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Mathematical strategies and emergence of socially mediated metacognition within a multi-touch Dynamic Geometry Environment

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Abstract
Our research aimed to investigate the potential learning benefits to young children of implementing digital interactive multimodal technologies that provide both visual and haptic experiences in elementary mathematics classrooms. We studied the ways in which fourth-grade students collaboratively create collective strategies for solving mathematical problems utilizing dynamic geometry software with multi-touch interfaces, a combination we call a *multi-touch Dynamic Geometry Environment*. We examine in-depth two case studies each illustrating how mathematical strategies, collaboration, and socially mediated metacognition emerge in the small groups of children while working on an activity using the Geometer's Sketchpad® on the iPad to make sense of an intuitive idea of covariation. We found that children's interactions with their peers, the interviewer, and the mDGE favored the emergence of varied collaborative behaviors and socially mediated metacognitive processes that fostered the co-construction and development of mathematical strategies over a short period of time.

Keywords Mathematical strategies · Collaboration · Socially mediated metacognition · Dynamic geometry software · Multi-touch · Young children

1 Introduction

Digital multimodal technologies combine software tools and hardware platforms that allow users to interact through two or more communication channels or input and/or output modalities. Multimodal human-computer interaction has evolved and spread out in various research areas with increasing use in education (Jamies & Sebe, 2007). Our study is situated within the confluence of mathematics education and developmental psychology

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and aims to examine the role of digital multimodal technologies such as dynamic geometry environments (DGEs) on the iPad in mathematical collaborative learning.

There already exists a deal of research on the benefits of implementing DGEs in secondary mathematics (e.g., Falcade et al., 2007; Mariotti, 2009). Research also has shown how utilizing DGEs such as the *Geometer's Sketchpad*® (henceforth Sketchpad) influences mathematical learning in elementary grades. Sketchpad experiences offer representational affordances to allow young children to investigate complex mathematical ideas in short periods of time (Ng & Sinclair, 2015; Sinclair et al., 2013). Moreover, research has suggested that multi-touch interfaces can allow students to work together in small groups to solve mathematical problems while fostering peer collaboration, direct physical interaction, higher gestural expressivity, and motivation (Dillenbourg & Evans, 2011). Multimodal experiences combining visual and haptic cues have been highly valued in mathematical learning (Arzarello et al., 2014; Hegedus, 2013; Sinclair & Heyd-Metzuyanim, 2014). However, little is known about how utilizing DGEs on multi-touch interfaces allow elementary-school children to have access to challenging mathematics.

We believe that digital multimodal technologies integrating visual dynamic output and multi-touch input have the potential to impact young children's mathematical learning under specific design conditions that enhance the representational and communicational affordances of both technologies. Our research implemented the use of Sketchpad on the iPad in elementary classrooms, which we call a multi-touch Dynamic Geometry Environment (henceforth mDGE), to examine how young children can explore different mathematical concepts. We analyzed the learning benefits of utilizing an integrated activity-software-hardware approach through which sensations in multiple sensory modalities, such as seeing and touching, offered to young children the opportunity to intuitively grasp an embodied mathematical model of these concepts, including *covariation*.

Furthermore, mathematics education researchers have shown how collaboration in small groups can enhance team-oriented problem solving and creation of collective strategies (Goos et al., 2002; Kim et al., 2013). In turn, they have shown how collaborative activities can promote what they label *socially mediated metacognition* (Goos et al., 2002), or *social metacognition* (Kim et al., 2013). However, little is known about how collaboration and metacognition from a *social practice* theoretical perspective can emerge in mDGE-supported problem-solving situations and can facilitate the co-construction and evolution of mathematical strategies. Although there exists research on high-school students' understanding of covariation through DGEs (e.g. Falcade et al., 2007), exploiting the mDGEs' potentialities to facilitate young children's collaborative gesturing which in turn fosters group coordination and collective embodied meanings of covariation remains unexplored.

We believe that organizing small teams around one iPad could promote collaboration and metacognitive processes as the social interaction and the technological affordances heighten the opportunity for enhancing children's monitoring of their strategies to solve the dynamic activities and regulation of their learning in the mathematics classroom.

The goal of this paper is to examine how young children working in small groups co-construct body-grounded mathematical strategies in making sense of a multi-input activity about the intuitive idea of covariation, and how these strategies evolve over short periods of time through the emergence of collaboration and socially mediated metacognition within the mDGE. We address two questions: How do the young children in the small groups strategize together in making sense of the underlying mathematics of the EtchASketch™ activity while utilizing the mDGE? How do collaboration and socially mediated metacognition emerge within the technological learning environment influencing the co-construction and improvement of children's strategies during the activity?

2 Theoretical Background

Our study draws on sociocultural theories of semiotic mediation. Mariotti (2009) defines *mediation* as "the potentiality that a specific artifact has with respect to fostering the education process" (p. 428) and *semiotic mediation* as a knowledge-construction process that is "consequence of instrumented activity where signs emerge and evolve within social interaction" (p. 428). Accordingly, we consider mDGEs as cultural artifacts that mediate young children's mathematical thinking and problem-solving (Hegedus & Moreno-Armella, 2011; Ng & Sinclair, 2015), and have the *semiotic potential* of developing both personal and mathematical meanings (Bartolini-Bussi & Mariotti, 2008).

Sketchpad offers point-and-click Euclidean tools to construct geometric objects and mathematical configurations. Children can click on hotspots and drag the objects to manipulate and continuously transform them but the configurations always preserve their underlying mathematical principles. We see pointing, clicking, and dragging parts of the geometric constructions as embodied actions that offer a continuously dynamic and enactive perspective on geometry, enabling a semiotic mediation between the visual-dynamic representations of the objects and the child who is developing the *mathematical invariance* (Hegedus, 2013; Sinclair et al., 2013). The DGE's dragging function mediates children's mathematical reasoning and creation of new meanings about the geometrical configurations (Mariotti, 2009).

