoning is based on a theoretical framework which is widely used to highlight possible sources of students' difficulties in mathematics: Tall and Vinner (1981) lens, focusing on learners' grasp of mathematical concepts, with an extension, suggested by Tsamir and Tirosh (2023).

Tall and Vinner (e.g., 1981) provided a theoretical tool to analyze learners' grasp of mathematical concepts. They coined the term concept image that entails knowledge segments, ideas, and perceptions that are connected in a person's mind with a specific concept. Concept images develop gradually with time, by way of engaging in different mathematical and daily experiences. The formation of concept images accords with Tall's notion of met-before, namely, the effect of earlier experiences on one's learning (McGowen & Tall, 2010). Concept images are setting-sensitive, and those that are activated under specific circumstances are labeled evoked concept images. Some concept images are expressed verbally, portraying the concept in a personal concept definition (Tall & Vinner, 1981). These personal concept definitions are also considered parts of one's concept image. Due to the phenomenon of compartmentalization, learners sometimes believe in the correctness of incompatible concept images, where either one or more (all) are mathematically wrong (Vinner, 1990). According to Tall and Vinner, the formal definition of a mathematical notion, that is, the concept definition, is the minimal set of necessary and sufficient critical attributes of the notion, which are acceptable by mathematicians at a given time. Since it was published, Tall and Vinner's framework has been continually used in academic studies that investigated learners' conceptions of various mathematical concepts (e.g., Mason, 2006; Ng, 2021; Ulusoy, 2021; Tirosh & Tsamir, 2022; Tsamir et al., 2008).

The definition of a mathematical notion determines two mutually exclusive sets—the related set of examples and the set of non-examples. However, one's concept images might cause errors in this respect, as examples or non-examples might not be regarded as such. In this study, we distinguish between cases where students mistakenly include non-examples in the set of examples, referred to as *mis-in examples*, and cases where students mistakenly take examples out of the set of examples to address them as non-examples, referred to as *mis-out examples* (Tirosh & Tsamir, 2022; Tsamir & Tirosh, 2023). Various *mis-in* and *mis-out examples* have been reported, without marking them as such, for example, Vinner's (1991) findings that students *mis-in* more than one tangent at a point on a graph (*mis-in examples*) and Fujita (2012) findings that students reject, *missing-out* squares from being rectangles (*mis-out examples*).

## 1.2 What does research tell us about students' conceptions of extrema points?

Four types of errors were identified in studies addressing students' conceptions of extrema points:

### 1.2.1 Extrema points require continuity

Shepherd et al. (2012) assigned eleven university students to read the textbook definitions and answer some questions. One student, who examined whether a graph of a function with a jump discontinuity has minima values, repeatedly and unsuccessfully tried to apply the definition. The student was certain that a graph with a jump discontinuity could not represent a single function.

### 1.2.2 At an extreme point M, $f'(m) = 0$

Zaslavsky and Shir (2005) reported on four 12th graders who discussed the statement that the derivative at a maximum point must be zero. One student tried to convince the other three that the statement was false, by presenting $f(x) = \begin{cases} 1 \; for \; x = 0 \\ 0 \; for \; x \neq 0 \end{cases}$ as a counterexample. His peers rejected this solution, refusing to consider a discontinuity point as maximum.

### 1.2.3 Extrema points are absolute

Shepherd, Selden, and Selden, (2012) reported on a university student who erroneously viewed a local maximum of the function as absolute. The student explained:

> It said to find any relative minimum, relative maximum, absolute minimum, and absolute maximum. But… I wasn't quite sure why they were asking me to find possibly four different things if they're supposed to be just the same thing… synonyms. (p. 241)

### 1.2.4 $f(x)$ and $f'(x)$ have extrema points at the same places

Students claimed that the extrema points of $y = f(x)$ and those of $y = f'(x)$ shared the same $x$ values (Orhun, 2012; Ubuz, 2007). For example, Orhun (2012) presented 102 12th graders with a graph of $y = f'(x)$, a continuous function that intersected the $x$-axis at $x = 1$ and at $x = 3$ and had a maximum point at $x = 2$. Most participants (72%) erroneously wrote that the original function also had a local maximum at $x = 2$ and that there was no minimum.

By and large, most publications report on small-scale studies that examined students' conceptions of extrema points as a small part of a wide-ranging research. The extrema-related items were usually, a scarce collection of tasks, often just one, given in one representation. In our study, in contrast, we included several rich context tasks and used a sizable sample of students. We also used the ideas and the terminology offered by Tall and Vinner (e.g., 1981), and in our previous publications (e.g., Tirosh & Tsamir, 2022), to answer the research question: What patterns of student thought evoke mis-in and mis-out examples of minima points?

## 2 The study

### 2.1 Participants

128 students, who had successfully completed at least two calculus courses, two linear algebra courses, and a course of differential equations in mathematics or a mathematics-oriented faculty (computer science, computer engineering, and electronic engineering) participated in this study. In the Calculus 1 courses that they studied, the syllabuses recommended textbooks such as Thomas et al. (2004) or Zorich (2004). The definitions of local extrema presented in Zorich (2004, p. 221) are:

> Definition 1 A point $x_0 \in E \subset \mathbb{R}$ is called a *local maximum* (resp. *local minimum*) and the value of a function $f: E \to \mathbb{R}$ at that point a local maximum *value* (resp. *local minimum value*) if there exists a neighborhood $U_E(x_0)$ of $x_0$ in $E$ such that at any point $x \in U_E(x_o)$ we have $f(x) \le f(x_0)$ (resp. $f(x) \ge f(x_0)$).
> Definition 2 If the strict inequality $f(x) < f(x_0)$ (resp. $f(x) > f(x_0)$) holds at every point $x \in U_E(x_o) \setminus x_0 = \mathring{U}_E(x_0)$, the point $(x_0)$ is called *strict local maximum* (resp. *strict local minimum*) and the value of the function $f: E \to \mathbb{R}$ a *strict local maximum value* (resp. *strict local minimum value*).

