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Teachers' use of rational questioning strategies to promote student participation in collective argumentation

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Abstract
Teachers' questioning plays an essential role in shaping collective argumentative discourse. This paper demonstrated that rationality dimensions in teacher questions can be assessed by adapting Habermas' three components of rationality. By coordinating Habermas' construct with Toulmin's model for argumentation, this paper investigated how two secondary mathematics teachers used rational questioning to support student participation in collective argumentation. This paper identified various ways in which two participating teachers used rational questioning to support student participation in argumentation via contributions of argument components. The results establish a theoretical connection between the use of rational questions and students' contributions of components of arguments. The results indicated that not all rational questions were associated with a component of argument, and rational questions may additionally support argumentation in general for the development of a culture of rationality. The study has implications in terms of theory and professional development of teachers.

Keywords Collective argumentation · Teacher questioning · Habermas' theory · Toulmin's model · Secondary mathematics

It is widely agreed that teachers play a pivotal role in orchestrating mathematical argumentation in classrooms and should promote students' engagement in productive classroom-based collective argumentation (Forman et al., 1998; Gomez Marchant et al., 2021; Hunter, 2007; Yackel, 2002). Teacher questioning is an essential component of developing a classroom context that is conducive to mathematical argumentation, justification, and reasoning (Kazemi & Stipek, 2001; Martino & Maher, 1999; Wood, 1999). In the USA, the National Council of Teachers of Mathematics (NCTM) asserted in *Principles to Actions: Ensuring Mathematics Success for All* (2014) that effective mathematics teachers are expected to use purposeful questions to access students' conceptual understanding, prompt critical thinking, and advance students' reasoning and sense-making of

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mathematical ideas. Nevertheless, teachers often have a limited understanding of what types of questions are appropriate, and they have difficulties with incorporating questioning strategies to scaffold argumentation (Kosko et al., 2014; Sahin & Kulm, 2008). Teacher questioning has the potential to invite students into a conversation and promote participation, but the types of questions can have a significant impact on students' engagement in productive argumentation.

While much research (e.g., Boaler & Staples, 2008; Foster et al., 2020; Kazemi & Stipek, 2001; Sahin & Kulm, 2008) has been invested in documenting current situations or difficulties that teachers experienced with using questions to regulate argumentative discourse, few studies addressed classroom implementations or interventions with the purpose of focusing specifically on improving the use of collective argumentation in teaching. The main goal of this paper was to investigate how teachers used *rational questioning* to support student participation in collective argumentation, as evidenced by contribution of argument components.

1 Teacher questioning in supporting collective argumentation

In this paper, collective mathematical argumentation refers to a cyclic process of making mathematical claims by teachers and students (or a small group of students) in which they support claims by providing reasons and data or challenging claims or rebutting those reasons and all other related activities that are aimed at constructing or responding to argument components (e.g., claims, warrants, data, rebuttal) (Conner et al., 2014; Staples et al., 2017). Supporting student participation in argumentation is pedagogically demanding and challenging for many teachers. Teachers must have the requisite pedagogy skills, such as being able to pose questions for elaboration, explanation, and justification (Sahin & Kulm, 2008), establish appropriate classroom social norms and sociomathematical norms (Yackel, 2002), respond to students' ideas in ways that develop arguments (Lampert et al., 2013) , and choose appropriate tasks to foster understanding (Rogers & Kosko, 2018). In this paper, we concentrate on questioning strategies that may support the development of these requisite pedagogies.

Teacher questioning plays an essential role in facilitating classroom-based mathematical argumentation (Kazemi & Stipek, 2001; Wood, 1999). Some researchers (e.g., Conner et al., 2014; Franke et al., 2009; Sahin & Kulm, 2008) have classified teacher questions in order to provide insight into how different levels or types of questions were used to lead classroom discussions. For example, Sahin and Kulm (2008) found that most teachers' questions were factual and lecture-based. Franke et al. (2009) found that although teachers frequently asked students to explain their answers or to offer justifications, these questions did not always produce further explanations. They argued that a single specific question was not sufficient to elicit a complete explanation or justification. Conner et al. (2014) examined types of questions that were specifically linked to individual argument components rather than to the general promotion of students' participation in argumentation. However, these questions may not capture the full picture in terms of understanding teachers' facilitation of argumentative discourse.

An understanding of current situations or difficulties that teachers may have in using questions is not enough to address teachers' difficulties in using questioning strategies to scaffold collective argumentation. McCarthy et al. (2016) stated that

"Identifying 'good' and/or 'effective' questioning strategies is a major challenge to mathematics teachers" (p. 80). Stylianides et al. (2016) called for more research to design didactical tools that would address teachers' difficulties in supporting argumentation. In this paper, we investigated how teachers used questions to support student engagement in argumentative discourse; and we draw on the work of Habermas (1998) and Toulmin (1958/2003) .

2 Theoretical frameworks

2.1 Teacher rational questioning framework

Habermas' (1998) theory of communicative action has been applied to a variety of fields, including psychology, economics, and political science. Although Habermas did not write directly about education, his work has provided transformative perspectives that inspired educators to seek reform in education systems (e.g., curriculum, educational research). Boero (2006) proposed that parts of Habermas' theory of communicative action about rationality (i.e., Habermas' construct of rational behavior) in discursive practices can be used as a theoretical tool for addressing students' proving and argumentation practices in mathematics education. Following Boero, some researchers have used Habermas' theory to conduct studies that centered on students' proving and argumentation practices (e.g., Cramer & Knipping, 2018; Guala & Boero, 2017; Morselli & Boero, 2011; Zhuang, 2020; Zhuang & Conner, 2018, 2020).