The iPad facilitates new multimodal interactive experiences that mediate mathematical learning (Hegedus, 2013; Sinclair & Heyd-Metzuyanim, 2014). Multi-touch tablets are horizontal surfaces utilized as input and as output to/from a digital environment, where input can be provided by users' fingers pointing directly on different locations of the screen (Dillenbourg & Evans, 2011). Such characteristics offer *co-location*, as various users can be located around the device interacting face to face, and *multi-input*, as the interfaces support synchronous detection of multiple inputs (Dillenbourg & Evans, 2011). As the iPad allows for several simultaneous inputs, children can produce multiple finger gestures simultaneously over the platform favoring coordinated enactive representations and embodied meanings of the geometrical objects (Abrahamson & Bakker, 2016; Hegedus, 2013).

Mathematical strategies are students' methods to solve problems with mathematical content (Yerushalmy, 2000). A *strategy* constitutes an intellectual approach to achieve a task goal, which can emerge while children explore the problem, so that they harness their thinking resources at hand, their own prior knowledge, and the meanings that emerge through task exploration to create solutions, whether correct or not. Within a multi-touch activity, mathematical strategies can be grounded in the body (Abrahamson & Bakker, 2016). As small groups solve the same problem around the iPad, it also mediates team collaboration, awareness and negotiation of more effective strategies while thinking mathematically (Hegedus, 2013; Mercier & Higgins, 2013). Moreover, multiple strategies can emerge, change, and evolve over short time periods (Siegler, 2007). Therefore, strategies represent varied and evolving ways of thinking revealing the children's capacity to flexibly exploit cognitive, social, and environmental means to solve the problems.

A semiotic mediation perspective on strategies allows educators to understand how children explore and exploit psychological, social, and environmental means within the mathematics classroom (Goos et al., 2002; Kim et al., 2013). Small-group work with the mDGE facilitates children's harnessing of social resources including their peers' prior knowledge, experiences, and ways of thinking, which can be shared through peer interaction. The teachers' feedback is another relevant social means. Children also exploit environmental means such as the haptic and visual dynamic feedback from the task when they and their peers interact with the technology. Therefore, such a supportive classroom could promote social and technological interactions that potentially mediate as a whole the co-creation and improvement of collective strategies to solve problems (Goos et al., 2002).

Collaboration in small-group problem-solving is critical for learning mathematics (Forman, 1989; Kim et al., 2013). Wells (2000) defines *collaboration* as the process involved in joint activity where the participants work as a learning community toward shared goals so that understanding is achieved as the different participants contribute to the group and harness others' contributions. Thus, small-group collaboration mediates the social construction of signs. Children may benefit from sharing their ideas with their peers, mainly when they have a different perspective of a problem (Goos et al., 2002; Wells, 2000). Building on Wells (2000), we call this process of developing collective strategies within the mDGE the *co-creation of mathematical strategies*.

Based on Goos et al. (2002) and Kim et al. (2013), we focus on the notion of *socially mediated metacognition*. These authors endorse the psychological model of metacognitive functions as *monitoring* and *regulation* of the thinking processes, and include the process of *awareness* and *assessment* within the monitoring function but conceptualize metacognition as a social practice. From a sociocultural perspective, metacognition emerges in small-group work as mediated by three sources: the individual's psychological resources, social means through collaborative interaction with peers and teachers, and environmental means through the interaction with tasks and technologies (Goos et al., 2002; Kim et al., 2013).

3 Methods

3.1 Context and participants

Our research project was implemented in seven fourth-grade classrooms of a suburban elementary school of Massachusetts, USA. Participants were 9/10-year-old students, and came from varied cultural backgrounds. The data we present here come from six of these students, including girls and boys. The children's parents provided written informed consent and the children provided written informed assent.

3.2 Research design

We utilized a qualitative research approach (Creswell, 2007) that entailed the design and implementation of nine activities within a mDGE in regular mathematics classrooms, facilitating visual-dynamic and touching experiences. The mathematical activities were designed in Sketchpad and presented to students through the app *SketchPad®Explorer* on the iPad. These activities aimed to allow young children to access complex topics that they had not been taught before, such as symmetry, geometric transformations, covariation, and area measurement. This paper focuses on an activity about covariation, a central notion that early learners could begin to develop prior to formal instruction on functional dependence and that is fundamental for learning algebra in the future.

Classroom lessons Implementation of each activity involved a whole-class presentation of the task, 20-minute small-group work, and a whole-class presentation of the groups' strategies, in the context of the regular mathematics classrooms. There were approximately five to seven groups of three or four children within each classroom. Each small group worked around one iPad to solve the activities. These groups were the same students used to form their regular classes, and teachers would rearrange them each class; groups were not combined based on mathematical ability or prior use of iPads (rarely used in these classrooms). This paper presents data from two groups of three students, and each group was part of a different classroom.

Role of research team The principal investigators designed the activities in collaboration with teachers to ensure that the learning objectives and questioning strategies were legitimate in their classrooms. The lead researcher assisted the cooperating teacher during the implementation. The research team was focused on interviewing and recording the small groups of children. As the activities required group discussion, the interviewers were asked to encourage think-aloud whenever possible for us to collect evidence of how the children were thinking about each other's thought processes.

Multiple-case study design We constructed selected case studies utilizing a multiple-case study design (Creswell, 2007; Yin, 2009). This paper reports on findings from two case studies involving two small groups of three children each solving the EtchASketch activity designed to introduce the covariation concept in the mathematics classroom. For each case, we analyzed the participants' conversations and interactions (Forman, 1989; Wells, 2000) with the technology, their peers, and the interviewer while solving the activity within the mDGE in the small group, focused on how they co-created collective strategies and how these strategies changed mediated by collaboration and the technological affordances. We selected these two groups for the case studies from among the others because these students greatly interacted within the small groups. Moreover, both groups displayed contrasting approaches to solve the problem, allowing us to illustrate commonalities and differences across the cases. These approaches were representative of the variability among the groups.

3.3 EtchASketch activity about covariation

The goal of the EtchASketch activity for students is to trace with their fingers a colored circle around a fixed black circumference (Fig. 1). To solve the task, one student can drag and control the horizontal-moving *Point 1* located at the bottom of the screen while another student (or second finger input) can drag and control the vertical-moving *Point 2* located at the right of the screen. A third student (or third finger input) can adjust the blob's color to make a rainbow around the circle by dragging the point on the spectrum, or can change the blob's size by dragging the point *H*.