The textbooks include definitions of global extremum and extremum at the edge of a closed segment, but the participants in this study referred only to local extrema, so we present only these definitions. Moreover, during their studies, students commonly did not attend to the recommended textbook; they were satisfied with summaries of their lectures, weekly exercises that they had to submit, and the series of Shaum publications that included a generous collection of worked-out examples (e.g., Ayres & Mendelson, 2012). In all the above-mentioned references connections to $f' = 0$ were made in *theorems*. For instance,

> Theorem 13.1:
> If $f$ has a relative extremum at a point $(x_0)$ at which $f'(x_0)$ is defined, then $f'(x_0) = 0$. Thus, if $f$ is differentiable at a point at which it has a relative extremum, then the graph of $f$ has a horizontal tangent line at that point.
> (Ayres & Mendelson, 2012, p. 98)

Commonly, the definition itself got only little (if any at all) attention. Students' interest was mainly in the algorithmic aspect of solutions to related tasks. This is quite understandable, as they were never directly asked about definitions, neither in their weekly assignments nor in the final exam. Also, no emphasis was put on the difference between definitions and theorems, it was dealt with as a known, previously acquired issue. Taking all this into account, we accepted Definition 1 and Definition 2 as correct definitions.

The participants' ages ranged between 25 and 35 years old. All expressed interest in answering the questionnaire; 15 agreed to take part in individual interviews. We approached participants, who wrote some interesting or puzzling solutions and invited them to an individual follow-up interview. For this reason, we did not give identical items to the interviewees as our aim was to better understand the ideas that underlined their written solutions that we found either vague or perplexing. It should be noted that students' interest in being interviewed was much lower than their interest in answering the questionnaire, and thus, not all invited students agreed to be interviewed. Despite this, we were able to interview over 10% of the participants.

## 2.2 Tools and procedure

The research tools were questionnaires and individual, oral interviews. We present six tasks from the questionnaires, which refer to extrema points (Tasks 4 and 5) or to minima points (Tasks 1, 2, and 3).

Task 1: Define: What is a minimum point?

Task 2: Draw two graphs of functions with minima points and mark them

Task 3: True or False?

Statement 1: Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function.

If $K(x_0,f(x_0))$ is a minimum point, then $f'(x_0)=0$.
True / False Prove

Statement 2: Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function.

If $K(x_0,f(x_0))$ is a minimum point,
then $f(x)>f(x_0)$ for all $x\in\mathbb{R}$
True / False Prove

Statement 3: Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function.

If $f'(x_0)=0$, then $K(x_0,f(x_0))$ is an extreme point.
True / False Prove

Task 4: Find (if possible) the extrema points of the given functions and specify for each point its type, minimum or maximum:

$$\text{(a)}\ f_1(x) = x^6 + 15 \quad \text{(b)}\ f_2(x) = x^4 + 2x^3 \quad \text{(c)}\ f_3(x) = |\, x \,|$$

$$\text{(d)}\ f_4(x) = \begin{cases} -x^2, x \neq 0 \\ 10, x = 0 \end{cases} \quad \text{(e)}\ f_5(x) = \begin{cases} -x^2, x \neq 0 \\ -10, x = 0 \end{cases}$$

Task 5: Label all minima-points, Ni (N1, N2, N3, …) on each of the eight graphs in Fig. 1.

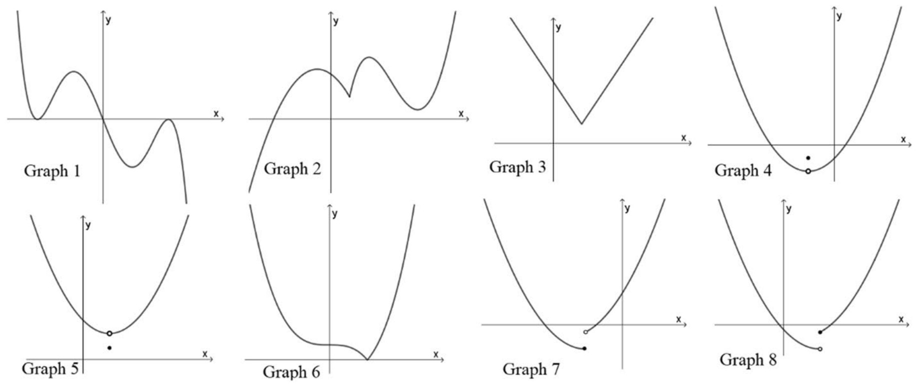



**Fig. 1** The graphs presented in Task 5

Studies indicate that students' solutions are sensitive to problem representations (e.g., Leikin et al., 2013). Occasionally, students provide incompatible solutions to different representations of the same mathematical problem (e.g., Tirosh & Tsamir, 1996). Several erroneous conceptions of function-related concepts were identified with reference to specific problem representation (e.g., Stylianou, 2002). When presenting tasks in graphic, algebraic, and verbal representations, certain incorrect concept images were evident in all three representations, while others were only in a particular representation (Tsamir & Ovodenko, 2013). Aiming to examine a spectrum of students' conceptions of extrema points, we formulated the tasks in three representations: verbal (Tasks 1 and 3), algebraic (Task 4), and graphic (Tasks 2 and 5). The solutions could be provided in any representation.

The questionnaires were distributed and answered in various classes during 90-minute-long sessions. After reading the written solutions to the tasks, we invited students to take part in 1–1 interviews. During the interviews, each participant was shown one of their solutions and asked to clarify the underlying ideas. The interviews, which lasted 30–45 min, were audio-taped and transcribed.