Habermas (1998) defined a person as a rational being if a person has the ability to "give account for his orientation toward validity claims" (p. 311) with respect to three core structures of rationality: in the proportional structure of knowledge (knowing), in the teleological structure of action (acting), and in the communicative structure of speech (speaking). In other words, the assumption taken by Habermas is that for a rational being, discourse and reflection are integrated and that "the three rationality components—knowing, acting, and speaking—combine, that is, form a syndrome" (p. 311). Knowledge, action, and speech constitute what Habermas called *epistemic*, *teleological*, and *communicative* components of rationality. Building on Habermas' theory, Boero and his colleagues articulated that the three rationality components may be applied to mathematical argumentation practices in which students are expected to strategically choose specific tools (teleological rationality) to achieve the goal (i.e., validity claims) on the basis of mathematical knowledge (epistemic rationality), such as rules, theorems, axioms, and principles and communicate in a precise way with the goal of being understood by the classroom community (communicative rationality) (Boero, 2006; Boero & Planas, 2014). To clarify, communicative rationality involves the use of precise mathematical language and appropriate mathematical representations that follow standard notations, but also criteria for easy reading and manipulation of expressions (Morselli & Boero, 2011). The three aspects of rationality in argumentation practices correspond to what policy documents and mathematics educators suggest as important characteristics of classroom argumentation: mathematics classrooms should rely on mathematical evidence for verification; teachers should focus on reasoning and proving; and students should engage in conjecturing, problem-solving, and communication (Boero et al., 2010; NCTM, 2000; National Governors

Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

According to Douek (2014), the teacher should support students to meet the three requirements of rationality (i.e., epistemic, teleological, and communicative) when organizing argumentative discourse. In order to reach such aims, Douek further proposed the idea of using *rational questioning* as a teaching method to promote students' fitting epistemic, teleological, and communicative requirements of argumentation practices. According to Douek, the role of rational questioning is described as below:

> Rational questioning should favour student's maturation (from the ability to act and develop a discourse about acting towards the ability to organise strategies and express them a priori, when the situation is mastered enough) by supporting—in between—going back and forth from 'action' to its rationalization, accounting for validity of statements and strategies, and autonomously producing a conclusive rational discourse. (p. 211)

The idea of using rational questioning as a process of enculturating rational discourse into the practice of argumentation corresponds to Boero et al. (2010)'s argument that rationality in argumentation must be guided and promoted by teachers. Thus, rational questioning can be regarded as a didactical tool for developing a culture of rationality in the classroom, as discussed by Rodríguez and Rigo (2015):

> How the teacher negotiates her own rationality practices—an objective that, by way of dialogical exchange, involves the students by means of constant questions, not only about what but also about why—and how this enculturates her students in that rationality. (p. 93)

Considering the idea of rational questioning as a potential way to enrich collective argumentation through multiple perspectives, we studied how rational questioning supported collective argumentation. The *Teacher Rational Questioning Framework* (TRQF) was developed to understand how teachers engage students in participation in argumentation with different kinds of rationality (see, Table 1; preliminary versions of this framework can be found in Zhuang, 2020; Zhuang & Conner, 2018, 2020). Our concept of rational questioning is situated in a perspective in which we view teachers as rational beings who are capable of providing an account of their strategic choices of didactical tools (e.g., types of rational questions) to achieve their instructional goals (e.g., adequate epistemic reasons, efficient teleological choices, rules of communication). We define a *rational question* as a question that contained at least one component of rationality. If a question contains an epistemic rationality component, we label it as an epistemic rational question (*ER*). Thus, if a question contains all three components of rationality, we label that question as an epistemic, teleological, and communicative rational question (*ETCR*). Note that a rational question could contain multiple components of rationality when the teacher provides opportunities for students to engage in more than one dimension of rationality (e.g., a question that justifies the effectiveness of means or tools should contain both an epistemic and teleological rationality component, *ETR*). In this sense, some rational questions include two or three components of rationality, and others may involve only one.

Unlike other teacher questioning frameworks (e.g., Boaler & Humphreys, 2005; Hufferd-Ackles et al., 2004; Sahin & Kulm, 2008), TRQF does not divide teachers' questions into high- or low-level questions, open or closed questions, or categorize

Table 1 Teacher Rational Questioning Framework (TRQF)

Rationality component	Features	Description of questions	Examples
Epistemic rationality (*ER*)	Questions address the epistemic validity of arguments according to shared mathematical propositions, theorems, axioms, and principles	•Facilitate/elicit students to reason and justify their arguments and ideas for their own benefit and for the class •Clarify/challenge students to reason and justify their arguments and ideas when they give unclear or incorrect responses	•Can you tell me why? •Why do you agree or disagree with his/her/their claims?
Teleological rationality (*TR*)	Questions address the conscious choice of means/tools to obtain the desired arguments (i.e., acceptable arguments in context)	•Allow students to show or reflect on the strategic choices of means/tools that they used to achieve their arguments or ideas •Point students toward a specific means or tools	•How did you figure that out? •Can I combine these two math terms?
Communicative rationality (*CR*)	Questions address the conscious choice of means of communication within a given community	•Allow students to communicate or reflect on their steps of reasoning and final claims of argumentation to ensure that their use of mathematical language (e.g., oral language, written language, visual representations, symbolic notation) conform to the norms of and are understandable in the given mathematical classroom community •Guide students to correct use of mathematical terminologies, representations, and phrases to form legitimate ways of reasoning	•What is it called when a triangle has two equal sides? •How would we write this equation mathematically correct?

questions from an analysis of practice. Instead, we categorize a teacher's question by its components of rationality based on Habermas' (1998) theory of rationality. This framework has the potential to allow teachers to plan and manage a classroom situation that is based on the rationality they wish to develop, rather than a high-level question, an open question, or exactly what teachers should say in context. According to Boero et al. (2010), awareness of the epistemic, teleological, and communicative requirements of rationality is inherent in expert argumentation practices and is consistent with the field's suggested orientation toward proving and argumentation in mathematics education. In addition, Boero et al. argued that the rationality requirements of argumentation must be taught to students through the guidance of the teacher by using specific didactical devices.