We claim that implementing this task in small groups of students within the mDGE has *semiotic potential* for an initial access to the covariation concept (Bartolini-Bussi & Mariotti, 2008). Mathematically, the *Point 1* has been constrained to move along an invisible horizontal line segment and the *Point 2* along an invisible vertical line segment orthogonal to the horizontal one. To construct the circle, we have *parameterized* two input functions using the software's tools and hidden the perpendicular lines. The only visual-dynamic output is one colorful blob, which moves in the directions of the two inputs stemmed from point 1 and point 2's movements, leaving one continuous colorful trace on the screen. Using the sketchpad on a touchscreen allows the group of children to provide multiple inputs with their fingers simultaneously on the horizontal

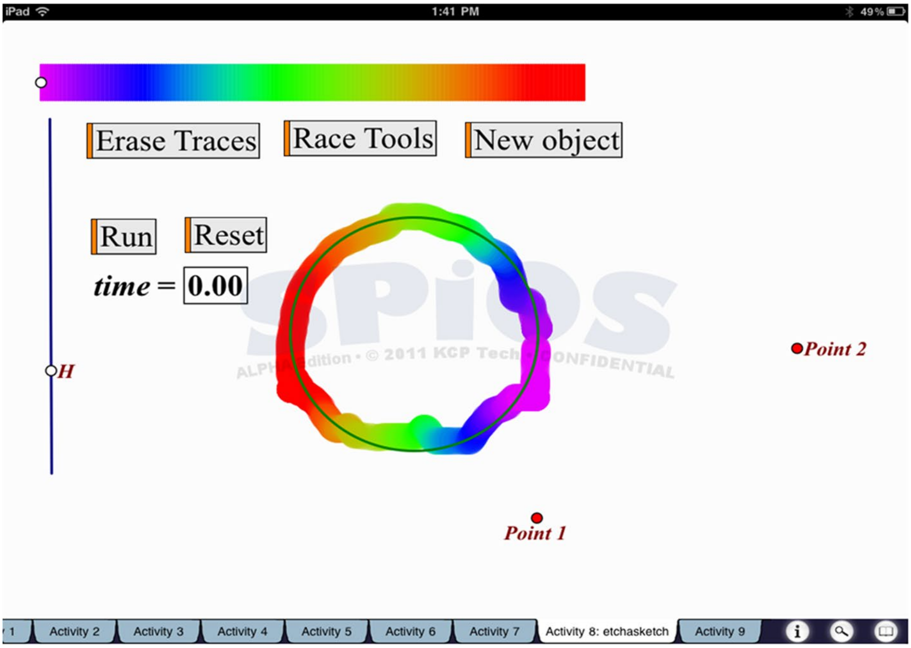

Fig. 1 EtchASketch activity about covariation

screen (Arzarello et al., 2014) to manipulate the two sliders. The children have to find out the underlying mathematical principle of the activity and coordinate their dragging actions with their partners to solve it. In this sense, the dragging actions over the direct visualizations emerge as collaborative interaction.

Regarding the cognitive potentialities, the EtchASketch activity within the mDGE allows young children to grasp a notion of *covariation* from a dynamic bodily-grounded perspective, as they can intuitively make sense of the covariation principle that using two inputs in mutually dependent ways, or dependent actions with their fingers, can produce only one output. Covariation is a rich mathematical idea, closely related to the concept of function and the mathematics of change in later grades. Confrey and Smith (1995) propose a correspondence approach to *covariation*, implying coordination of changes in two variables. They advocate an operational, action-guided covariation approach as an entry point to functional thinking in elementary school. For Thompson and Carlson (2017), covariational reasoning implies envisioning changes in two quantities' or variables' values as happening simultaneously, smoothly, and continuously. They emphasize the quantitative nature of covariation rather than just numerical, and highlight a recursive, dynamic, and continuous view of the concept. Thompson and Carlson (2017) propose that covariational reasoning can evolve through six potential developmental levels: *no coordination*—image of one or another variable's variation, without coordination of values; *precoordination of values*—image of two variables' values varying, without synchronous coordination; *gross coordination of values*—image of quantities' values varying together, with no links between the individual values of quantities; *coordination of values*—image of coordination of one variable (x)'s values with another variable (y)'s values with an emerging discrete collection of pairs (x, y); *chunky continuous covariation*—image of changes in one variable's value in simultaneous coordination with changes in another variable's value, and both variables varying with chunky continuous variation; and *smooth continuous covariation*—image of changes in one variable's value in simultaneous coordination with changes in another variable's value, and both variables varying smoothly and continuously (p. 441). We envision this developmental path as part of a hypothetical learning trajectory and conjecture that our EtchASketch activity is an instructional environment that could help young children to create an intuitive notion of covariation focused on no coordination of values, and advance through all or some of the more sophisticated developmental levels involving a bodily-grounded covariation concept as an increasingly smooth continuous coordination of values.

By solving the EtchASketch problem through the mDGE, children can take advantage of such dynamic continuous aspects of the covariation concept, particularly trying to discover the embedded mathematical principle of the task through the dynamic continuous and collective exploration of the points' movements with their fingers as guided by the task constraints (Abrahamson & Bakker, 2016; Abrahamson & Sánchez-García, 2016). Although solving the activity does not require formal knowledge, within this action-oriented environment, the children can learn "to move in a new way through a representational system that has become central in mathematics, the Cartesian system of perpendicular x- and y-axes" (Abrahamson & Bakker, 2016. p. 6). They also could make sense of the multimodal dynamic experience by harnessing prior intuitions related to functional dependence to generate strategies. These emergent types of coordinated actions that children display constitute new foundations of covariation on the body so that when they begin learning this concept formally, they can harness their embodied intuitions. Complementarily, children may use both prior knowledge about the circle properties and the visual continuous feedback to coordinate their actions simultaneously, and not just produce squares.

3.4 Data collection methods

The data collection strategy for the small-group work was the task-based interview. After the teacher-researcher introduced the task prompt for all the groups, one research assistant focused on each small group, observing children's actions during the solution process, listening to their discussions, and interviewing them about what they were thinking and doing, including their strategies. Task-based interviews lasted approximately 20 minutes. One video camera captured the whole class and two video cameras focused on the two small groups. Children's utterances and actions were fully transcribed for analysis.