The analysis of the data was conducted in three rounds, (1) identifying students' conceptions of extrema points, (2) discussing possible sources for the common errors, and (3) categorizing *mis-in* and *mis-out examples*. Rounds 1–2 were completed by the first and second authors and round 3 by the first and third authors. Each round included two major steps: (a) individual work where each of the engaged authors individually analyzed the materials and (b) pair meetings to examine the classification that was offered by each of the involved researchers and to negotiate (minor) differences. Then, we further met to decide on the segments that best illustrated the main ideas of our findings.

A portrayal of the timeline we followed in our examination of the data would first point to our initial reading of questionnaires (partly even before their distribution ended) for choosing, inviting, and interviewing suitable participants. When we finished collecting the data (questionnaires and interviews) and transcribed the oral input, our work got quite sequential. We started round 1 by individually analyzing students' conceptions, as expressed both in the written assignments and in transcripts of the interviews. This stage took about 3 months. Authors 1 and 2 met three times, each time for about 2 h, discussed their categorization, and then presented the findings in distribution tables. We continued directly working on round 2 for approximately another month, including two 2-h meetings we held to conclude this stage. The analysis we made in this round was based on our products from round 1. As the idea of mis-in and mis-out examples emerged much later, the first and the third authors reviewed the data again and performed round 3. We examined the data summaries from round 1, and once again, first individually, then together, identified and analyzed mis-in and mis-out examples.

In the report of the findings, all names are pseudonyms; students' emphasized dictions (e.g., voiced louder) are written in capital letters.

## 3 Findings

Tables 1, 2, 3, 4, and 5 present the distribution (in percentages out of 128 responses) of all the coded questionnaire responses to Tasks 1–5. All the participants correctly solved two graphically represented tasks, Task 2 and Task 5-1. In task 2, the participants correctly marked the minima points on the graphs that they drew. They usually drew pairs of smooth graphs (91%), where either both were "parabola–like" with a single minimum (50%), or the first had one minimum and the second had two. Other examples included a cusp-minimum (6%) or a discontinuity minimum point (2%), all percentages refer to the

**Table 1** Distribution of students' answers to Task 1: "Define minimum point"

| The definition | $N = 128$ |
|---|---|
| The lowest point in a neighborhood* | 47% |
| The lowest point of $f$ | 22% |
| Change in monotonicity | 12% |
| $f'(x_0) = 0$ plus, other conditions | 11% |
| No answer | 8% |

*Regarded (almost) correct

**Table 2** Percentage of students' responses to the statements in Task 3 ($N = 128$)

| **Judgment**<br>Justification | $f$ - continuous<br>$(x_0, f(x_0))$ - minima<br>$\Downarrow$<br>$f'(x_0) = 0$ | $f$ - differentiable<br>$f'(x_0) = 0$<br>$\Downarrow$<br>$(x_0, f(x_0))$ extrema | $f(x)$ - continuous<br>$(x_0, f(x_0))$ minima<br>$\Downarrow$<br>$f(x_0) < f(x)$ for all x |
|---|---|---|---|
| **False*** | **33** | **63** | **72** |
| **Correct Justification** | 20 | 53 | 59 |
| No justification | 13 | 10 | 13 |
| **True** | **63** | **28** | **21** |
| It's the definition | 25 | 7 | 13 |
| Algorithmic considerations | 7 | 3 | - |
| Supporting example | 5 | - | - |
| No Justification | 26 | 18 | 8 |
| **No answer** | **4** | **9** | **7** |

The numbers in bold are the sum of the non-bold numbers below them

*Correct judgement

**Table 3** Distributions of students' solutions to "extreme points", Task 4a–c ($N = 128$)

| | $f_1(x) = x^6 + 15$ | $f_2(x) = x^4 + 2x^3$ | $f_3(x) = \|x\|$ |
|---|---|---|---|
| | **Min $x = 0$** | **Min $x = -1.5$** | **Min $x = 0$** |
| **Correct result** | **66** | **60** | **53** |
| **Correct Justification** | **58** | **60** | **45** |
| **Incorrect Justification** | **8** | **-** | **8** |
| $f'(x_0) = 0 \rightarrow$ extrema | 8 | - | 8 |
| | **No extrema** | **$x = -1.5$ , $x = 0$** | **No extrema** |
| **Incorrect result** | **18** | **22** | **34** |
| **Incorrect Justification** | **18** | **22** | **34** |
| $f'$ & $f'' = 0 \rightarrow$inflection no extrema | 18 | - | - |
| $f'(x_0) = 0 \rightarrow$ extrema | - | 12 | - |
| $f'(x_0) \neq 0 \rightarrow$ no extrema | - | - | 2 |
| No differentiability $\rightarrow$ no extrema | - | - | 22 |
| No Justification | - | 10 | 10 |
| **No answer** | **16** | **18** | **13** |

