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Mis-in and *mis-out* concept images: the case of even numbers

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Abstract
This paper reports on concept images of 38 secondary school mathematics prospective teachers, regarding the evenness of numbers. Written assignments, individual interviews, and lesson transcripts uncover salient, erroneous concept images of even numbers as numbers that are two times "something" (i.e., *2i* is an even number), or to reject the evenness of zero. The notion of concept image serves in the analysis of the findings, and the findings serve in offering two refinement notions: *mis-in* concept images that mistakenly grant non-examples the status of examples (e.g., *2i* is an even number), and *mis-out* concept images that mistakenly regard examples as non-examples (e.g., zero is not an even number). We discuss possible benefits in distinguishing between these two refinement notions in mathematics education.

Keywords Concept image · *Mis-in* concept images · *Mis-out* concept images · Even numbers

There are two critical attributes in the definition of an even number—being an integer and fulfilling the demand of $n = 2\ k$. To the best of our knowledge, studies have usually focused on the requirement that $n = 2\ k$, presenting the research subjects with whole numbers (e.g., Zazkis, 1998), or addressing the special nature of zero (e.g., Levenson et al., 2007). We regard the study of students' knowledge about the critical attribute *integers* as important. Accordingly, the first aim of our study was to examine university students' grasp of the evenness of numbers with special attention to this attribute.

In this paper, we analyze students' conception of even numbers by means of Tall and Vinner's theory of concept image–concept definition (e.g., Tall & Vinner, 1981). Throughout the data analysis, we felt a need for some theoretical refinement. Consequently, this paper offers a fresh angle for examining concept images, which leads to our second aim: To use our findings for delving into types of learners' concept images. We elaborate on this theoretical extension when concluding the theoretical background.

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1 Theoretical background

1.1 Concept image–concept definition

The construct concept image was coined and introduced by Vinner and Herschkowitz at a PME conference (1980), and then, it was addressed by Tall and Vinner in a manuscript, published in *Educational Studies in Mathematics* (Tall & Vinner, 1981). In that publication, Tall and Vinner provided detailed terminology around the construct, highlighting facets of humans' mathematical reasoning.

The image of a concept entails the total of knowledge segments, ideas, indications, and intuitions that are connected in a person's mind with a specific notion.

> It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures (p. 152).

The formation of concept images accords with Fischbein's notion of *primacy effect*, that is, a person's tendency to consider early learnt material, while ignoring later, updated information (e.g., Fischbein, 1987); and with Tall's notion of *met-before*, namely, the effect of earlier experiences on one's learning (McGowen & Tall, 2010).

Parts of the concept image are triggered in a certain setting, while others may come up in a different setting. These are *evoked* concept images (e.g., in verbal settings, students erroneously wrote that $f'=0$ at inflection points; but on graphs they erroneously marked peak points because "the graph keeps increasing, but the slope changes dramatically" (Tsamir & Ovodenko, 2013)).

Some concept images are expressed in a way of verbal phrasing, describing or pseudo defining the concept in a *concept definition image*.

> It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image (p. 152).

Vinner further pointed to the prevailing phenomenon of compartmentalization, that is, situations in which individuals believe in the correctness of two (or more) incompatible images (Vinner, 1990). It may happen that one of the concept images is correct, or that both (all) are mathematically wrong.

Since it was published till now, Tall and Vinner's theoretical framework has been continually used in studies that investigated learners' conceptions of various mathematical constructs (e.g., definite integral-Serhan, 2015; complex numbers-Jalkh, 2020; equation-Tossavainen et al., 2012; radian-Akkoc, 2008; triangle, height, quadrilateral, circle, cylinder-Levenson et al., 2012; Ng, 2021; Tsamir et al., 2015; Ulusoy, 2021; Vinner & Herschkowitz, 1980; function-Beach, 2020; Maull & Berry, 2000; inflection point-Tsamir & Ovodenko, 2013; tangent, limit, continuity-Fonseca & Henriques, 2020; slope-Hoffman, 2015). A number of publications highlight the benefit of using the concept image–concept definition theoretical framework for the analysis of learners' conceptions of mathematical notions and in formulating instructional tools:

> The notion of a concept image (Tall & Vinner, 1981) has been taken up by many generations of mathematics educators. The reason, I think, is because of the useful distinction it makes between formal definitions and the gradual development of a sense of what the term means to them [*learners*]… the label… opens up the possibility for probing in more detail the constituents of concept images and the ways

in which these arise and are integrated (to some degree or other) in individuals and through their interaction with peers, with expert-others, and through reflection on their own experience. (Mason, 2006, p. 154–155)

According to Tall and Vinner, the formal definition of a mathematical notion, that is, the *concept definition*, is the minimal set of necessary and sufficient critical attributes of the notion, those that are acceptable by the mathematicians at a given time. The definition of a mathematical notion determines two mutually exclusive sets–the related set of examples and the set of non-examples. However, one's concept images might cause some errors in this respect, as examples or non-examples might not be regarded as such.