3.5 Data analysis methods

The unit of analysis was the children's interactions with their peers, the interviewer, and the technologies within each small group during the task solution. We constructed the analytical framework for this study during the data analysis phase and included three categories: (a) mathematical strategies, (b) collaborative behaviors, and (c) socially mediated metacognition, all emergent processes during the time of the activity, as a product of children's social interaction and technology use. Indicators of these processes were the children's utterances, actions on the iPad including gestures, and configurations produced by their actions on the screen. We developed the codes for the three categories during the data analysis process, grounded on the theoretical definitions presented in the second section of the paper. Our perspective on mathematical strategies is grounded in Abrahamson and Bakker (2016) and Siegler (2007). Our perspective on collaboration is grounded in Goos et al. (2002) and Wells (2000). We adapted some of the Goos et al.'s collaborative behaviors. Our perspective on socially mediated metacognition is based on Goos et al. (2002) and Kim et al.'s (2013). And for this category, we adapted Kim et al.'s (2013) framework of metacognitive functions and sources to the specificities of our learning environment and theoretical perspective. To analyze the semiotic mediation of the mDGE, we draw on Hegedus (2013), Mariotti (2009), and Sinclair et al. (2013). Figure 2 presents the specific mathematical strategies, collaborative behaviors, metacognitive functions, and metacognitive sources we coded within each category, developed during the analytical process.

The data coding method consisted of a step-wise iterative process of seeking redundancy, using first a *process* coding cycle and second a *pattern* coding cycle (Saldaña, 2013). For the coding process, one of us reviewed the transcripts from each group's conversations/interactions line by line looking for instances of each category and coded them according to the theoretical and operational definitions stemmed from the literature review. Results of this coding stage were used to establish the initial analytical coding tree. Both of us widely discussed this coding scheme and solved any disagreement. Then, both utilized the coding tree to analyze independently the transcripts from all the groups. As more small groups' transcripts were codified and discussed, the first coding tree was refined until the final coding tree was configured (Fig. 2). The patterns allowed us to describe and analyze the two case studies.

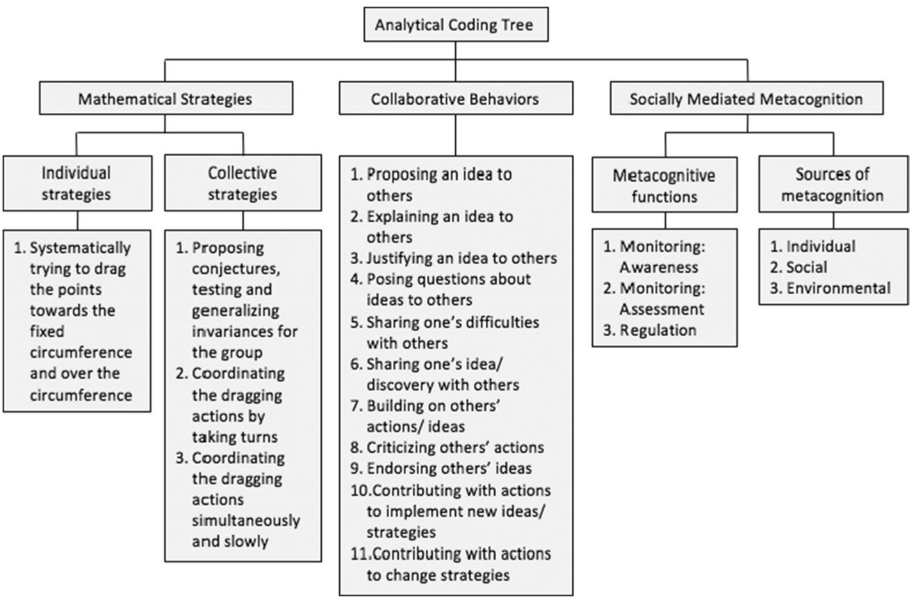

Fig. 2 Analytical coding tree

4 Results

For each case narrative, mathematical strategies are marked in **<u>bold underlined lower case</u>**, collaborative behaviors in **bold lower case** and metacognitive functions in ***italic bold lower case***.[1]

Case study 1: Ava, Mia, and Jacob's co-creation of strategies

Ava, Mia and Jacob's group utilized four strategies to solve the task (Fig. 3), three of them created collectively.

To start, Jacob located the spectrum dot on green. Then, Mia put her finger on Point 1 (hereon P1) and Ava put her finger on Point 2 (hereon P2). Their first strategy entailed individual systematic attempts to **<u>drag the points diagonally towards the fixed circumference,</u>** <u>and then to</u> **<u>drag them over the circumference</u>**. Because P1 moved only horizontally and P2 moved only vertically, the blob did not move, and there were no traces. The girls repeated these actions reiteratively without success, while Jacob observed their actions. Three collaborative behaviors emerged: First, Ava **shared her own difficulties with others,** by saying "This one does not move." Then, Jacob **criticized others' actions,** by saying "You are using the wrong thing", referring to the girls' dragging. And then, Mia **posed a question to others** about which would be a good strategy by saying "How will we get it?". Such collaborative behaviors helped the children to begin making explicit that something was wrong about their strategy, based on the visual feedback on the screen, as they did not produce a circle. The Excerpt 1 shows the subsequent interactions evidencing the children's awareness and assessment of their initial strategy and the co-creation of a second mathematical strategy:

Excerpt 1

1	**Ava**	This is hard! *{Dragging P2 toward different directions; P2 moves only when she drags vertically tracing vertical lines. Observes the screen. Drags P2 up-and-down systematically, tracing vertical lines as she drags}* (Fig. 3a).
2	**Mia**	*{Observes Ava's actions on the screen and drags P1 horizontally; P1 moves tracing horizontal lines}* (Fig. 3a).
3	**Ava**	Ah!, now I got it! *{Observes Mia's actions on the screen. Repeats three times the same action of moving P2 vertically and observes on the screen how the blob moves tracing vertical segments. Then she drags P2 horizontally, the point does not move. She drags again P2 vertically, the point moves. Then she drags Mia's P1 vertically and the point does not move. Observes the screen}*

End of excerpt 1

The Excerpt 1 reveals a conjunction of individual, social, and environmental sources mediating the children's metacognitive awareness, assessment, and regulation of their actions. Just when Ava **shared again her own difficulties with others**, she changed the way of moving P2; because of her repeated dragging of P2, the blob eventually moved and small green segments were traced (line 1). Ava began intentionally observing and analyzing the visual feedback product of their actions on the screen. Thus, she noticed the regularity that P2 just moved vertically producing vertical traces without forming any shape (line 1, Fig. 3a). Similarly, after observing Ava's actions, Mia began moving P1 towards different directions

[1] Students' names are pseudonyms.