The numbers in bold are the sum of the non-bold numbers below them

**Table 4** Distributions (%) of students' solutions to "extreme points", Task 4d–e ($N = 128$)

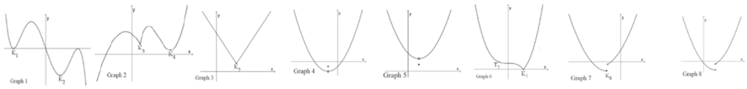
| | $f_4(x) = \{\begin{matrix} -x^2, x \neq 0 \\ 10, x = 0 \end{matrix}$ | $f_5(x) = \{\begin{matrix} -x^2, x \neq 0 \\ -10, x = 0 \end{matrix}$ |
|---|---|---|
| | **Max $x = 0$** | **Min $x = 0$** |
| **Correct result** | **56** | **11** |
| **Correct Justification** | **42** | **11** |
| **Incorrect Justification** | **14** | - |
| Investigating $f(x) = -x^2$ | 14 | - |
| | **No extrema** | **No extrema** |
| **Incorrect result** | **34** | **58** |
| **Incorrect Justification** | **34** | **58** |
| No continuity- no extrema | 32 | 30 |
| For $x = 0$ $f$ is undefined | 2 | 2 |
| Drawing the graph | - | 26 |
| | | **Max $x = 0$** |
| **Incorrect result** | | **14** |
| **Incorrect Justification** | | **14** |
| Change in monotonicity | | 8 |
| $f'(x_0) = 0 \rightarrow$ extrema | | 2 |
| Drawing the graph | | 4 |
| **No answer** | **10** | **17** |

The numbers in bold are the sum of the non-bold numbers below them

entire population of 128 students. Similarly, all students marked $K_1$ and $K_2$ as minima points on Task 5-1 (Table 5). These two cases might have led us to assume that the participants had a satisfactory concept image of minima points (Tall & Vinner, 1981). This, however, was not the case.

In this section, we report on the students' concept images that were identified in explicit, verbal expressions (i.e., students' definitions, evaluation of statements, and justification), as

**Table 5** Distributions of students' minimum points—task 5 (% of 128)



| Solution | Graph 1 | Graph 2 | Graph 3 | Graph 4 | Graph 5 | Graph 6 | Graph 7 | Graph 8 |
|---|---|---|---|---|---|---|---|---|
| **Correct** | $K_1$, $K_2$ **100** | $K_3$, $K_4$ **75** | $K_5$ **80** | None **70** | $K_6$ **53** | $K_7$ **74** | $K_8$ **68** | None **70** |
| **Partial** | - | $K_4$ **25** | None **20** | - | None **37** | None **18** | None **28** | - |
| **Incorrect** | - | - | - | **30** | **10** | **8** | **4** | **30** |
| Fill the hole | | | | 26 | 10 | - | 4* | 4#+5 |
| Vicinity point | | | | 2 | - | - | - | 2 |
| Discontinuity P. | | | | 2 | - | - | - | 19 |
| Bending point | | | | - | - | 8* | - | - |

The numbers in bold are the sum of the non-bold numbers below them

* marked the correct point

\# marked both edges

well as in implicit clues conveyed in their solutions (e.g., marking mis-in extrema points on graphs). We organized this section with a clear delineation of the representation in which each concept image was identified, to indicate how different representations occasionally shaped students' evoked concept images. Moreover, due to the role allotted to examples and non-examples in the investigation and formation of mathematical concepts (e.g., Bills et al., 2006; Chick, 2009; Mason & Pimm, 1984; Petty & Jansson, 1987; Watson & Mason, 2005), putting a spotlight on the categorization of mis-out and mis-in examples may grant added value to our reading of the data. Thus, we specified in each case mis-out and mis-in examples that were identified.

### 3.1 Students' concept images of extrema points: mis-out examples

Our data indicates several erroneous *concept images*, that may result in *missing examples*, hence incorrectly narrowing the set of examples of extrema points.

#### 3.1.1 Extrema points require continuity

This conception was identified in solutions of Task 4, in which 32% of the students wrote that $f_4(x) = \begin{cases} -x^2, x \neq 0 \\ 10, x = 0 \end{cases}$ has no extrema points, and commonly explained that "there is no extreme point when there is no continuity," or "the only possibility is at $x = 0$, but there the function is discontinuous, thus there is no extremum" (Table 4). Similar solutions were provided (58%) to $f_5(x) = \begin{cases} -x^2, x \neq 0 \\ -10, x = 0 \end{cases}$. In their solutions to Tasks 5-5 and 5-7, presented in a graphic representation, participants (37% and 28%, respectively) missed-out the minima points (Table 5), with several students writing: "there are no minima points." In both representations more students *missed-out* extrema, removable discontinuity points than jump discontinuity points (Tables 4 and 5).

### 3.1.2 Extrema points require differentiability

This line of reasoning dictates that not only discontinuity points, but also cusp points are prevented from being extrema points. Participants (34%) incorrectly wrote that there are no extrema points to $f_3(x) = |x|$ in Task 4, which was presented in the algebraic representation (Table 3). Many (22%) added: "It's a cusp point" or "If its not differentiable, there are no extrema points." Anat, who wrote: "$y = |x|$ could have had minimum at $x = 0$, but it's not differentiable there," elaborated in her interview:

Anat: Cusp points have no derivative. As a rule, there is no extremum at cusp points.
Interviewer: What do you mean by "as a rule"?
Anat: I mean – NEVER. Extrema points are not acute.

In reaction to the graphic representation of Tasks 5-2, 5-3, and 5-6, students (25%, 20%, and 18%, respectively) missed-out the minima cusp points (Table 5). They explained, for instance, that "a minimum point does not look like this, the derivative is not zero!"

### 3.1.3 Extrema points implies $f'(x_0) = 0$

This conception means that $f'(x_0) = 0$ is a necessary condition for extrema points. Students explicitly expressed this idea in reaction to the verbal representation of Task 1 (define) and Task 3 (true or false statements). In Task 1, 11% of participants mentioned $f'(x_0) = 0$ in their definitions, for example, "a function has a minimum in the shift from decrease to increase where $f'(x_0) = 0$." In Task 3, 63% of students incorrectly judged statement 1 as true (Table 2). Their justifications were: (i) "It's the definition" (25%), (ii) algorithmic considerations, such as, "to find extrema points, we calculate where $f'(x_0) = 0$, therefore the statement is correct" (7%), (iii) a supporting example (5%), presenting a drawing of a "parabolic" function, or a quadratic expression as a validating proof (e.g., "for instance, if $y = x^2 - 1$, $y' = 0$ for $x = 0$, there's a minimum"). In his interview, Danny explained:

Danny: It's the definition… like going downhill, reaching a valley, catching your breath, and advancing to climb the following hill.
Interviewer: What do you mean by 'catching your breath'?
Danny: The zero slope is like momentarily resting between the down and the up.
Interviewer: Is the 'zero slope' between descent and ascent a must?
Danny: Sure. It's a split of a minute…. In mathematics, this is how one finds a minimum value, when investigating a function, I mean…$f'$, then $f'(x_0) = 0$… the $X$s found are $X$s of extrema points.