In this study, we offer some distinctions within the notion of concept image. We examine *mis-concept images*, separating *mis-in concept images*, that mistakenly grant non-examples of even numbers the status of examples, from *mis-out concept images*, that mistakenly deprive examples of their essence, to regard them as non-examples. That is, we distinguish between cases where students mistakenly infer to non-examples as examples; or mistakenly include non-examples in the set of examples (*mis-in* concept images), and cases where students mistakenly take examples out of the set of examples to address them as non-examples (*mis-out* concept images) (see also, Tirosh & Tsamir, 2022).

In the literature, *mis-in* concept images have been widely reported, without marking them as such. For instance, Zazkis (1998) depicts her students' tendencies to assign evenness to numbers in base 5, as if they were numbers with the critical attribute of base 10 integers (in their case, whole numbers); Tsamir and Ovodenko (2013) outline students' tendencies to mark inflection points "where the graph bends" (e.g., the graph keeps increasing, but in a different steepness), ignoring reference to "change of convexity-concavity"; and Vinner (1991) details students' tendencies to draw more than one tangent at certain points on a graph.

Mis-out concept images were also reported, untitled, in various cases, limiting the scope of examples by extending critical attribute restrictions. For example, adding an unnecessary critical attribute of $f'(a)=0$ to inflection points, thus limiting the concept to horizontal inflection points (e.g., Tsamir & Ovodenko, 2013); viewing different sized length and width as critical attributes of rectangles, thus rejecting squares from being rectangles (e.g., Fujita, 2012); and regarding a tangent line as "touching the curve at one single point", perhaps, due to past experiences with circles (e.g., Tall, 1986; Vinner, 1982).

1.2 Learners' conceptions of even numbers

In the literature, we identified three types of concept images related to even numbers: (a) even numbers are multiples of 2, (b) even numbers are numbers with an even unit digit, and (c) zero is not an even number.

1.2.1 Even numbers are multiples of 2

These concept images were identified among elementary school teachers and high school students. Zazkis (1998) reported on a tendency of 73 prospective elementary school teachers to base their argument for evenness on the existence of the factor 2 in the given expression. For instance, the evenness of 2^{100} and $1{,}234{,}567\times 2^{40}$, in which 2 is explicitly evident in the representation of the number, were obvious for most prospective teachers, and they mentioned the existence of the factor 2 in their justifications. However, while the factor 2 doubtlessly determined evenness, the lack of it did not necessarily determine oddness.

Moreover, an erroneous variation of the "factor 2" idea was found in the solutions of 15 prospective teachers, who confused an even exponent with an even factor. For example, 3^{100} was marked even, because "100 is even," and 3^{99}marked odd, because "both 3 and 99 are odd." Similarly, in a master's thesis, supervised by the first author of this paper, Dorani (2014) documented a tendency of 100 high school students to base their decisions regarding evenness on "divisibility by 2" or "being a multiple of 2" considerations. For example, all 100 participants used this argument to explain that 8 is an even number and 72 used it for 5430. However, students erroneously used this line of reasoning to conclude that also $\frac{2}{4}$ is even (29 students), 3^6 is even (18 students), and $2\sqrt{8}$ is even (23 students). Another interesting, yet erroneous adaptation of the "divisible" consideration, was expressed in Ball's study (1993), where a child, Sean, was documented claiming that six can be both even and odd because it is divisible both by 2 and by 3.

1.2.2 Even numbers are numbers with an even unit digit

Zazkis (1998) further found that preservice teachers tended to refer to the "last digit on the right" when explaining that 1234567 is odd. However, they were not aware that the "last digit" strategy does not hold, for instance, for base 5 representation of integers. Consequently, 34_5 and 12_3 were also regarded by some students as even. She concluded that her findings indicate a tendency to erroneously determine the parity of numbers according to their given presentations (see also, Zazkis & Gadowsky, 2001).

1.2.3 Zero is not an even number

Learners' difficulties to regard zero as an even number have been widely reported in the literature. The findings indicated a tendency to claim that "zero is just nothing" (e.g., Blake & Verhille, 1985; Catterall, 2006; Reys & Grouws, 1975; Seidelmann, 2004; Tsamir & Tirosh, 2003; Wheeler & Feghali, 1983). Furthermore, two master's theses supervised by the second author (Dolev, 1989; Gliksman, 2017), reported that students tend to argue that "zero is special, it is neither even nor odd" or "zero is neutral, not even, not odd."

Our paper has two foci: (a) analyzing university students' grasp of even numbers by Tall and Vinner's theory; and (b) using the findings as a base for examining the refinement of *mis-in* versus *mis-out* concept images to the theory. More specifically, the research questions are:

a. What are students' concept images of even numbers?
b. Can *mis-in* and *mis-out* concept images of even numbers be identified?