2.2 Integration of teacher rational questioning framework and Toulmin's model

Toulmin (1958/2003) model has been widely used in mathematics education research to study the structure and functional elements of argumentation (i.e., claims, data, warrants, rebuttals, and backing) (e.g., Inglis et al., 2007; Krummheuer, 1995). In this paper, we drew on an extended version of Toulmin's model (Conner, 2008) (see Fig. 1) to examine how teachers use questions to manage collective argumentation. An individual argument component may serve multiple functions in an episode of argumentation. For instance, an argument component may serve as a rebuttal (a statement that describes circumstances under which the warrant would not be valid) in a previous sub-argument and as data in a subsequent sub-argument, or it may serve as a claim in a sub-argument and warrant in another argument. We labeled these argument components as a rebuttal/data or warrant/claim. Through Toulmin's lens, we explore whether a teacher's question directly prompts an argument component (e.g., claims, data, warrants, rebuttals) and how rational questions are associated with various argument components. The teachers' questioning actions are

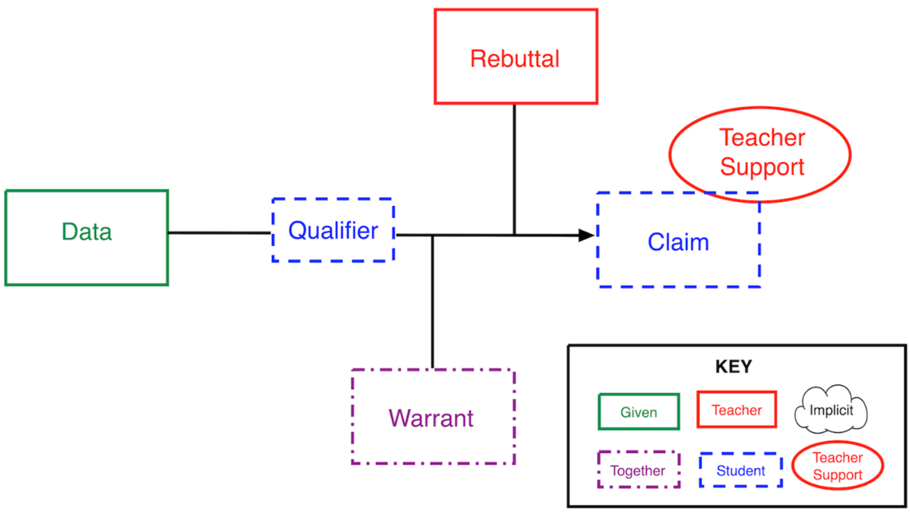

Fig. 1 Components of an extended Toulmin diagram. Adapted from Conner (2008)

represented by red ovals that connect to argument components and are labeled as "Teacher Support" to denote teachers' contributions and actions that prompt parts of arguments. Sometimes, parts of an argument may not be explicitly stated by the teacher or students but can be inferred from the context of the argument in the classroom community; we label these implicit parts with a surrounding cloud (black).

The purpose of this paper was to investigate how teachers used rational questioning to facilitate student participation in argumentation. For this paper, we integrated Habermas' (1998) construct with Toulmin (1958/2003) model in order to understand how teachers' questions might be constrained in relation to the three rational components (Habermas' lens) and frame the role of teacher questioning with respect to scaffolding the process and product of argumentation (Toulmin's lens). We build on Conner et al. (2014)'s classification of questions that directly connect to argument components on the basis of Toulmin's diagrams. For this paper, we explored teachers' questions that not only supported students to contribute argument components (Toulmin's lens) but also teachers' questions that promoted students' engagement in the rationality dimensions of argumentation (Habermas' lens). Thus, this paper analyzed teacher questioning as situated in argumentation practices that were and were not connected to argument components. The following research questions guided this paper: How does rational questioning support student participation in collective argumentation in a mathematics classroom? In particular, what combinations of rational questions support different components of arguments in a mathematics classroom?

3 Methods

3.1 Participants and data

Two secondary mathematics teachers, Jill and Susan (all names in this paper are pseudonyms), participated in this study. Both Jill and Susan had a good understanding of mathematical argumentation because they had learned about supporting argumentation in methods and pedagogy courses during their teacher education programs, and they participated in individual argumentation-focused professional development (PD) during their first 3 years of teaching. The courses included units that were designed to develop prospective secondary teachers' conceptualizations of mathematical collective argumentation and prepare teachers to develop students' argumentative skills in school. The PD involved intentional reflection on teaching using stimulated recall interviews that prompted analysis and discussion of participants' supportive teaching actions in their classrooms, with respect to collective argumentation. Although the data for this paper were collected early in these teachers' experiences, the two participants' rich experiences with argumentation created conditions that increased the potential of identifying productive questioning strategies as the teachers facilitated collective mathematical argumentation.