Danny's description portrays a concept image, embedded in daily, "downhill, reaching a valley, catching your breath, and advancing to climb the following hill" views, and in the mathematical investigation of differentiable functions.

In their responses to $f_3(x) = |x|$ in Task 4, presented algebraically, among the students (34%) who erroneously claimed that there are no extrema points, 2% explained: "if $f'(x_0) \neq 0$, then there is no extreme point" (Table 3). It should be noted that several students who wrote "continuity required" or "differentiability required" justifications, expressed later the view that $f' = 0$ is required. For instance, Tal wrote in Task 4 that $f_3(x) = |x|$ has no extremum, because "the only possibility is a cusp point, but it has no derivative" (accordingly, we categorized her answer: "differentiability requirement"). She elaborated in her interview:

> At extrema points, functions go like that (signals with her finger a 'parabolic' wave with a maximum) or like that (signals another wave with a minimum). So, at an extreme point the derivative has to be zero. It must EXIST and BE ZERO. At cusp points there can't be an extremum because there is no derivative. A non-existing derivative can't be equal to zero.

### 3.1.4 Extrema points are absolute

This conception was explicitly conveyed only in students' solutions to the verbal representation of Task 1 and Task 3-2. Here we had 22% of students (Table 1) define a minimum point as "the lowest point" or "the lowest point of the function." However, in his interview, Eden explained:

> Eden: A minimum point is the lowest point… minimum means lowest, smallest, the least.
> Interviewer: So... can a function have only one, single minimum, as it has to be THE LOWEST point?
> Eden: [giggled] Definitely NOT, it's the lowest IN ITS NEIGHBORHOOD.

Eden wrote a non-rigorous answer to task 1, taking the assignment "define" too lightly. Still, students also erroneously marked statement 2 in task 3 as being correct (21%, Table 2).

### 3.1.5 $f'(x)=0$ and $f''(x)=0 \Rightarrow$ No extreme point

In their solutions to the algebraic representation of $f_1(x)=x^6+15$, 18% of students wrote that "there is no extreme point" and explained that "when $f'(x)=0$ and $f''(x)=0$ it's an inflection point, not an extreme point" (Table 3). Students tended to regard these two conditions as necessary (thus, mis-out non-horizontal inflection points) and sufficient (thus, mis-out extrema points like for $f(x)=x^6$).

## 3.2 Students' concept images of extrema points: mis-in examples

Our data indicate some erroneous lines of reasoning that could give rise to *mis-in examples*, hence overextending the set of examples of minima points:

### 3.2.1 Change in monotonicity

The characterization of minima points as "points, shifting from decrease to increase," is valid for points that are placed in a continuous segment of the function. In our study, however, students (12%) verbally defined minima as "points where a function changes from decreasing to increasing" with no reference to continuity (Table 1), and some further explained in their interviews: "it's like reaching a valley when hiking, and then turning to climb the next hill," or "it's like the bottom of a bowl," providing daily "continuous" examples. This line of reasoning was also, inappropriately applied to discontinuity points in Task 4, overextending the examples set to include four types of mis-in examples, discontinuity, hole-filling, vicinity, and absent points. These are described next.

### 3.2.2 Discontinuity points

As specified, it is possible to have a discontinuity extreme point, but basing its being an extremum on "change in monotonicity" consideration is invalid. Nevertheless, in their solutions to the algebraic representation of Task 4, students (14%) erroneously wrote that $f_5(x) = \begin{cases} -x^2, x \neq 0 \\ -10, x = 0 \end{cases}$ has a maximum at $x = 0$ and 8% justified this stating "because at $x = 0$ the function changes from increasing to decreasing" (Table 4). Several students (14%) who correctly wrote that $f_4(x) = \begin{cases} -x^2, x \neq 0 \\ 10, x = 0 \end{cases}$ has a maximum at $x = 0$, also provided this explanation. They reached this solution after an investigation of $f(x) = -x^2$, mentioning "change of monotonicity" considerations, with no reference to the given $f(0) = 10$. In the graphic representation, students (19%) marked a non-minimum, jump discontinuity point, located between the descent and ascent of the function, as a minimum in Task 5-8 (Table 5).

The three types of mis-in examples, hole-filling, vicinity, and absent points were far more surprising because the choices of minima points violated basic function-graph definitions.

### 3.2.3 Hole filling

In reaction to the graphic representation of Task 5, 26% of students marked a minimum by filling the hole in Task 5-4, 10% did it in Task 5-5, and 9% in Task 5-8 (Table 5). Bar, who filled the hole on Graph 4, explained in her interview:

Interviewer: You marked a minimum here…
Bar: Yes. Where the graph changes from decreasing to increasing.
Interviewer: But the point is actually here (shows on the graph).
Bar: Can't be… It's not on the graph.
Interviewer: Yes, IT IS… A bit special, a discontinuity point. A removable discontinuity point that BELONGS TO THE GRAPH.
Bar: Oh… then, there is no extreme point. Certainly NOT HERE… not at all...
Interviewer: Why?
Bar: Because the ONLY OPTION is HERE (points at the discontinuity point), but it's out…
Interviewer: So? Is it a problem?
Bar: It's not A PROBLEM. It's just NOT AN EXTREMUM.
Interviewer: What kind of point is it then?
Bar: Just a regular point.