2 The study

The data derived from an academic, semester-long, methods course taught by the first author of this paper at Tel Aviv University. In the weekly, 1.5 hour face-to-face sessions that took place at the university, we discussed various mathematical concepts, using theories, and reported publications to analyze incorrect ideas. The topic *even numbers* was discussed during five meetings, usually as part of a lesson with some worksheets, group discussions, and whole class sessions.

2.1 Participants

The participating 38 prospective secondary mathematics teachers graduated from academic, mathematically rich programs (e.g., B.Sc or master's degrees in mathematics or computer science), which included elementary number theory, where definitions of "divisibility" explicitly addressed integers. In the Teaching Certificate Program, where this study took place, evenness, that is, even numbers and even functions, are some of the concepts that are commonly discussed.

2.2 Tools and procedure

The data arise from three sources: Written assignments, individual interviews, and lesson transcripts. Two written assignments about even numbers were submitted to the students in the first lesson that was dedicated to this topic, as a starter to class discussions: Assignment 1: "(a) write an example of an even number; (b) write another example of an even number; (c) write a third example" (see, Watson & Mason, 2005). Upon submission of Assignment 1, students were given Assignment 2, "Identify the even numbers":

Examine the following numbers, circle the even ones, and explain your choices.

How many even numbers did you find?

1. 470	5. $\frac{3}{5}$	9. $3e$	13. 5π
2. 1275	6. $2i$	10. 1942	14. 0.6
3. $4e$	7. 0 (zero)	11. 0.264	15. $\frac{2}{3}$
4. 8π	8. $7i$	12. 135768	16. 0.5

To examine learners' concept images of the construct even number, we presented in Assignment 2 a collection of examples and non-examples (e.g., Petty & Jansson, 1987). The examples included 1942, 470 and zero. We assumed that 1942 would be easily identified as an even number, since it obeys to the various criteria reported to be given by subjects for evenness; zero was commonly reported as challenging, and we wondered about students' concept images regarding the evenness of 470, since it has zero as a unit digit. The non-examples included 1275, that is an odd number, as well as some rational (e.g., 0.6), irrational (e.g., 8π), and complex (e.g., $2i$) numbers that have some appearance of "two times." We also included numbers that visually insinuated oddness (e.g., $7i$). Students' concept images are setting dependent (Tall & Vinner, 1981). Thus, we chose items to allow examining the sensitivity of their concept images to guises of "divisible by 2" (e.g., $2i$) or "an even unit digit", that they might view in "a last even digit on the right" (e.g., 0.6).

Ten students volunteered to be individually interviewed. The interviews were conducted by the two authors, following the session in which the worksheets were submitted, and prior to the class discussions. Each student was interviewed once, for 15–30 min. Interviews were semi structured. Each interviewee was first asked to explain one or more solutions to Assignment 2, then we posed additional, follow up questions. The choice of the follow up questions specifically related to what each interviewee had said beforehand. The interviews were audio recorded and transcribed. One researcher led the session in each interview, while the other wrote notes in real time (e.g., the student hesitates, giggles).

The lessons were audio recorded and transcribed, and after each lesson some notes "to keep in mind" were added (e.g., a certain student is puzzled, interrupts another student, speaks with certainty). Every week, the two authors met to review students' ideas and to adjust the original teaching plan, to ensure that reference to the participants' solutions be included in class discussions.

The analysis of the data was conducted by the two authors in two rounds: round 1—identifying students' concept images of even numbers—and round 2—identifying *mis-in* and *mis-out* concept images. Each round included two major steps: (a) individual work—each author, separately read and categorized the concept images that were identified in the materials (the assignments, lesson segments, and the interviews), (b) teamwork—meetings of both authors to examine the categorization of the data that was offered by each of us, and to negotiate (minor) differences. Then, we further met to decide on the segments that best illustrate the main ideas of our findings.

The discussion of the findings is presented in Sects. 3 and 4, by addressing the two research questions.

3 What are students' concept images of even numbers?

An examination of the examples that were provided by the students of even numbers yielded the following: All students provided correct examples. Their first examples were 2, 4, 6, or 8. The second examples were the following natural numbers: 6, 8 (8 students); 2-digit natural numbers (12 students), 3-digit (9 students), and 4- to 6-digit numbers, with 123,456 as the largest (9 students). The third examples were mostly natural numbers: 2-digit natural numbers (16 students), 3-digit numbers (9 students), and 4- to 6-digit numbers (11 students). Two students gave non-typical third examples, that is, zero and (-6). Had we stopped here, one might assume that the students had quite a good grasp of even numbers.

Students' solutions to the "Identify the even numbers and explain your choices" assignment varied. Twenty students answered correctly that there are four even numbers (470, 1942, 135768, and zero), and four students found three even numbers (470, 1942, and 135768). These 24 students seem to refer, at least in an implicit manner, to the demand of a specific set of numbers. For 20 of these 24 students, it was probably *integers* or *whole numbers*, yet four students erroneously excluded zero. It seems that for them evenness applies only to *natural numbers,* and indeed some more evidence for this concept image is reported further on.