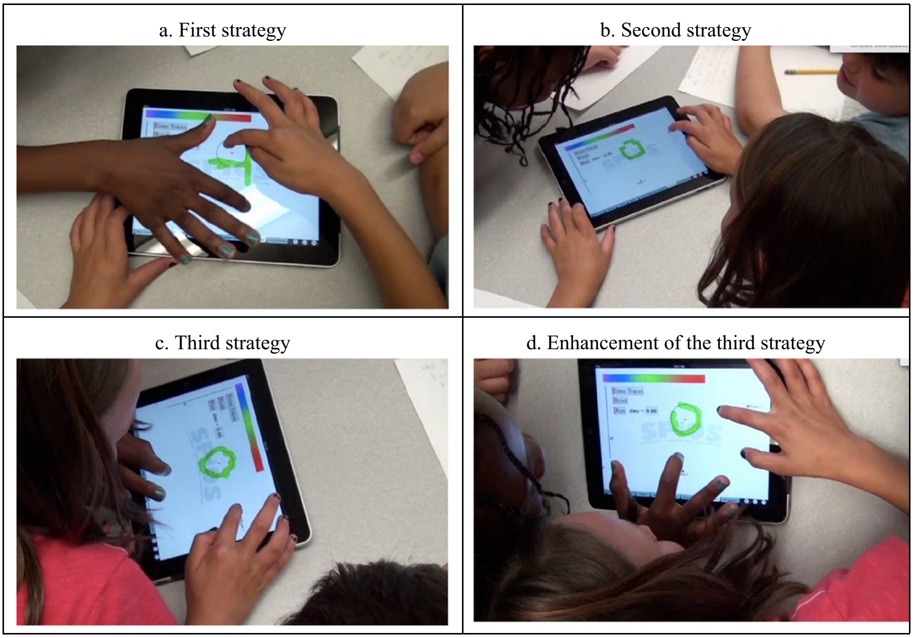

Fig. 3 Evolution of mathematical strategies in Ava, Mia, and Jacob's group

and analyzing the visual feedback product of her own actions. She noticed the regularity that P1 just moved horizontally and began dragging side-to-side (line 2, Fig. 3a). In turn, by observing Mia's actions and visual feedback, and testing her own actions, Ava seems to have inferred the invariance that each point moved only in one dimension, enabling her to trace a vertical or a horizontal segment (line 3). Therefore, Mia and Ava did not copy each other; instead, they **built on the nature of the other's actions.** Hereafter, both girls began systematically dragging the points in their predetermined directions. The utterances and actions in line 3 can be interpreted as Ava's thought experiment accompanied by new collaborative behaviors. Through analyzing visual feedback, Ava **proposed a conjecture** about how the points moved, and implicitly **shared her discovery with others** by saying "Ah! Now I got it!" Then, she **tested her idea** by moving P2 vertically three times, always observing the visual feedback on the screen. She noticed how P2 moved up-and-down tracing vertical lines. Then, she dragged P2 horizontally and noticed that it did not move so there were no traces. Then, she explored P1, dragging it vertically but it did not move and there were no traces. Ava's thought experiment formulating a conjecture and testing it to generalize the results is a strategy facilitated by the mDGE's affordances; she discovered in which dimension the points moved and mathematized the activity by **generalizing the invariance** that her two actions of dragging P1 and P2 produced horizontal and vertical segments, respectively.

The prior collaborative behaviors preceding Ava's thought experiment may have triggered the group members' ***metacognitive awareness*** as a whole, as all the children began being conscious of the constraints of their initial individual strategy. It was also triggered by the children's reflection on the visual feedback product of the girls' actions on the screen. Jacob's initial criticizing of the girls' actions indicates the first attempt to consciously ***assess the accuracy*** of their initial strategy, building on this visual feedback.

Then, Ava and Mia began ***regulating their actions*** as a team, **contributing with actions to change the strategy** and solve the task. Finally, ***metacognitive awareness*** of the current state of the task and knowledge about the points' movements led Ava to ***assess*** their embodied strategy, ***regulate her actions*** to rethink the whole approach to the task, propose a conjecture and test, and generalize it to the group using the technologies.

Emergence of collaboration related to socially mediated metacognition described above favored the co-creation of a third mathematical strategy, product of the children's conscious joint actions, the interviewer's feedback, and the mDGE's feedback. After observing the events of the Excerpt 1, the interviewer realized that the children figured out how to drag the points so he confirmed how each point moved by saying "This point only moves this direction *{Drags P1 horizontally}*, not this direction *{move the finger vertically}*, and that point only moves in that direction *{Drags P2 vertically}*." The children observed the screen; Ava restarted the task and assigned the control of P1 to Mia showing ***regulation of the joint activity***. New collaborative behaviors emerged as Ava and Mia began working together **contributing with their actions to implement a new strategy**: Mia dragged P1 horizontally tracing a horizontal segment and then Ava dragged P2 vertically tracing a shorter vertical segment. Then, Ava dragged P1 and P2 by taking turns and repeated this sequence several times leaving horizontal and vertical traces (respectively), some of them longer than others. This sequence of actions can be interpreted as Ava's attempt to test the new strategy. The product on the screen resembled two sides of a square with small staircases on the corners. The Excerpt 2 presents how the interviewer's scaffolding fostered the girls' collaborative creation of this third strategy:

Excerpt 2

1	Interviewer:	So… Can two people do it at the same time one person on each point?
2	Ava & Mia:	*{Ava restarts the task. Mia drags P1 horizontally leaving a horizontal segment and then Ava drags P2 leaving a vertical segment, then Mia drags P1 horizontally leaving a horizontal segment, then Ava drags P2 leaving a vertical segment. The product is a square}* (Fig. 3b)
3	Ava:	Oh No!
4	Mia:	*{Restarts the task}*
5	Ava:	Ok. Now!
6	Ava & Mia:	*{The girls start drawing the shape again by taking turns. Mia drags P1 horizontally leaving a horizontal segment, then Ava drags P2 leaving a vertical segment, then Mia drags P1 horizontally leaving a horizontal segment and then Ava drags P2 leaving a vertical segment. The product looks like a square. Both children keep on tracing the shape by taking turns}*

End of excerpt 2

In Excerpt 2, the interviewer encouraged the children to trace the shape between "two people" (line 1), so two collaborative behaviors emerged: First, Ava and Mia **contributed with actions to implement the new strategy** so they **<u>coordinated their dragging actions by taking turns</u>** to trace the shape. Thus, with the interviewer's guidance the girls created a new collective strategy (line 2, Fig. 3b). Because they did not coordinate their actions simultaneously, the new shape looked like a square instead of a circle (Fig. 3b). Second, Ava **shared her own difficulty** through her surprise expression (Line 3), so Mia restarted the task (line 4), indicating that they analyzed the visual feedback and realized they did the wrong shape. Collaboration increased the girls' ***metacognitive awareness*** of the current state of the task and ***assessment*** of the product of their embodied actions, leading them to

regulate their joint activity. Although the girls restarted the task, they repeatedly coordinated their actions by taking turns producing a square again (lines 5 and 6)**.**

The Excerpt 3 presents the co-creation of a fourth more advanced mathematical strategy:

Excerpt 3

1	Jacob:	*{Observes the Ava and Mia's square}* You did a square. This is to do together.
2	Interviewer:	So how does your circle look?
3	Ava & Mia:	Like a square! *{Laugh}*
4	Interviewer:	It looks like you are taking turns. Are you doing that on purpose?
5	Ava:	Yes.
6	Jacob:	It was supposed to do it as a circle, together!
7	Interviewer:	Do you think you can, do you think you can do it together at the same time?
8	Jacob	Do it at the same time.
9	Ava & Mia:	This is so quick! *{Ava and Mia restart the activity. They drag the respective points at the same time, coordinating their movements very fast. They observe the product. It looks like a circle but it is not a well-defined circle}* (Fig. 3c)
10	Ava & Mia:	*{Ava and Mia restart the task and drag their respective points simultaneously, coordinating short movements slowly. The product looks like a well-defined circle}* (Fig. 3d)

End of Excerpt 3

In Excerpt 3, various collaborative behaviors emerged and were important social sources for the transition to a new approach. Jacob, who had been observing the girls' actions, **criticized their joint activity** in terms of not fitting the goal and **proposed systematically (three times) the idea** of doing the shape together at the same time (lines 1, 6, and 8). Even when he had not dragged the points, by observing his partners' actions and the visual feedback on the screen, he seems to have inferred the underlying principle of the task, and proposed a new strategy suggesting the girls should coordinate their movements to make "a circle, together" (line 6). These events meant that he analyzed the visual feedback focusing not only on the points' movements but also on the shape's properties. After the interviewer encouraged the girls to move the points simultaneously (line 7), Jacob **endorsed this idea**, and also **proposed** to move the points simultaneously (line 8). The interviewer was also a social source. First, he asked the children what the shape looks like (line 2), calling their attention on the shape's properties rather than on the points' movements. The girls answered that the product was a square and laughed (line 3), indicating that they were ***aware*** the shape was not the expected outcome and **shared this difficulty** through a genuine emotional way. The interviewer also asked the girls about the strategy (line 4) and Ava confirmed it (line 5). Similar to Jacob, the interviewer suggested working together, probably leading children to reflect on their actions and assess the outcomes of their strategy by analyzing the shape's properties (line 7). The word "together" in Jacob's and the interviewer's utterances could mean "simultaneously" and not by "taking turns"; however, Jacob did not make this idea explicit. In contrast, the interviewer emphasizes "at the same time" (Line 7), offering an important hint about the strategy. The girls understood this idea and **<u>coordinated simultaneously their dragging actions</u>**, one point vertically and one point horizontally, moving the blob simultaneously in two dimensions and tracing a circle (line 9, Fig. 3c). Initially, they implemented the strategy very fast so their product was an ill-defined circle (Fig. 3c). By interpreting visual feedback, the girls became ***aware*** of the strategy outcome and ***assessed its accuracy***, so that they restarted the task and **<u>coordinated</u>**

__their movements of the two points simultaneously, but slowly__, moving the blob in both dimensions and tracing a well-defined circle (line 10, Fig. 3d). Then the girls ***monitored their dragging actions*** and produced a circle revealing that they harnessed both visual and social feedback, and ***regulated their joint activity*** refining their strategy to reach the shared task goal.

Case study 2: Erin, Ian, and Sam's co-creation of strategies

Erin, Ian, and Sam's group co-created three collective strategies to solve the task (Fig. 4).

To start, Sam located the spectrum dot on purple. Then, Ian dragged P1 and Erin dragged P2 reiteratively, observing and analyzing visual feedback on the screen. The children's initial goal was to figure out which point moved vertically and which horizontally. Ian noticed that Erin dragged P2 horizontally, without moving it. The Excerpt 4 shows how they created the first and second collective strategies.