Bar voiced a sequence of ideas regarding removable discontinuity points, first as "not being on the graph," then, accepting it being on the graph, but impossible for being "an extremum," and vaguely concluding that it is "just a regular point", that is, a point with no special characteristic.

Ron, who erroneously marked two minimum points at the same $x$ on Graph 8 by filling the hole, explained in his interview that "both are where the function stops decreasing and starts increasing." Consequently, the change in the monotonicity idea nullified Ron's knowledge about the concept function and allowed him to invalidly correspond two different $f(x)$ values to the same $x$.

### 3.2.4 Vicinity point

In reaction to the graphic representation of Task 5, a few students (2%) marked a minimum in the vicinity of the hole on Task 5-4 (Table 5). Aviv, who marked an arrow pointing to the hole and added "it's very close to this point", explained in his interview:

> Interviewer: Where EXACTLY is the minimum?
> Aviv: Where the graph stops decreasing and starts increasing. It's very, VERY close to… (points to the hole). The closest possible… I can't say where EXACTLY. It's almost here (points again).
> Interviewer: So… it IS actually here (points to the hole).
> Aviv: NO. NO. Here the limit of $y=f^{'}(x)$ from both sides is zero, so there's a minimum. BUT it's a DELETED LIMIT. This point (the hole) is not on the graph.

Once again, we encounter a student's conception that removable discontinuity points are not on the graph. Still, their overgeneralized "change in monotonicity" criterion is implemented by means of the limit of $f^{'}$ considerations to these discontinuous points. Moreover, the deleted limit of $y=f^{'}(x)$ disclosed anew the belief that $f^{'}=0$ is necessary for extrema points.

### 3.2.5 Absent point

In reaction to the graphic representation of Task 5-4, Gal, who marked a vicinity-minimum, concluded in the interview: "It's… like a deleted limit of the function when $x$ tends… (points to the graph). Actually, the minimum is IN THE GAP… that's the limit." While the previous vicinity point conception just insinuated a non-existing, hole extreme point, Gal overtly stated that the minimum is "in the gap." That is, an extreme point does not have to belong to the domain of the function.

### 3.2.6 $f^{'}(x)=0 \Rightarrow$ extrema points

This conception dictates that $f^{'}(x)=0$ is sufficient for extrema points, and it imposes, for instance, that horizontal inflection points be missed-in as extrema points. This was evident in 28% of students' true judgments of the verbal representation (Task 3-3, Table 2). Participants justified: (a) "It's the definition, $f^{'}(x)=0$ means a zero slope, thus, extreme point" (7%) and (b) algorithmic considerations (3%), (e.g., "to find extrema point, we calculate $f^{'}(x)=0$"). When solving Task 4, presented in an algebraic representation, 12% of the students only calculated where $f_2(x)=x^4+2x^3$ satisfied $f^{'}(x)=0$, then, concluded that there were two extrema points, at $x=-1.5$ and $x=0$. Similarly, in reaction to $f_1(x)=x^6+15$, 8%

of the students solved only $f^{'}(x)=0$, reaching the correct solution. This phenomenon is also expressed in Jemma's interview, who for $f1$ and $f2$ checked only where $f^{'}(x)=0$:

Jemma: To find extrema I find the derivative. If it's zero, there is an extremum.
Interviewer: Always?
Jemma: ALWAYS...
Interviewer: If the derivative is not defined at a point, can it still be an extremum?
Jemma: No. IT CAN'T.
Interviewer: Is there a case that the derivative is zero, but there is NO EXTREMUM?
Jemma: No. There isn't.
Interviewer: … and if there is a minimum…
Jemma: I already told you... The derivative there, is zero.

Apparently, for Jemma, $f^{'}(x)=0$ is both, a necessary and sufficient condition for extrema points. In fact, she does not appear to differentiate between necessary and sufficient conditions.

### 3.2.7 Bending points

A total of 8% of the students labeled a bending point given in the graphic representation of Task 5-6 (see drawing, Table 5) as the minimum. The explanation was in the form: "It's a minimum, you can see it, there is no lower point in the neighborhood."

## 4 Discussion

We would like to open this section by confiding with you the latest, and most striking incident that we faced. In the analysis of the data of this study, we identified the "bending point" in the students' reactions to Graph 6 (Task 5) as an erroneous concept image, and we even came up with an optional rationale for their (mistaken to us) explanation: "…there is no lower point in the neighborhood." However, even though the categorization was endorsed by the three authors, two experienced high school math teachers, a mathematician, and by the reviewers of the first version of the manuscript, we were wrong. Only when we worked on the manuscript to prepare this revised version, we noticed, that this solution is consistent with Definition 1 (see "Participants" subsection under "The study" section), and its validity can be easily identified in the reader-friendly definition:

> … $f$ is said to have a *relative minimum* at $x_0$ if $f(x_0)\leq f(x)$ for all $x$ in some open interval containing $x_0$ (and for which $f(x)$ is defined). In other words, the value of $f$ at $x_0$ is less than or equal to all values of $f$ at nearby points. (Ayres & Mendelson, 2012, p. 98)

We were shocked. There is no doubt that this solution should have been accepted as correct. The moral lesson learned here is momentous: Using the definition is not at all self-evident even for experts. In line with the reported phenomenon that only concept images are commonly used in problem-solving (e.g., Vinner, 1991; Vinner & Dreyfus, 1989), we all were trapped in a concept image that misled us. In future classes, we certainly are going to use this item to initiate related discussions.