In their explanations of why the chosen numbers were even, the students referred to having 2 as a factor, being a multiple of 2, being divisible by 2, or having an even last digit. These correct considerations were given by the students who provided the correct solution, yet also by 14 students who erroneously allotted the attribute of parity to rational numbers, to irrationals, and to complex numbers; or by those who deprived zero of its evenness.

In their written explanations, only three students added the attribute "integers," identifying the four even numbers (470, 1942, 135768, and zero); another student wrote "natural numbers" and identified 470, 1942, and 135768, excluding zero. Commonly, all 38 students sporadically wrote one or two explanations and then added that the explanations hold for the other choices as well. The class discussions regarding which of the numbers is even, led to the "big" question: "What is an even number?" Special attention to this question was paid in the second session that dealt with even numbers in the course, as described ahead.

Students' concept images were evident in their written answers, their contribution in class discussions, and during the interviews. In the following sections, we present the identified, erroneous concept images as they evolved in the three settings. It should be noted that all names mentioned in the quoted segments are pseudonyms, and that words that are written in capital letters stand for expressions that were voiced by the participant with emphasis.

3.1 An even number is something divisible by/a multiple of/two

In their written responses, 14 students included $4e$ and 8π as even numbers, and 11 of them marked $2i$ as well. They explained that these are "multiples of 2," "divisible by 2," "have 2 as a factor," or "are two times something." Two of them just wrote: "it's 2 times."

In the following lesson, the class was asked to write "what is an even number?" Half of the students (19) correctly wrote that it is "an integer divisible by 2" (some added with no remainder), or "an integer that is a multiple of 2," or "an integer that 2 is one of its factors." Others offered incorrect, personal concept definitions. Four students wrote that it is a natural number divisible by two, eight students wrote that it is "a number that is divisible by 2," or "a number that is a multiple of 2," with no reference to integers, and seven students just wrote "it is a multiple of 2," with no reference to any kind of numbers.

In his interview, Danny explained that $4e$ and 8π are even numbers:

Danny: These are multiples of 2.
Interviewer: Let's look at $2e$
Danny: Well, it is TWO TIMES e [pause], and the same goes for 8π.
Interviewer: The same??
Danny: It's TWO TIMES 4π.
Interviewer: So, a number is even when…
Danny: It's a multiple of 2.

Gad explained why $2i$ is an even number.

Gad: I SEE that it ($2i$) is divisible by 2.
Interviewer: What do you mean by see?
Gad: It's written TWO TIMES.
Interviewer: What does "divisible" mean to you?
Gad: [hesitating] what do you mean by what does it mean? We have here TWO TIMES. It's divisible by two. It's even.

This line of reasoning was also presented to explain that $\frac{2}{3}$ and 0.264 are even numbers. For example, Gali explained in her interview:

Interviewer: Gali, you marked that $\frac{2}{3}$ (two thirds) and 0.264 (zero point two hundred sixty-four) are even numbers, why?
Gali: $\frac{2}{3}$ is two times a third and 0.264 Ehhh… EVERYTHING is even, the 2, 6 AND the 4, you can also say that it is two times 0.132.
Interviewer: So, $\frac{5}{7}$ (five sevenths) would be…
Gali: Odd
Interviewer: and 0.135 (zero point a hundred thirty-five)
Gali: Also… Odd I mean.

Abby opened by offering her strategy for determining evenness:

Abby:	I simply checked if it can be written as two times something, then it's even. [goes over her solutions] *4e* is 2 times *2e*, *8π* is 2 times *4π*. Would you like me to continue? [goes on] $\frac{2}{3}$ is 2 times a third…
Interviewer:	And 1972?
Abby:	I can show that it's 2 times something [pause]. It is. But no need to bother, because the last digit is even.
Interviewer:	What about 2i?
Abby:	It's 2 times *i*.
Interviewer:	…and what would you say about *2x*?
Abby:	Also. Even, I mean.

In our study, the tendency to examine evenness by the existence of an even factor, that is, being divisible by or a multiple of 2, seems to be overgeneralized to the cases of *2i*, $\frac{2}{3}$, 0.6, *4e*, *8π*, and even *2x* (see also, Dorani, 2014; Zazkis, 1998).