Data for this paper were collected as part of a larger project that examined how teachers learn to support collective mathematical argumentation. The data that were analyzed for this paper were obtained during the two teachers' last year of participation in the project. Video recordings of two sets of two consecutive days of instruction from each of the teachers (i.e., four lessons from each teacher) and accompanying transcripts were chosen as the main data source. These lessons consisted of approximately 209 min of instruction for Jill

Table 2 Participants' school context and lesson topics

Teacher	Years teaching	Grade level	Course	Lesson topics	Length of lesson	Student population
Jill	3	9th	Algebra I	Days 1 and 2: Factorization Days 3 and 4: Arithmetic and Geometric Sequences	90 min 119 min	Some students in class had specific learning disabilities in math
Susan	2	10th	Algebra II/ Geometry	Days 1 and 2: Exponential Functions Days 3 and 4: Partitioning Line Segment	176 min 167 min	Gifted and interested 9th and 10th grade students

and approximately 343 min of instruction for Susan. Field notes, students' written work, and post-lesson interviews served as additional data sources that helped us to learn more about the collective argumentation that was under analysis and the role of teacher questioning in supporting argumentation. Table 2 provides each participant's teaching background and instructional lessons that were observed.

3.2 Data analysis procedure

An argumentation episode was located by identifying the final claim of an argument and looking forwards and backwards to identify related data and warrants; an episode may contain multiple sub-arguments that support or refute parts of the initial argument. An episode of collective mathematical argumentation in a classroom context often ends with an answer to a problem that the teacher and students are working toward. Within each argumentation episode, the TRQF (see Table 1) was applied to classify moment-by-moment questions asked by the teacher. This paper drew from Cotton's (2001) definition of a question as "any sentence which has an interrogative form or function" (p. 1); in classroom settings, the teachers' questions included "instructional cues or stimuli … and directions for what they are to do and how they are to do it" (p. 1). Because context is critical in capturing the essence of what teachers do strategically (Jacobs & Spangler, 2017), an effort was made to use not only the verbal content of the classroom discourse, but also non-verbal cues and other visual indicators that were derived from the videos as well as from post-lesson interviews (if applicable) when we interpreted the types of questions that the teacher employed.

Next, we used the extended Toulmin model (see Fig. 1) to diagram every episode of argumentation and noted teachers' questions that directly prompted an argument component (i.e., claims, warrants, data). Finally, by integrating TRQF and Toulmin's model, we investigated what combinations of rational questions were associated with different parts of arguments and what kinds of rational questions may not have prompted an individual argument component, but supported students' participation in argumentation in general.

As an illustration, let us consider an argumentation episode from Susan's lesson on day 2. During this episode, the whole class was discussing the percent change and the growth factor of the function $q(x) = 84 \bullet 1^x$ (see Table 3 for transcript). At the beginning of this episode, S1 provided the correct answer that $q(x)$ was "*not growth or decay*" (line 2). Instead of giving direct feedback to acknowledge the correctness of a student's argument, Susan challenged the student's statement by asking him to provide an explanation: "*Why is this one not growing or decaying*?" (line 3). We coded this question as a rational question that contained epistemic rational components (*ER*), which required students to provide epistemic reasons to justify their arguments. In other words, this type of question focused students' attention on key mathematical concepts (i.e., epistemic rationality) by asking students to discuss the validity of their claims. In this context, the students were expected to make connections with the concept

Table 3 Teacher Rational Questioning Framework (TRQF) applied to argumentation episode from Susan's class

Transcript (questions are in bold font)		Rational question code
1	Susan: Okay. Let's talk about this one. **Who wants to help me out with this one?**	Question without a rational component (*NR1*)
2	S1: It's not growth or decay	
3	Susan: [Wrote 'no growth/decay' on board] Wait, I'm confused. I told you all of these were exponential. **Why is this one not growing or decaying?**	Contains epistemic rational component (*ER1*)
4	Multiple Students: Because one…it doesn't change	
5	Susan: Okay. **Because what's one?**	Contains epistemic and communicative rational components (*ECR1*)
6	S2: The growth decay factor	
7	Susan: Yeah, [the growth or decay factor it doesn't really matter because it's] not actually brewing either of those things. It's equal to 1[wrote on board]. **So the percent change is what?**	*NR2*
8	Multiple Students: [The percent change is] zero	
9	Susan: Wait, **how do I know that?**	Contains teleological rational components (*TR1*)
10	Multiple Students: Because the *y* values don't change	
11	Susan: So, it's not changing, **so what did this look like? What graph did this look like?**	Contains teleological and communicative rational components (*TCR1*)
12	Multiple Students: A straight line	
13	Susan: A straight line, okay. **Did it have a table?**	*TCR2*
14	Multiple Students: No, [it did not have a table]	
15	Susan: **If you made a table, what would the table look like?**	*TCR3*
16	Multiple Students: 84 for all the *y's*	
17	Susan: Yeah, excellent 84 for all the *y's*. Okay I think we are done with that one. **Is there any other information on this one?**	Contains all three rational components (*ETCR1*)
18	Multiple Students: No	

of percent change. However, the student's initial warrant, "*Because one…it doesn't change*" (line 4), was ambiguous and incomplete from Susan's perspective. Therefore, Susan asked a follow-up question that contained both epistemic and communicative components, "*Because what's one*?" (line 5). In this context, a follow-up epistemic rational question combined with a component of communicative rationality (*ECR*) requested that the students use an appropriate mathematical term (communicative rationality of their explanations), which supported the students toward providing a more comprehensible warrant (epistemic rationality of their explanations).