In the following sub-sections, we summarize the highlights of our findings, offer some next-step research, and a selection of instructional ideas.

### 4.1 Concept images, mis-in or mis-out examples, and possible sources

In this study we identified several concept images of extrema points, some validating and strengthening previous research results, while others are new lines of reasoning. Our findings indicate that the "$f'(x)=0$ is a must" together with the "change in monotonicity" concept images, fiercely dominated the students' formal knowledge, leading them to violate definitions of "function," "graph," and "discontinuity point," going as far as accepting a "non-existing point" as an extreme point. These are novel, remarkable findings, especially because the subjects were post-secondary, mathematics-oriented students. We also found some unique links between the identified concept images and related mis-in and mis-out examples. These connections are significant as students' reactions to various exemplary and non-exemplary items may open a window to their conceptions (e.g., Tirosh & Tsamir, 2022). Our findings may further serve in formulating meaningful instruction (e.g., Tsamir & Tirosh, 2023).

Tall and Vinner's theoretical framework offered clues for possible sources of the identified concept images. On the whole, early mathematical, daily, and linguistic experiences, are potential sources of students' conceptions that are not in line with the expected conclusions (Tall & Vinner, 1981). In Israel, extrema points are first introduced in middle school algebra lessons. In these pre-calculus discussions, applicable, "change in monotonicity" considerations are highlighted when referring to extrema points. No attention is given to the special circumstances that the functions are continuous. In fact, even in high school calculus the "change in monotonicity" is often applied to determine, or to be deduced from extrema. Thus, it seems only natural to witness students' tendency to overgeneralize "change in monotonicity" attributes to discontinuous functions. In high school, students routinely find $X$s of extrema points of functions by calculating $f'(x)=0$. Commonly, the investigated functions are either smooth or discontinuous due to vertical asymptotes. Students rarely discuss jump or removable discontinuity points, and those are never extrema points. Extrema points are normally within a smooth neighborhood, or scarcely at the edge of a closed domain. Furthermore, in many cases, the test cases of $f'(x)=0$ indeed results in extrema points. These continual experiences may lead to the intuitive conception that $f'(x)=0$ is necessary, and perhaps even sufficient for extrema points. Here, like in many other cases, students learn to recognize concepts by experience and usage (Tall & Vinner, 1981).

### 4.2 Second thoughts and follow-up studies

An overview of the data portrays a complex picture that ignited our second thoughts regarding three issues:

(a) Our categorization of "continuity required," "differentiability required," or '$f'=0$ required," was based on a careful analysis of the students' written solutions. These concept images were identified in the students' mis-out examples, that were conveyed in their justified exclusion of extrema cusp points and discontinuity points. However, it seems that students did not necessarily differentiate between "continuity" and "differentiability" (see also, Duru et al., 2010; Juter, 2017; Sevimli, 2018), or may have rightfully considered a certain warrant sufficient for a specific task. We got some indications that even those students who wrote "continuity required" or "differentiability required" occasionally believed that the stronger condition $f'=0$ was required (e.g., Tal, Section 3.1.3). Consequently, reference made to "continuity" or "differentiability" may bare pointers to $f^{'}=0$ and perhaps to "change in monotonicity" too. Also, the mis-in examples of "filling the hole," "vicinity points," and "the hole itself" cohere with these concept images and express a complementary grasp of "smoothly settled" extrema points, where $f^{'}$ exists, equals zero, and possibly, related changes in monotonicity are evident as well.

(b) The students' grasp of minimum as "the lowest point of the function" (Table 1) would be expected to result in a tendency to exclude additional local minima in the other tasks. However, no related *mis-out* examples were observed (e.g., in Task 5-1 all participants marked the two minima). While our assumption regarding the connection between the students' written solutions (i.e., concept images) and related mis-out examples are logical interpretations, responses to tasks are often setting-dependent and inconsistent. Thus, due to possible compartmentalization (e.g., Vinner, 1990), certain, declared lines of reasoning do not necessarily yield the related exclusion of examples. Moreover, in Eden's interview (section 3.1.4) it seems that rather than an inappropriate definition of extrema points, we witness an offhanded definition, addressing the most eye-catching characteristic of the concept, which is supported by related, daily terminologies (Pimm, 1987).

(c) We carefully separated between the *concept images*: $f^{'}(x)=0 \Rightarrow$ extrema points and extrema points $\Rightarrow f^{'}(x)=0$, as these two signify references to a necessary attribute on one hand and a sufficient attribute on the other. In certain cases, it seemed clear which of the two accounts for the participants' solutions (e.g., calculating only $f^{'}(x)=0$ to find extrema points of $f_2(x)=x^4+2x^3$ in Task 4 indicates $f^{'}(x)=0 \Rightarrow$ extrema points conception). But, it turned out that occasionally, students did not differentiate between these two conditions (e.g., Jemma, Section 3.2.2., see also Nardi, 2007).

We believe that our second thoughts and dilemmas call for further, fine-tuned research.

### 4.3 Possible instructional implications

This study investigated the students' conceptions of extrema points with special attention to the verbal, algebraic, and graphic representations of the tasks. The representation-related findings impart the data extra merit because beyond the information regarding the students' various concept images, we get to know in what settings specific conceptions might be elicited, identified, and consequently, addressed in teaching. Several concept images were found in a single representation (e.g., extremum is absolute), and others in two (e.g., change in monotonicity), yet no concept image in this study appeared in all three representations. As concept images are setting-dependent, knowledge of connections between the students' ideas and the representations of the tasks that elicited them may contribute to designing meaningful instruction (e.g., Pape & Tchoshanov, 2001).