We would like to note that in some cases during class discussions students came to re-examine their erroneous conceptions. In one instance, two students, Steffi, who defined even numbers as integers that are divisible by 2, and Sam who viewed 0.6 as even, were asked to discuss their opinions and try to convince each other, in front of the class:

Steffi:	I'll go first. I'll start where we agree, OK? Is 1942 even?
Sam:	Sure.
Steffi:	and 1275…
Sam:	Odd.
Steffi:	What about 0.6 (zero point six) …
Sam:	Even.
Steffi:	and $\frac{3}{5}$?
Sam:	[hesitates] No. It's odd…
Steffi:	BUT… but 0.6 IS $\frac{3}{5}$ (three fifths).
Sam:	Oh [pause] It is. It's…

Another student, Shuli, who knew that even–odd attributes apply to integers, continued this line of reasoning:

Shuli:	I can show that ALL decimals are even.
Sam:	What do you mean?
Shuli	0.7 (zero point seven), if you say that [pause] it's odd, then I ask about 0.70 (zero point seventy). I can always add zeros without changing the decimal… So, it's ALWAYS even. Perhaps that's why even and odd are valid only for integers.

At this point, Sapir, who erroneously believed that *2i* is even and *3i* is odd, asked:

Sapir:	But with complex numbers we saw that it's OK [pause] Right?
Shuli:	What do you mean by OK? What about *0.7i*?
Sapir:	Gosh…

Shalom: Wow, I knew that fractions can't be even or odd, but I did mark *2i* as even… and *2e*, and *8π*.

Up until this instance, Shalom lived in peace with his mistaken view of even numbers. Shalom had a certain recollection of evenness being irrelevant to rational numbers, but this knowledge was compartmentalized, to allow him to attribute real irrationals, and complex numbers with this property. Sapir, on the other hand, did share her confusion about the complex numbers during an earlier part of the lesson. However, she was somehow reassured that "with complex numbers it's OK." So, she might have stayed in an awkward state of mind if this discussion would not have happened.

3.2 Last digit on the right determines parity

The "last digit on the right" test for examining whether an integer is even is a valid method. When addressing the question: "What is an even number?" no student suggested that an even number is an integer having 2, 4, 6, 8, or zero as a unit digit. Still, during the interviews, this line of reasoning served as a prevalent explanation for the evenness of 135,768, 470, and, erroneously, for 0.6, and 0.264 (see also Zazkis, 1998). Abby clearly expressed her view regarding the efficiency of the last digit criterion, while she generally followed (in an invalid manner) her personal, multiple of 2, definition.

In the following subsections, we first address the tendency to view natural numbers with an even unit digit as even numbers. While providing a seemingly correct answer, students occasionally explained that 470, 1942, or 135768 are even numbers, using phrasings that may imply erroneous concept images. In the second subsection, we address the incorrect tendency to view *any number* with an even last digit on the right as an even number.

3.2.1 An even number is a natural number with an even unit digit

All the students marked 470, 1942, and 135768 as even numbers. Among their explanations were "the unit digit is even" or "a number with an even unit digit." That is, pointing to the unit digit, with no specification of the type of the number. One student, Sarina, wrote: "it's a natural number with an even unit digit." She also defined later: "an even number is a natural number divisible by two." As it will be evident later in her explanations, Sarina meant that even numbers are natural numbers. In the lesson, when addressing the task "Identify the even numbers," Sarina elaborated on her solutions. In the following lesson segment, students other than Sarina are marked as S1, S2 etc.

Sarina: I only had to examine 470, 1942, 1275 and 135768. Natural numbers. 1942 and 135768 are even, they end with even digits. 1275 is odd because the unit digit is odd.
S1: Must it be a natural number?
Sarina: YES
S2: What about 470?
Sarina: It's a natural number, divisible by two.
S2: The unit digit is zero…
Sarina When the unit digit is zero, it's even too. [pause] You can check.

Sarina is considering a set of numbers that can be even numbers. Unfortunately, the set that she mentions is partial.

Bar, who marked 470, 1942, and 135768 as even numbers, also argued, during the interview, for the requirements of natural numbers with an even unit digit.

Interviewer: How did you decide which number is even?
Bar: [looks at her worksheet] 1942 is a natural number and it has 2 as a unit digit. So, it is even. [pause]. 135768 has 8 [points to it]. So, it is even. It must be a natural number, and the last… the units, it's even, yes…
Interviewer: 470?
Bar: Even.

Sarina and Bar followed their personal definitions. Bar's ongoing mathematical experiences lay foundation to her concept image that even numbers are definitely natural numbers.

3.2.2 An even number is any number with an even number on the last, right digit

In their solutions to the task "Identify the even numbers," 10 students marked 0.6 and 0.264 as being even numbers. Sagit, presented the class with her explanation:

Sagit: Well, you can SEE that actually all the digits of 0.264 are even [pause]… most important is the LAST digit on the right. It's even.

Another student, Ben, commented on that:

Ben: I used different ways. 470, 1942, 0.264, 135768 and 0.6 end with an even digit [pause] and $4e$, 8π and $2i$ are multiples of 2.

These findings indicate an overgeneralization of the attributes of divisibility and evenness to any number and the "last digit consideration" to decimals (e.g., Zazkis, 1998; Zazkis & Gadowsky, 2001).