When the students said that the percent change of $q(x)$ was zero, Susan posed a teleological rational question, "*How do I know that*?" (line 9) to request that the students describe the method that they used to achieve the answer. We coded this question as a rational question that contained teleological rational components (*TR*), not epistemic rational components, because this type of question helped students to notice and articulate a pattern (their means of solution) that would lead to a generalizable solution (when percent change of a function was zero, the function was neither growing nor decaying). In this context, Susan focused on the students' words, which included that "*the percent change of* $q(x)$ *was zero*" in her question to allow students to be able to better articulate their problem-solving strategies. Later, Susan used students' strategies as topics for further investigation for the class (another request for teleological rationality). The follow-up combinations of teleological and communicative rational questions (lines 11, 13, and 15) were used to guide the students to use a variety of visual representations, such as graphs and tables, to represent the function. At this point, the teacher's rational questioning helped the students to understand what constituted a mathematically different representation. At the end of this episode, Susan asked, "*Is there any other information on this one*?" (line 17). Although this question did not prompt the students to directly contribute an argument component (as identified through Toulmin's lens), this question provided students with opportunities to reflect on epistemic, teleological, and communicative dimensions of argumentation practices. Thus, we coded this question as a rational question that contained all three components of rationality.

As shown in Table 3, each question was categorized as having either zero, one, two, or three components of rationality (i.e., *NR*, *ER*, *TR*, *TCR*, or *ETCR*) according to TRQF. Next, we used the extended Toulmin's model to recreate the argument diagrammatically and investigated whether each question was associated with a component of argument and, if so, with which component (see Fig. 2). For instance, the rational question "*Did it have a table*?" (*TCR2*) prompted students to construct "Claim 3"; we indicated that this question was associated with "Claim 3" in the diagram by "attaching" it diagrammatically. A question was associated with a component if the component was contributed in response to the question, even if it was not what the teacher expected. Through Toulmin's lens, some rational questions supported students' construction of warrants. The epistemic rational questions "*Why is this one not growing or decaying*?" (line 3) and "*Because what's one*?" (line 5) prompted students to construct a warrant (i.e., warrant 1) for their claim "It's not growth or decay" (claim 1). The questions that contained combinations of teleological and communicative rationality (lines 11, 13, and 15) in this episode were used to support the students' claims about different mathematical forms of representation of the function $q(x)$ (i.e., data/claim 1 and claim 3). Not all of the teacher's questions in this episode were categorized as rational questions (labeled as *NR*, lines 1 and 7), but these questions related to the argumentation by prompting the students to construct what we identified as claims (i.e., claims 1 and 2) through Toulmin's lens.

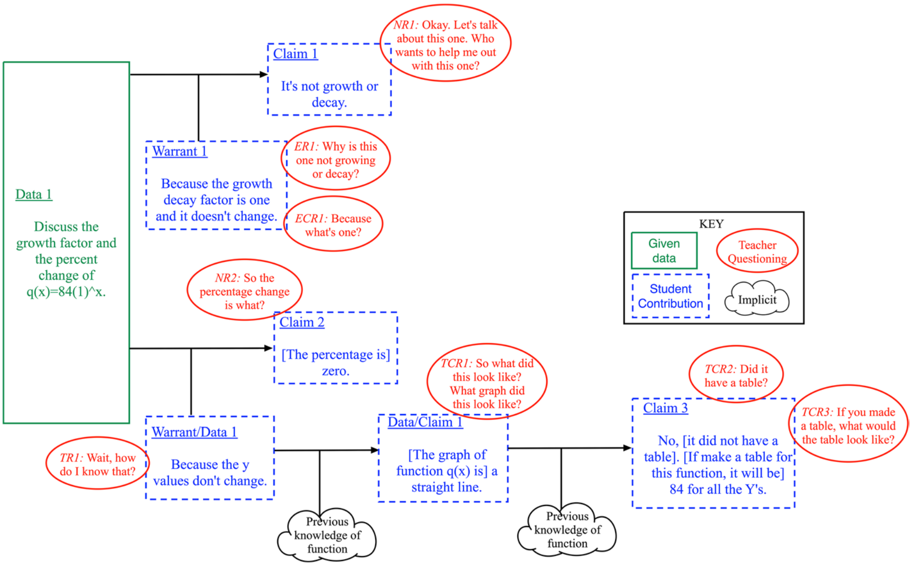

Fig. 2 An illustrative episode of argumentation from Susan's day 2 lesson

4 The use of rational questioning in supporting collective argumentation

In this section, we describe how two beginning secondary mathematics teachers used rational questioning (Habermas' lens) to prompt student participation by contributing argument components (Toulmin's lens). Table 4 summarizes the numbers and percentages of rational questions for each participating teacher and each lesson topic. The counts of teachers' questions show the extent to which the use of rational questions is pervasive within these teachers' practices. As shown in Table 4, within argumentative discourse, nearly 90%

Table 4 Numbers and percentages of rational questioning in argumentation episodes

Days of instruction	Number of rational questions	Number of non-rational questions	Number of argumentation episodes
Jill days 1 and 2	136 (92%)	12 (8%)	23
Jill days 3 and 4	182 (93%)	14 (7%)	25
Susan days 1 and 2	196 (89%)	23 (11%)	39
Susan days 3 and 4	123 (88%)	16 (12%)	25

of the questions that the two teachers asked involved at least one rationality component. This result provides evidence for Boero et al.'s (2010) assertion that teachers expect epistemic, teleological, and communicative dimensions of rational behavior to appear in students' mathematical argumentation practices.

4.1 Use of rational questioning during collective argumentation

Through the use of TRQF and the extended Toulmin's model, we identified how each rational question within an argumentation episode was associated with an argument component, as illustrated in Table 5. All but 22 rational questions were associated with argument components. Based on the definition of epistemic rational questioning (see Table 1), it is not surprising that epistemic rational questions were often used to prompt warrants (89 out of 100 prompted warrants, as shown in Table 5). The teachers used epistemic rational questions to encourage their students to justify why their arguments held (e.g., Jill asking, "*You are correct; it's not three, but why*?") or provide reasons for their teleological choices of means, such as Susan asking, "*Wait. 12.5 divided by four. Why did you divide by four*?" Epistemic rational questions addressed whether students' explanations for their choice of tools or means supported the production of logico-deductive warrants that met the standard of mathematical theorems, axioms, and principles.