For instance, the conception that "extremum is absolute" which was expressed only in participants' answers to the verbally represented tasks (Definition-Task 1, Statement-Task 3-2), negated their solutions to algebraically or graphically represented tasks (finding more than one extreme for a function). The subsequent incompatibility can be used in the design of a sequence of tasks aiming to raise students' awareness of the inconsistency. A potential pair of tasks is the verbally presented statement, Task 3-2 (extremum is absolute) and the graph in Task 5-1 (all students labeled two minima). Moreover, Eden was quite forgiving, when he corrected in his interview his "extremum is absolute" definition and added "in an environment" (see “Extrema points are absolute” subsection under the “Findings” section). The two tasks combined may promote Eden’s motivation and lead to refine their definition and to examine what it means to define in mathematics.

The concept image “$f' = 0 \Rightarrow$ extremum” was identified in the students’ solutions to two representations, the verbal (Task 3-3) and the algebraic (Tasks 4-1 and 4-2). A way to encourage students to review this *concept image* is to present the following tasks:

Exercise 1:Statement 3, Task 3 from the questionnaire

Exercise 2: Find the extrema points of the following functions:

(a) $f_1(x) = x^4 + 2x^3$ (from Task 4)
(b) $f_2(x) = x^3$

In Exercise 1 and Exercise 2(a) some students are likely to express this concept image, but$f_2(x) = x^3$, with which students are quite familiar, illustrates a case in which $f'(0) = 0$, but at $x = 0$ there is no extremum. A discussion of the students’ solutions offers an opportunity to reflect on their reasoning,

Another aspect that was considered in the analysis of the data was the apparent mis-out examples and mis-in examples (e.g., students missed-out the minimum of $f(x) = |x|$ because extremum requires differentiability; students missed-in $x = 0$ as an extremum in Task 4-2 because $f'(x) = 0$ is sufficient for extrema, see Table 6). We believe that the information regarding the students’ mis-in and mis-out examples can be helpful in both identifying the learners’ conceptions and promoting their knowledge. Mis-out examples differ from mis-in examples from an instructional point of view. In the case of mis-out examples, the attributes that the learner considers are consistent, but the domain (fundamental set) is insufficient. Thus, consulting the definition is a must. It is the only option for convincing the need to add the missing parts of the example space (Vinner, 1991). On the other hand, mis-in examples contradict certain critical attributes of the concept. For instance, Exercise 2 may produce cognitive conflict and follow-up discussions. Evidently, there is no guarantee that students will correctly resolve the conflict, thus, the teacher’s role here is crucial (Tall & Vinner, 1981). We would like to note that our instructional ideas illustrate ways of using errors as a springboard for inquiry (also, Borasi, 1996).

We would like to conclude this section by sharing with you an anecdote we experienced years ago. During a PME conference, by the end of a vivid presentation of a study regarding the students’ common errors in a certain topic, a colleague raised his hand and asked: “why are you engaged in ‘disaster research,’ pointing to students’ disabilities?” Our simple answer was and still is: “We do not regard such studies as ‘disaster research.” We address errors, for instance, like Borasi (1996), who explained in her book:

> The approach to errors information in this book recognizes the potential of errors to provide the source of valuable opportunities for mathematical exploration, problem solving, and reflection, for students as well as teachers. (p. 4)

We are well aware that in every field there are periodic trends that dictate thematic as well as linguistic guidelines. However, no matter how we choose to label the phenomenon (e.g., errors, mistakes, misconceptions, alternative conceptions), we should bear in mind that

> some schools of thought have been aware that errors are not only inevitable, but also a healthy part of one’s education – as suggested by the popular motto “You learn from your mistakes”. (Borasi, 1996, p. 3)

**Table 6** Connections between concept images, representations, and examples in all tasks

| Concept image | Examples | | Representation of task | | |
|---|---|---|---|---|---|
| | Mis-out | Mis-in | Verbal | Algebraic | Graphic |
| Extremum require continuity | + | − | - | Task 4 ($f_4(x)$, $f_5(x)$) | Task 5 (graphs 5 and 7) |
| Extremum require differentiability | + | − | - | Task 4 ($f_3(x)=\lvert x\rvert$) | Task 5 (graphs 2, 3, and 6) |
| Extremum require ($f'=0$) | + | − | Task 1 (define)<br>Task 3 (statement 1) | Task 4 ($f_3(x)=\lvert x\rvert$) | - |
| Extremum is absolute | + | − | Task 1 (define)<br>Task 3 (statement 2) | - | - |
| No extrema if $f'$ and $f''$ are zero | + | − | - | Task 4 ($f_1(x)=x^6+15$) | - |
| Change in monotonicity | − | + | Task 1 (define) | - | Task 5 (graphs 4, 5, 7, and 8) |
| Bending point | − | + | - | - | Task 5 (graph 6) |
| $f'=0$ sufficient for extremum | − | + | Task 3 (statement 3) | Task 4 ($f_1(x)=x^6+15$) and ($f_2(x)=x^4+2x^3$) | - |

+connection found

−no connection found

Indeed, human beings are constantly influenced by intuitive, coercive, not necessarily correct ideas, and in mathematics classes, it is the teachers' responsibility to encourage inquiry-based learning environments:

> Because errors, by definition, are results that do not meet expectations, they can be considered a prototypical example of an anomaly. Thus they, too, can be viewed as a natural stimulus for reflection and exploration and as a means to support inquiry. (Borasi, 1996, p. 28)

Thus, equipping teachers and educational-material-developers with information regarding what is easy and what is challenging for students seems an essential prerequisite (e.g., Borasi, 1996; Fennema et al., 1996; Ryan & Williams, 2007; Santagata, 2005; Willingham et al., 2018).

**Data availability** The datasets are available from the corresponding author upon reasonable request.

## Declarations

**Competing interests** The authors declare no competing interests.

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