3.3 Zero is special

In their responses to the task: "Identify the even numbers," eight students did not mark zero as an even number. In class, when addressing the question: "is zero an even number"? the students mainly communicated two erroneous concept images regarding zero and evenness: (a) zero is neither even nor odd, and (b) evenness is irrelevant for zero.

3.3.1 Zero is neutral, neither even nor odd

In the class discussion about: "is zero an even number?" Ami, a student who wrote that 470, $4e$, 8π, $2i$,1942, 0.264, 135768, 0.6, and $\frac{2}{3}$ are even numbers, but zero is not, explained:

Ami: Zero is not even [pause] BUT it is not odd as well. It is SPECIAL.
T: Special?

Ami: Zero is like a test case, [pause] for extremum points, for inflection points, undefined in division. SPECIAL.

T: What about evenness?

Ami: When it comes to zero, you must always think differently… It's neutral. Special. Not negative not positive [pause], Minus zero is zero. Is it then negative too? Zero is different. not even and not odd. Neutral.

Similar concept images were reported in Dolev (1989) and Gliksman (2017). Here, Ami addresses zero's special nature, overgeneralizing from cases where zero plays an exceptional role, like the investigation of functions.

3.3.2 Evenness is irrelevant to zero

Indications for this concept image were expressed by Sarina and Bar who claimed that even numbers are necessarily natural numbers. Thus, when referring to zero, they said:

Sarina: It is not a natural number [pause] and neither are the other numbers... NO. It's like what we had with the complex numbers… I believed that *5i* is larger than *3i*, but we learnt that being "larger than" is not applicable to complex numbers [pause]. I remember well the example that clarified it for me. Remem ber?

Some voices…

We had to look at photos and to point to photos with a brown dog. The first photo had a black dog. We all agreed that it was not a brown dog. Then, the photo had a yellow dog. Again, we knew that it was not a brown dog. THEN, the photo had a CAT. We giggled. But since then, I carefully pay attention to BOTH being a dog and being brown. I mean, here, being a natural number AND being divisible by two. So, the other numbers, they're the cat. They are NOT NATURAL numbers. Evenness is irrelevant to them.

Bar: No. It's not a natural number. Even or odd are irrelevant here. Since elementary school, whenever we talked about even or odd numbers it was natural numbers. Divisible by two.

Several students who claimed that zero is not even, occasionally added that the attribute of evenness is irrelevant to zero. Here are two excerpts taken from different, non-sequential segments of a lesson. The students are marked as S5, S6:

S5: No. No. In my opinion, zero is not even. No. [pause]. Being even or odd isn't applicable to zero…

S6: [in reaction to a student's explanation that zero is even, because it's divisible by 2] I disagree. You can't talk about zero being divisible or being even. It's just not right. [pause]. You could also say that zero is negative, but being positive or negative, even, or odd, are irrelevant to zero. It's just nothing. Can nothing be even? [giggles] it can then be red as well…

Our attention in this study was paid to concept images of even numbers with special attention to their being integers. Data regarding students' concept images of non-even, and particularly, odd numbers is of no less interest. Our initial observations regarding evenness may provide fruitful seeds for further studies.

4 Can *mis-in* and *mis-out* concept images of even numbers be identified?

A brief answer to this question would be, yes. Here, like in a previously reported study about learners' grasp of parallelograms (Tirosh & Tsamir, 2022), we identified the two types of concept images: *Mis-in* concept images, where learners overstretched the boundaries of the example-space for even numbers, to allow non-examples to bear the title of examples; and *mis-out* concept images, where some examples were deprived of their exemplary status. In our case, allotting rational, irrational, and complex numbers the attribute of evenness are pointers to *mis-in* concept images. On the other hand, the persistent grasp of zero as a non-even number, and the restriction of evenness to natural numbers are clear indications of *mis-out* concept images. We will elaborate and go into specifics for each of the two cases.

4.1 *Mis-in* concept images of even numbers

It seems that many learners preferred one critical attribute (e.g., divisibility by 2) over the other (being an integer). These *mis-in* concept images of evenness revealed a connected net of *mis-in* concept images regarding related constructs, such as divisibility and factorization. For the latter, the applicability to a restricted set of numbers was neglected. It should be noted that in Hebrew, the term even for "an even number" (mispar zugi) is the same word that is used for pairs in daily context (to work in pairs, a pair of shoes–laavod bezugot, zug naalaim). In everyday language, ever since kindergarten (and even earlier), evenness is mentioned with reference to all kinds of countable "things" that go in "twos." Intuitively, this might have set the ground for "*something* multiplied by two" (e.g., Tall & Vinner, 1981).

Moreover, "like much informal talk, spontaneous discourse about mathematics is full of half-finished and vague utterances, such as 'it,' 'this,' and 'something'" (Pimm, 1987, p. 22, see also, Vinner, 1997, 2018). Pimm describes how a kindergarten teacher explained "evenness" in terms of "being able to share *it* into two groups" (ibid., pp. 90). Rather than referring to the number of "things," the sharing refers to the "things" (it) and may yield "something that can be divided by 2," "something that is 2 times," or just "two times" concept images for evenness.