Our results showed that teleological rational questions were used most frequently by both teachers to prompt students to contribute claims and intermediate claims (185 out of 244 prompted claims, as shown in Table 5). For instance, Jill's lesson on day 1 included Jill's introducing "area model methods" to teach students how to factor trinomials (see Fig. 3). Jill asked a large number of teleological rational questions to strategically orient students with respect to filling in an area model (e.g., "*Alright now I have the inside of my area model filled out. How do I get the outside*?") and finding the greatest common factor in each row and column of the area model in order to obtain the factored form of trinomials (e.g., "*What is the greatest common factor of the bottom row*?"). These teleological rational questions supported students' construction of intermediate claims (e.g., "5*x and* 2*x are the linear terms*", data/claim) before the students recognized that the factored form should be $(x + 5)(x + 2)$.

Table 5 Rational questions associated with argument components

Components of rationality in the question	Data	Warrant	Claim (& data/claim)	Rebuttal	Total argument components
Epistemic (*E*)	1 (1%)	89 (89%)	6 (6%)	4 (4%)	100
Teleological (*T*)	6 (2%)	44 (18%)	185 (76%)	9 (4%)	244
Communicative (*C*)	2 (8%)	5 (22%)	16 (70%)	0 (0%)	23
Epistemic and teleological (*ET*)	0 (0%)	81 (82%)	3 (3%)	15 (15%)	99
Epistemic and communicative (*EC*)	0 (0%)	30 (84%)	3 (8%)	3 (8%)	36
Teleological and communicative (*TC*)	1 (1%)	9 (12%)	61 (84%)	2 (3%)	73
All three rational components (ETC)	1 (2%)	18 (45%)	17 (43%)	4 (10%)	40

If a rational question prompted a data/claim, we counted it as associated with a claim; we treated data/claims as intermediate claims. If a rational question prompted a warrant/data or a warrant/claim, we counted it as associated with a warrant. If a rational question prompted a rebuttal/data or a rebuttal/claim, we counted it as associated with a rebuttal

Factored Form	Area Model	Final Product
	x^2 | __x __x | 10	$x^2+7x+10$

Factored Form	Area Model	Final Product
$(x+5)(x+2)$	| x | +2 x | x^2 | 2x +5 | 5x | 10	$x^2+7x+10$

Fig. 3 Example of Task in Jill's class. A worksheet that contained this scaffolded problem (and others) was given to the students and displayed on the board. Note. Students were given the Final Product (right-most column) and were expected to fill in the blanks in the Area Model (center column) and then write the answer in Factored Form (left-most column). The second copy presents the completed problem after the discussion

The teachers mainly used a single component of communicative rational questions to ensure that the students used and wrote mathematical notations, terminologies, and representations that conformed to the norms that exist in the shared mathematical community. These questions usually occurred toward the end of argumentation episodes (e.g., Jill asking, "*I need to write it in the factored form. So, tell me what to write*.") to support the students' construction of mathematical claims with precise mathematical language and appropriate mathematical representations (16 out of 23 prompted claims, as shown in Table 5).

As displayed in Table 5, the teachers often used combinations of components of rational questions to support collective argumentation. In our analysis of questions in Toulmin's diagrams, we observed that a single epistemic rational question may prompt a student's initial warrant, but these initial warrants were often incomplete, incorrect, or ambiguous, and the students did not initially build a complete argument component to support their claims. Therefore, the teachers often asked a follow-up question that contained a combination of components of rationality, which clarified for the class the specific parts of warrants that the students had not made explicit in their original explanations. For example, following a student's answer to an epistemic question, Jill might have asked a question that contained a combination of epistemic and communicative rational components (*ECR*), such as "*So we just look at, what do you mean just look at the exponents*?" Such questions that contained combinations of epistemic and communicative rational components pushed students to use newly learned language to talk about newly learned content and express their ideas clearly. Most questions containing combinations of epistemic and communicative rational components were associated with warrants (30 out of 36 prompted warrants, as shown in Table 5) that were initially not complete or were ambiguous (i.e., these questions enabled the students to supplement an initially incomplete contribution of a warrant) (see Zhuang & Conner, 2022, for more details about teachers' use of incorrect answers).

When students engaged in classroom-based argumentation, they frequently made mistakes and constructed partially correct claims in the process of constructing and justifying conjectures, and this process sometimes resulted in invalid or not completely mathematically correct claims or warrants. Thus, the teachers had to efficiently manage incorrect answers to direct the argumentation practices toward mathematically acceptable arguments (Zhuang & Conner, 2022). Questions containing combinations of teleological and epistemic rational components were used to follow up students' initially incorrect argument components and guide students to construct additional warrants based on their initially incomplete reasoning (81 out of 99 prompted warrants, see Table 5). For example, Susan asked, "*So where is the seventeen in this function that allowed you to conclude that the percent change is seventeen percent*?" (*ETR*) to provide opportunities for students to reflect on and reconsider their initially incomplete reasoning.