Excerpt 4

1	Ian	*{To Erin}*. No, like this. *{Drags P1 horizontally. Observes the blob. It moves tracing a horizontal segment. Drags P1 vertically. Observes the blob. It does not move}*. That one moves up *{Points out P2}* So, move that one up.
2	Erin	*{Drags P2 vertically. Observes the blob. It moves tracing a vertical segment. Drags P1 vertically. Observes the blob. It does not move}* Ah! You have to try to move this all of this way to go around the circle at the same time *{Points out P1 with one index finger and moves it horizontally; simultaneously points out P2 with the other index finger and moves it vertically, joins the two fingers and points out over the circumference}*
3	Ian	I'll take P1.
4	Erin	I took this *{Points out P2}*. Please take P1, this goes side-to-side *{Points out P1},* and this goes up-and-down *{Points out P2}* so that way we are making the circle and both make this to go around.
5	Ian	*{Drags P1 horizontally, tracing a short horizontal segment. Drags P1 vertically, there is not trace. Observes the screen}* This goes back-and-forth *{Points out P1}*
6	Erin	*{Drags P1 vertically, tracing a small vertical segment on the shape Ian drew; waits for Ian's next action}* You have to press it and then we both have to go around at the same time, pressing it, because two guys together make a circle.
7	Erin & Ian	*{Ian drags P1 horizontally, then Erin drags P2 vertically. They move their respective points taking turns once at time. The product looks like a polygon of multiple short sides}* (Fig. 4a)

End of excerpt 4

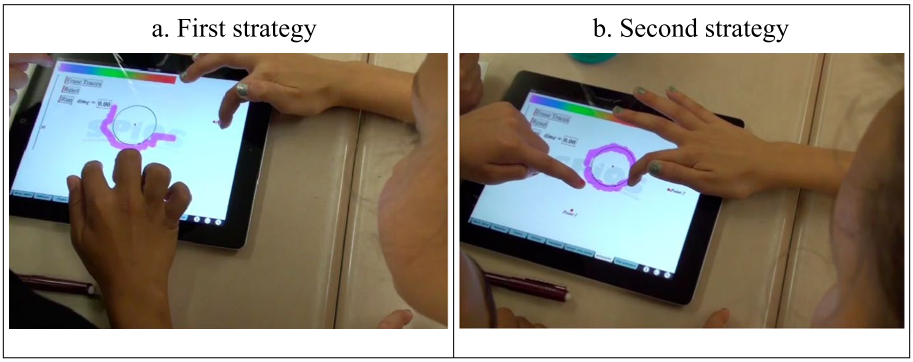

Fig. 4 Evolution of mathematical strategies in Erin, Ian, and Sam's group

The Excerpt 4 shows the interplay among individual, social, and environmental sources of metacognition.

The first six collaborative behaviors emerged when Ian saw Erin moving P2 horizontally without success so he **criticized her actions** (line 1)**.** Ian **proposed a conjecture** about how the points moved and **tested** two potential P1's movements. Noticing that P1 moved horizontally, he **generalized the invariance** that P2 moved vertically and **shared his discovery** with Erin, and simultaneously, **proposed a new idea** about how the points moved (line 1). **Building on Ian's idea**, Erin also **conjectured** about each point's movement and **tested** them, once at each direction **contributing with her actions to implement a new strategy** (line 2). Two new collaborative behaviors emerged as Erin also **shared her discovery** with Ian, and simultaneously, **explained the idea** of how to move the points at the same time to trace the circle (line 2). Erin's gesture in line 2, extending and moving her two index fingers in the direction of the points' movements and joining the fingers to move them together over the circumference indicates she is **making sense of the underlying principle** that two inputs make one output, and these meanings are embedded in her fingers; both utterances and gestures suggest she is not only talking about how to move two points, but also about how to coordinate two actions to draw only one circle. The prior collaborative behaviors and the strategic activity in which children engaged generating conjectures, implementing actions to test them, analyzing visual feedback on the screen, and drawing generalizations, are thought experiments facilitated by the affordances of the mDGE. This led both children to find out in which dimension each point moved and realize that they have to **coordinate their dragging actions at the same time by taking turns**. New collaborative behaviors emerged: Erin **explained again her idea** to Ian, but now **justifying it** in terms of reaching the goal (line 4). Ian traced a side of the shape and tried to trace another side by dragging P1 in the wrong direction. As P1 did not move, Ian **explained to Erin his own idea** showing that he is **building on her idea** (line 5). Erin traced a new side and waited for Ian to trace the next one. As he could not move his point, Erin called his attention and **explained and justified again the strategy** of making two actions at the same time to make one circle (line 6). Both Erin's statements are accompanied by actions and gestures pointing out the geometrical objects on the screen, and utilizing the technology to demonstrate her idea. However, the fact that each one made one movement at a time showed that they continue using the **taking-turns strategy**. Therefore, the collaborative behaviors influenced the children's ***metacognitive awareness of their strategy***, as they communicated that a good method is to coordinate their movements of the points at the same time. Thus, the children ***monitored and regulated their behaviors*** **coordinating their actions to implement their strategy**; however, they coordinated the points' movements by taking short turns, so the product was a multi-sided polygon (line 7, Fig. 4a). So far, "at the same time" seems to mean "by taking turns".

Over a short-time period, the team changed to a third collective more advanced mathematical strategy through emergence of new collaborative behaviors: Erin and Sam implemented the prior strategy coordinating actions by taking turns so they traced a shape that looked like a square. Erin **shared their own difficulties** with the joyful expression "[T]his is funny", restarting the task. This indicates that she **assessed** the visual feedback product of their actions, becoming ***aware*** that their actions were not producing a circle. Both children began **coordinating their**

dragging actions simultaneously, demonstrating that Sam also **assessed** the difficulty of their prior strategy. However, they did this coordination very fast producing an ill-defined circle. The Excerpt 5 shows how the children changed their strategy coordinating their dragging actions simultaneously and slowly, and explained it:

Excerpt 5

1	Erin & Sam:	*{Sam drags P1 horizontally while Erin drags P2 vertically. They move their respective points simultaneously and slowly. The product is a well-defined circle}* (Fig. 4b)
2	Ian	But… Do we have to go like around the circle?
3	Interviewer:	Yes, you should try make a whole circle.
4	Ian	Can we like to go inside?
5	Interviewer:	No, just one tries the edge.
6	Erin:	I have to go back-and-forth *{Erin explains to the interviewer}*
7	Interviewer:	How do you go back-and-forth?
8	Erin:	I have to like go back-and-forth, so when this one is going up *{Moves right-hand finger up-and-down}*, so when this one is going up is moving you *{Points out P2 with left-hand finger moving it up-and-down and simultaneously points out with right-hand finger over the circumference},* but in order to trace it, this would have like go back and like forth *{Points out P1 with right-hand finger moving it side-to-side}*
9	Interviewer:	Where is the back and forth helping you?
10	Erin:	Tracing around the circle.