All in all, our findings corroborate *mis-in* concept images that were reported, untitled in the literature (e.g., Tsamir & Ovodenko, 2013; Vinner, 1991; Zazkis, 1998).

4.2 *Mis-out* concept images of even numbers

The second phenomenon, that of *mis-out* concept images, may also be rooted in learners' daily and mathematical past experiences. Many students who claimed that zero is not even, based this view on repeated engagements with "zero is a special number" practices. They mentioned some correct, yet inappropriate instances of zero being an unusual number (e.g., neither negative nor positive). Then, with a pseudo-connecting sway, they concluded that zero is neither even nor odd, as well. Another well-known, erroneous grasp of zero as “nothing” that was voiced, may also evolve from daily or early-years mathematics encounters. None of these students made any attempt to examine the applicability of *multiple of 2*, or a similar, critical attribute that they themselves mentioned in their definitions. This might signal an instance of compartmentalization (e.g., Vinner, 1990); or of the coercive power of ones’ intuitive grasp of zero (e.g., Fischbein, 1987). Some students only omitted zero from the set of examples; others excluded zero, but, also, invalidly identified irrational and complex numbers as even numbers. The latter had both *mis-in* and *mis-out* concept images.

The second occurrence of *mis-out* concept images was expressed by learners, who declared evenness to be applicable just to natural numbers, impressions galvanized by years of engagement with the notion in different settings. Indeed, in everyday language, as well as in the mathematical instances, even numbers are commonly mentioned with reference to natural numbers. Thus, it may implicitly cause an inappropriate, seemingly self-evident jump to conclude that even and odd are relevant *only* to natural numbers (Fischbein, 1987; Tall & Vinner, 1981).

In our study, when asked to write even numbers, almost all the given examples were correct, natural numbers. This might indicate that within the set of examples, these are prototypical examples, intuitively accepted as representatives of evenness. However, prototypical examples both foster and hinder the formation of concepts. While being easily recognizable and paving the way to the initial formation of concepts, prototypes may formulate limited concept images. Students even tend to regard *only* prototypical examples as examples of the concept (e.g., Hershkowitz, 1989; Wilson, 1990). This might be the case also for those who generalized the limitation of even numbers to natural numbers.

Again, our findings endorse *mis-out* concept images that were reported uncategorized in the literature (e.g., Fujita, 2012; Tall, 1986; Tsamir & Ovodenko, 2013; Vinner, 1982).

5 What are the merits, if any, of differentiating between *mis-in* and *mis-out* concept images?

A question that comes to mind is: Is the mathematics education domain going to gain from a differentiation between *mis-in* and *mis-out* concept images? Is it not sufficient to regard all those as erroneous concept images? After all, our field has been criticized, from way back, time and again for its exploding bulk of terminology. Several terms address *almost* the same entities or phenomena by means of different, yet needless terms (E. Fischbein, personal communication, January 6, 1996). Our impression is that this is not the case. We claim that the terms *mis-in* and *mis-out* concept images offer a distinction that may be beneficial to teachers, to researchers and eventually, and most significantly, to students.

5.1 Implications for mathematics teachers

A major interest of mathematics teachers would be to find out whether information regarding students’ *mis-in* and *mis-out* concept images can be helpful in (a) identifying learners’ difficulties, related to a topic at hand, during teaching–learning communications, and (b) promoting their knowledge. We believe that teachers, who are the ones who have to “respond to situations as they emerge,” may gain valuable proficiency by being knowledgeable; knowledgeable, in the sense of being scholarly, well-informed both mathematically and pedagogically. The more teachers know about the kind of difficulties students encounter, the better they can notice, and perhaps also respond to students' needs in the moment.

> To develop your professional practice means to increase the range and to decrease the grain size of relevant things you notice, all in order to make informed choices as to how to act in the moment, how to respond to situations as they emerge. (Mason, 2002, p. xii)

That is, distinguishing between *mis-in* and *mis-out* concept images, and possessing the wording to address these two types of common difficulties, might be helpful in noticing.

For example, familiarity with the two types of *mis-conceptions* (*mis-in* or *mis-out*) might be productive in the design of instruction. Teaching that aims to address the two phenomena may call for distinctive instructional approaches. The *mis-in* concept images contradict certain aspects of the construct (as determined by its definition). This may play a role in delineating instruction because it allows inducing, and then, being guided to resolve cognitive conflict. For example, Steffi's emphasis that 0.6 equals $\frac{3}{5}$, while Sam believed that 0.6 is even but $\frac{3}{5}$ is odd, evoked the conflict factors simultaneously, and Sam's conflict is evident.