Our analysis showed that in most cases, communicative rationality was strongly intertwined with epistemic and teleological rationality; these questions helped students to express their reasoning (epistemic rationality, 30 out of 36 prompted warrants) or use of tools clearly (teleological rationality, 61 out of 73 prompted claims and intermediate claims). For example, Jill used questions that contained combinations of teleological and communicative rationality (e.g., "*How did you know to put 5 at the side of the box*?", *TCR*)

Table 6 Use of combinations of components of rational questioning in supporting collective argumentation

Components of Habermas' (1998) rationality in the question	Components of Toulmin (1958/2003) model addressed most often	Description of use
Epistemic (*E*)	Invite students to be explicit about *warrants*	•Encourage students to justify their arguments •Elicit students' reasoning (may be incorrect or incomplete)
Teleological (*T*)	The contribution of and linking of intermediate *data/claims* in support of the final *claims* (Relates to their choice of tools or methods of solutions)	•Explore choices of means or tools to solve the problem •Guide students' thinking when students are struggling with problem-solving
Communicative (*C*)	Make *claims* with correct mathematical representations	•Introduce mathematical terminologies or expressions •Focus on presenting arguments clearly in oral and in written form
Epistemic and teleological (*ET*)	Construct a more comprehensible *warrant*	•Clarify or emphasize specific aspect of the problem or one part of the students' reasoning •Guide students to reflect on and reconsider answers when answers are inappropriate/incorrect
Epistemic and communicative (*EC*)	Reconstruct/revise initial *warrants*	•Assist students with communicating their reasoning •Encourage students to use appropriate mathematical language and representations
Teleological and communicative (*TC*)	Construct valid *data* or *data/claims* to express legitimate ways of problem-solving	•Introduce mathematical representations as tools •Help students to focus on specific aspects of the problem and consider a problem from different perspectives
All three rational components (*ETC*)	Reflect on or construct public arguments as a whole. *Not usually associated with a specific argument component*	•Provide opportunities for students to make sense of other students' arguments •Move toward students' autonomy in constructing arguments

to encourage her students to follow the classroom conventions during their process of solution for the area model (see Fig. 3), so the students could easily find the greatest common factor in each row and column in order to obtain the factored form of trinomials.

Rational questions that involved all three rationality components (*ETCR*) were mainly used in two ways. One way was to provide students with opportunities to make sense of and evaluate other students' arguments. For instance, after a student provided his warrants for a mathematical claim, Susan asked, "*Wait, stop. Does everyone understand what he just said*?" Another way was to invite students to lead argumentation practices (e.g., Jill asking, "*Go to the board and tell us what you did*"). In this context, the students were expected to lead the whole classroom discussion and autonomously construct claims (teleological dimension) and provide warrants (epistemic dimension) to support their claims with appropriate technical expressions (communicative dimension). However, these questions did not always directly prompt any specific argument components in terms of Toulmin's lens; in fact, the 22 rational questions that related to argumentation but did not prompt specific argument components were all of this kind.

The results of this paper showed that both teachers used all possible combinations of the components of rational questioning to support collective argumentation across lessons. Table 6 summarizes the combinations of components of rational questions that were observed (Habermas' lens) with respect to the argument components that are most often associated (Toulmin's lens), and a description of the use of rational questions.

4.2 Use of non-rational questioning during collective argumentation

The results of this paper also indicated that not all questions in an argumentation episode were categorized as rational questions (these are labeled *NR* in Table 3). Some non-rational questions were used to prompt argument components; these questions usually generated intermediate claims (i.e., data/claims) at the beginning of an argument or gathered information at the end of an argumentation episode (i.e., contribute final claims). In total, 23 out of 65 non-rational questions (by Habermas' lens) prompted argument components. For instance, "*What do we think now*?" was often asked at the beginning of an argumentation episode to check students' current understanding. Questions such as "*So, what was your final answer*?" were used to request a final claim. Other non-rational questions were used to lead students through a method (e.g., "*Five is the greatest common factor of 5 × and 15, right*?"). The students' responses in this situation were usually nonverbal, such as head nodding or simply saying "*Yes*" to show their agreement with the teacher's way of thinking. Through Toulmin's lens, in this case, the teachers were constructing components of argument.

5 Discussion

Previous studies on teacher questioning have predominantly focused on descriptions of general categories or patterns of questions that teachers asked and perceived challenges or difficulties that teachers experienced with using questions to support classroom discussions. Drawing on Habermas' (1998) construct of rational behavior and Toulmin's (1958/2003) model for argumentation, this paper examined how teachers' questions that contained components of Habermas' rationality engaged students in collective argumentation. We viewed rational questioning as a long-term teaching intervention for the support of students' autonomy in producing rational argumentation practices. The results of this paper showed that the developed TRQF (see Table 1) captured most of the questions that the teachers asked during argumentation practices and provided insights into how a teacher's rational question related to (and prompted) argument components, which revealed aspects of student participation in argumentation.

The results of this paper indicated that most of the teachers' rational questions supported students toward directly contributing an argument component. Questions that contained different components of rationality promoted students' contributions of different components of arguments (see Table 6). Most of the epistemic rational questions (questions that contained an epistemic rationality component) were associated with *warrant*-related argument components that scaffolded students toward explicating adequate reasoning for their claims. Teleological rational questions were mainly associated with *claim*-related argument components that supported students in demonstrating their claims and achieving claims that were mathematically accurate and valid from the teacher's perspective. Communicative rational questions were associated with both warrants and claims; these kinds of questions helped students to present their arguments in comprehensible language to the classroom community.

In addition, we found that a rational question that contained a single component of rationality was often not sufficient to prompt a completely mathematically correct warrant or claim from a teacher's perspective. The role of teachers in supporting argumentation is to construct and sustain a mathematical community in which arguments are appropriately validated (Boaler, 1998). Thus, the use of follow-up questions with two rationality components served a critical role of inviting the students to engage in argumentation and prompted the students to revise their initially incorrect, incomplete, or ambiguous arguments. For instance, a question that combines epistemic rational questions with another component of rationality can be used to help students revise their initial warrants to make their explanations more complete and precise (intertwined with communicative dimension) or lead students to construct rebuttals for their initially incorrect warrants (intertwined with teleological dimension). In asking follow-up questions with multiple components of rationality, the teachers provided opportunities through which the students were able to deepen their conceptual understanding of mathematics and work through their errors. Despite their importance, these combinations of rational questions were not used exclusively; instead, they were useful in following up on students' ideas or arguments. This result provides additional empirical evidence to support Franke and colleagues' (2009) contention that sequences of questions are required to elicit a complete mathematical explanation or justification.