End of Excerpt 5

In Excerpt 5, Erin and Sam **coordinated their dragging actions continuously, simultaneously, and slowly** so that they produced a well-defined circle (line 1, Fig. 4b). A new collaborative behavior emerged as Ian, who was observing their partners, **posed questions to others** about plausible points' movements trying to understand the new strategy (lines 2 and 4). The interviewer provided feedback, discussing the task goal and explaining the points' movements (Lines 3 and 5). Erin showed a new collaborative behavior as she **explained their strategy** focused on how she moved one point (line 6). The interviewer helped again by asking Erin to explain more (line 7) and Erin **explained and justified their strategy** in a dynamic way focusing on how they should coordinate the two points' movements to produce a circle (line 8). Erin's utterances and gestures in line 8 representing both points' movements and the circle as one product revealed that she had made sense of the visual feedback on the screen mathematizing the activity. She grasped and embodied the intuitive operational view of covariation, **GENERALIZING** the underlying principle that two inputs make one output. Therefore, the strategy was embodied in her hands and fingers. The collaborative behaviors triggered ***metacognitive awareness and assessment of the strategy***, as the children were confident of the strategy accuracy. When the interviewer asked Erin to explain how their actions were helping her (line 9), she **justified their strategy** in terms of the task goal (line 14), implying that she had assessed how good their strategy was.

5 Discussion and conclusions

The primary contribution of our investigation provides insights into how learners can gain access to core mathematical ideas such as covariation through affordances of multimodal environments that foster *collaborative co-action*. Emergent technological and social *affordances* mediated the children's mathematization process as they engaged with their pairs and the interviewer's guidance in a dynamic-gestural *executability of the geometrical representations* (Hegedus, 2013). Despite differences in task exploration, we found that both teams created and improved collective strategies as a product of collaboration and socially mediated metacognition over a short-time period. The mDGE allowed the students to collaborate through collectively identifying, selecting/touching objects on a screen, gesturing, testing ways of action, and reviewing the results of these actions. Therefore, the semiotic potential (Bartolini-Bussi & Mariotti, 2008) of such an environment promoted engagement of learners to reason together through *co-action* (Hegedus & Moreno-Armella, 2011). Permanent reflection on the visual feedback product of their own and their partners' actions facilitated the emergence of participants' new *cognitive structures* (Abrahamson & Sánchez-García, 2016; Hegedus & Moreno-Armella, 2011). Moreover, children in both groups developed *thought experiments* (Sinclair et al., 2013) that allowed them to generate conjectures about their actions and test these actions to solve the task, inferring mathematical invariances from the task. Different to prior research on covariation through DGEs, our study is instrumental in showing how *collaborative co-action* can bring students from individual action-based strategies through mutual action coordination by taking turns (asynchronous) to a type of mutual action coordination that is increasingly simultaneous, smooth, and continuous (Thompson & Carlson, 2017), the enactive foundations of covariational reasoning. Therefore, the mDGEs' affordances can be mathematical and provide preliminary informal insights for learners to core mathematical ideas.

We argue that the children's individual strategy and their collective mathematical strategies are different in nature. Abrahamson and Sánchez-García's (2016) construct of *attentional anchors* from their ecological dynamics perspective was helpful to understand this difference. Initial individual attempts to solve the task were guided by immediate visual feedback. However, as the teams of children moved their fingers on the screen while testing systematically their actions, the task constraints emerged allowing for new *affordances for action*. The teams developed a new way of moving as they perceived these new affordances within the mDGE. A new relationship between the teams' actions and task constraints seemed to emerge, guiding them to novel ways of simultaneous, smooth, and continuous coordination that were increasingly improved in function of the task goal. Thus, solving the EtchASketch activity through the mDGE fostered new ways of moving as a product of motor-control coordination oriented by these emergent objects or *attentional anchors*, which in turn were mediated by the learning environment constraints (Abrahamson & Sánchez-García, 2016). These new forms of coordination among the children's chain of gestures were promoted not only by children-technology interactions but also by social interactions. Our study provides evidence that motor-control coordination emerged not only within a child but also among the children, turning the sense-making process into a collaborative creation of meanings about covariation.

We claim that an informal approach to functions results appropriate for young children to begin developing their early intuitive functional thinking as Confrey and Smith (1995) proposed. The participants' emerging ways of thinking align with Thompson and Carlson's (2017) developmental levels of *covariational reasoning*. However, our participants' strategies mainly entail enactive, mutually organized representations of covariation driven by the task constraints. We argue that Thompson and Carlson's levels are recursive; the same levels could be applied to varied, multilevel types of covariation representations such as enactive or symbolic. Our study shows how covariational reasoning, as proposed by Thompson and Carlson, could emerge and evolve rapidly in young children at the enactive level even during the task solution. But we propose that covariational reasoning could involve subsequent cycles of representational microdevelopment (Karmiloff-Smith, 1994) at more sophisticated symbolic levels during school years. However, as a limitation of our study, the EtchASketch task does not require to understand coordinated changes in numerical variables' values; therefore, we do not present evidence of how children would navigate through these developmental levels when using symbolic mathematical representations.

Regarding socially mediated metacognition, the visual-dynamic feedback from the mDGE favored group awareness, allowing the children to monitor their strategies and develop a more effective solution. The children explored and reflected on the outcomes of their own actions on the screen, conjecturing and testing, and analyzing the properties of the geometric shape through their own coordinated actions. Metacognition was a key process for children to co-create collective strategies as well as to transition from the initial to more advanced strategies.

Our design principles are intentional and underpin the co-variation structure embedded in the sketch; i.e., it is not by chance that the phenomenon of co-variation might well be discussed or attended to through communicative acts (e.g., abstract, metaphorical or gestural). Similar to Nemirovsky et al. (2013), we think of the mDGE in terms of material (the multimodal physicality) and the semiotic tool (accessing the mathematical structure through co-action and collaboration) and "fluent use of mathematical instruments allows for culturally recognizable creation[s]…" (Nemirovsky et al., 2013, p.373).

Our theoretical and analytical framework to examine students' interactions within mDGEs allow our results to be extended to a wider variety of problem-solving contexts; teachers can use our approach in mathematics classrooms.

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