Nevertheless, it is well-documented that there is no guarantee that the student will resolve the conflict in an adequate manner. The teacher's role is still crucial. Tall and Vinner (1981) referred to this issue:

> in certain circumstances cognitive conflict factors may be evoked sub-consciously with the conflict only manifesting itself by a vague sense of unease… it may be a considerable time later (if at all) that the reason for the conflict is consciously understood (p. 154).

However, the cases of *mis-out* concept images are different from an instructional point of view. All pieces of information that the learner has in mind, are consistent. The problem is that they are founded in a limited, insufficient realm. Thus, in the cases of *mis-out* concept images, consulting the definition is a must. There is no other option for convincing of the need to add the missing parts of the example space, or alternatively, to omit the redundant conditions.

> It is obligatory to remember that there are some contexts in which referring to the formal definition is critical for a correct performance on given tasks. (Vinner, 1991, p. 80)

Vinner (1991) argues that it is a common phenomenon, that only concept images are used in problem solving. Moreover, sometimes even undergraduate mathematics students do not use definitions, which they themselves correctly state and explain (e.g., Vinner & Dreyfus, 1989). In a conflict between the definition and the concept image, the latter tend to prevail (Edwards & Ward, 2004). However, in mathematics contexts, to ensure correctness, one should consult the definition (see, for instance, Vinner & Dreyfus, 1989; Rasslan & Tall, 2002). Vinner and Dreyfus (1989) advise to form an introductory, adequate concept image, by starting with various examples and non-examples. The formal definition should be only a conclusion of the various examples that were initially introduced. The choices of examples and non-examples are essential to convey the essence of a construct as well as its boundaries (e.g., Bills, et al., 2006; Watson & Mason, 2005).

Commonly, building well-planed instructional tools seems to be too big a task to be left solely for the teachers; especially if novice teachers are also expected to be knowledgeable and noticing. Indeed, mathematics education researchers and developers of educational materials are recruited to contribute their share in this significant undertaking.

5.2 Implications for mathematics education researchers

The research community aims to provide teachers with well-founded answers to challenging questions, such as What kind of (sequences of) tasks might have the diagnostic potential of revealing learners' understanding and fallacies? What kind of (sequences of) tasks might have the instructional potential of promoting learners' knowledge?

A possible guideline for identifying learners' ways of reasoning, is to design discriminating items that address each and every critical attribute of a construct. In the case of even numbers, being an integer, divisibility by 2. This decomposition is more demanding than it may appear at first glimpse. To examine learners' consideration of being an integer, rational (e.g., 0.6), irrational (e.g., 2π), and complex (e.g., $2i$) numbers, or numbers in another base (e.g., 2_5), that have some appearance of "two times" should be presented. To examine learners' consideration of the attribute "divisible by 2," other tasks might be in place (e.g., is 7^{100} divisible by 2?).

This is a necessary, yet insufficient starting point for designing research tools and instructional items, because difficulties regarding a certain attribute may vary (e.g., Nesher, 1987; Nesher & Peled, 1986). For example, all the above cases examine whether *mis-in* concept images are produced. However, in other cases, some critical concept images are *mis-out*. Awareness of these two types of difficulties, guide us in probing further into the critical attributes.

> It often helps to have names for complex ideas as Auden and others have noted: if we haven't labeled something, it escapes our notice. As an issue emerges, or a type of situation develops in which you would like to respond differently, then choosing a label for its playfulness to trigger awareness can be very helpful. (Mason, 2002, p. 79)

In our case, in order to investigate the possibilities of a *mis-out* concept image, we should first re-examine the constructs that underlie the critical attributes (e.g., integers, divisible, multiple, factor). "Integers" comprise of positive numbers, zero, and negative numbers. When designing this study, we somehow failed to consider the option of negative integers, so we included no such items in assignment 2. However, we do see indications for students' *mis-out* conceptions regarding the negative numbers. Surely, to examine learners' consideration of the full compass of this attribute, additional diagnostic tasks are in place (e.g., is (-2) even?).

In our study, we identified both *mis-in* and *mis-out* concept images. However, building a comprehensive body of knowledge regarding learners' conceptions is a gradual, spiral process. We revealed some hints regarding the concept images of even numbers that question the understanding of the attributes: *numbers* and *variables*. The participants here had a rich mathematical background, thus we assumed that those who claimed that $2i$, or $6e$ are even, meant that these are even *numbers*. However, following Abby's claim that $2x$ is also even (where x is certainly a variable), raises some doubts regarding our initial assumption. This is another line of research that might be interesting to follow. Furthermore, there were also some indications of students' *mis-in* conception of odd numbers (e.g., 0.135). Since the latter is well connected to evenness, researching it is relevant.

In sum, considering the types of *mis-in* and *mis-out* concept images may serve as an organizer in the search for, and in the formulation of discriminating items; it may also serve in highlighting and emphasizing specifics of the various critical attributes of the construct.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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