Questions that contained all three rationality components were primarily used to generate participation opportunities for student-led discussions. With limited supportive actions from the teacher, the students were expected to draw on their own rational behavior (epistemic, teleological, and communicative rationality) while making mathematical arguments and autonomously produce a rational discourse, which is a major goal of rational questioning (Douek, 2014). Although sometimes this type of rational question was not connected to any specific argument component, evidence exists from classroom dialogue and activities that a question such as "*Are there any questions about any of these patterns we just talked about*?" has the potential to provide students with opportunities to listen to and evaluate mathematical arguments that are made by other students. Students' participation in argumentation should not only involve the students as active contributors of argument components but also include students as active listeners, evaluators, or learners (e.g., Cramer &

Knipping, 2018; Krummheuer, 2007; Wood, 1999; Zhuang & Conner, 2020). The role of these rational questions is critical to trajectories of participation and distribution of responsibilities, and creates a culture of rationality in the classroom as described by Rodríguez and Rigo (2015).

Moreover, we also noted some questions without rationality components (non-rational questions) during argumentation episodes. The teachers mainly used non-rational questions to prompt the students to present task information or construct results for the final step of argumentation. Through Toulmin's lens, these non-rational questions prompted the students to provide data and final claims in the arguments, which aligned with the goal of promoting participation among the students. At other times, non-rational questions were used to lead students through a particular method that corresponded to what Wood (1998) called *funneling* questions. In this context, the teacher directs students to the answer that is desired by the teacher and does not provide students with opportunities to productively engage in argumentation practices.

6 Conclusions and implications

Drawing on Habermas' (1998) construct of rational behavior and Toulmin's (1958/2003) model of argumentation, we explored the rationality components of teacher questions as a way to promote student participation in collective argumentation. The conclusions of this paper are drawn from a limited number of lessons that were taught by two novice secondary mathematics teachers, which addressed topics in algebra and geometry. Despite the limitations of this data set, the data show the possibilities for the use of rational questioning in teachers' support of collective argumentation. We perceive Habermas' construct to be an important research perspective that offers a new and promising way to study different aspects of questioning strategies in terms of supporting collective argumentation. The results suggested that Habermas' construct, when used in conjunction with Toulmin's model, can be a powerful analytic tool for conceptualizing teachers' questions within argumentation practices. These two constructs are complementary as Habermas' construct captures teachers' intentions and consciousness that are inherent in argumentation (individual dimension), and the extended Toulmin's model frames the role of the teachers' questioning in order to facilitate the fundamental structure and steps of collective argumentation (social dimension).

Researchers (e.g., Boero et al., 2010; Cramer & Knipping, 2018) have previously applied Habermas' construct as a tool for analyzing how students engage in argumentation; however, this paper showed that Habermas' construct can also be used to analyze teachers' ways of facilitating argumentation in the classroom. The results of this paper also provided empirical evidence to support Boero et al.'s (2010) idea of integrating Habermas' construct of rational behavior with Toulmin's model as a powerful analytic tool to allow us to consider rational behavior in argumentation practices as facilitated by the teacher. It may be beneficial for future studies to continue to investigate the use of Habermas' theory to address relevant aspects of the complexity of the discursive activities, such as what constitutes good classroom-based collective argumentation. Additionally, future studies should examine the impact of the mathematical context (e.g., algebra, geometry, statistics) on teachers' use of rational questions in relation to student participation.

Some previous studies focused on the privilege of rationality components in students' proving processes and suggested that teachers should ensure that the epistemic and teleological components of rationality were always in the foreground (e.g., Morselli & Boero, 2009, 2011; Urhan &

Yüksel, 2019). Our study concerns the dynamic interplay between the components of rational behavior in teachers' questions. The results of this paper indicated that the teachers' use of combinations of components of rational questioning was beneficial to the facilitation of students' maintenance of a productive process of argumentation. We did not find that a single rationality component prevailed over or hindered another one; instead, we found that a combination of epistemic and teleological rational questioning was extremely helpful when students encountered a problem in teleological rationality. Communicative rationality was strongly intertwined with epistemic and teleological rationality in the process of constructing an argument that was accepted in the classroom community. We suggest that future studies continue to examine relationships between the components of rational behavior and how they interplay with each other for the purpose of improving student participation in proving and argumentation practices.

The use of rational questioning strategies in support of collective argumentation has implications for both teaching practice and teacher education. Firstly, the TRQF (see Table 1) that is developed in this paper may serve as a starting point to help teachers introduce their students to epistemic, teleological, and communicative rationality in mathematical argumentative discourse. The use of non-rational questions may be beneficial to some extent in supporting students' construction of data and final claims, even if they do not support students' rationalization of discourse. This result raises questions about whether mathematics teachers should limit these questions in order to successfully engage students in argumentation practices. Future study is needed to further investigate this issue. Secondly, although the significance of Habermas' construct of rationality as a tool for planning and managing teaching has been highlighted, it did not become an explicit tool for prospective teachers (as suggested by Boero et al., 2018). Teacher education programs may consider the introduction of Habermas' three components of rationality and the TRQF into pedagogy courses to provide teachers with additional ways of conceptualizing their questions during argumentation practices. Future research should continue to consider explicitly introducing Habermas' construct to teachers and making this construct more accessible to teachers.

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Declarations

Conflict of interest The authors declare no competing interests.

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