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Mathematics in the informal setting of an art studio: students' visuospatial thinking processes in a studio thinking-based environment

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Abstract
This study aimed to investigate seventh-grade students' visuospatial thinking processes in an art studio environment, where students were engaged with geometrically rich artworks. The students were asked to observe minimalist artworks, then create and critique their own and others' artworks based on the Studio Thinking Framework. Data were collected through interviews conducted with students, video recordings in the studio, and students' documents (sketches, artworks, and notes). The data were analyzed based on previous studies on spatial thinking and emergent data. The study's findings indicate that the Studio Thinking-based environment has the potential to elicit students' visuospatial thinking processes, mainly in recognizing shapes, decomposing and composing shapes, patterning, and transforming shapes rigidly and non-rigidly (scaling). The present study, which includes accounts of three studio works, suggests an emergent framework for the characterization of visuospatial thinking within a particular art-math-related environment. The findings of the study shed light on other studies on visual arts and mathematics education and on mathematical thinking and learning in informal learning settings.

Keywords Visual arts and mathematics · Visuospatial thinking · Studio thinking · Learning through arts · Informal mathematics education

1 Introduction

American artist Barnett Newman regarded the artistic act as a human birthright, arguing that human behavior has an inherently artistic nature in the 1947 essay "The first man was an artist"

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(O'Neill, 1990, p. 156). The arts have involved a variety of traditions and forms since antiquity (Hurwitz & Day, 2007), including mathematical approaches to visual arts (Danesi, 2020; Emmer, 2005; Gamwell, 2016). Visual arts provide opportunities for students to explore, discover, and express feelings and ideas that they consider important (Gardner, 1990), to think using an inquiry process in which sensibility and imagination are engaged, and to have aesthetic experiences (Eisner, 2002).

In a report commissioned by the Organisation for Economic Co-operation and Development on the impact of arts education, researchers discussed the methodological problems in studies on learning through the arts (e.g., measuring mediating factors, random assignment, lack of theoretical background) (Winner et al., 2013). How visual art is related to other disciplines is still open to debate. Seeking answers to the questions "What congruent cognitive processes exist between visual arts and mathematics?" and "How can students' thinking processes be explored in environments involving art-making?" is important for exploring the potential for collaborations between visual arts and mathematics.

In this study, we aimed to examine students' visuospatial thinking, which Goldsmith et al. (2016) suggested is a common point between the visual arts and geometry but has not yet been described in the interdisciplinary context of visual arts and mathematics. Visuospatial thinking is considered a crucial ability for the STEM disciplines as well as art and architecture (Newcombe, 2013) and is regarded as inherently related to mathematics or as a tool for supporting mathematical reasoning (Battista et al., 2018; Gutierrez, 1996; Hershkowitz, 1989; Mix & Levine, 2018; Presmeg, 1986).

This study investigated students' visuospatial thinking processes in the informal setting of an art studio. Informal learning settings are considered places where students have opportunities to apply mathematical knowledge, integrate what they have learned with other domains, and explore and learn from their experiences (Cooper, 2011). In art studios, students work on their projects, take risks, make mistakes, and envision new possibilities (Hetland et al., 2013). Seymour Papert explained how he was inspired by the nature of art studios, which led to his notion of constructionism (learning by making): "…the art room I used to pass on the way. For a while, I dropped in periodically to watch students working on soap sculptures and mused about ways in which this was not like a math class…" (Papert & Harel, 1991, p. 4).

Sensorimotor practices are fundamental to the arts, design, and crafts education according to the embodied cognition perspective (Groth, 2017), which claims the mind is naturally embodied and emotionally engaged (Lakoff & Johnson, 1999). Perceptual and motor systems play a significant role in shaping our concept of spatial relations (Lakoff & Johnson, 1999). Sensorimotor practices, such as drawing, talking, and using gestures, can improve students' spatial reasoning (Sinclair et al., 2018). Nemirovsky et al. (2013) also drew attention to the role of emergent tool fluency (perceptuomotor integration) in imaginary mathematical enactments by using the analogy of a musician playing the piano fluently. Despite this emphasis on the unity of perception and action, Sinclair (2009) argued that students are provided with limited exposure to the spatial, visual, and embodied aspects of mathematics compared to the numeric and algebraic aspects. Students' aesthetic and bodily experiences of mathematics in the informal setting of the art studio would thus offer new opportunities for mathematical thinking and learning.

This paper reports a case study involving three studio works that aimed to elaborate on students' emergent visuospatial thinking processes in the informal setting of an art studio. To make students' visuospatial thinking visible within environments involving art making, we designed three studio works through (1) the application of the Studio Thinking Framework

(STF), as it describes the nature and structures of the visual art studio based on habits of mind (Hetland et al., 2013); (2) studies on visuospatial thinking (e.g., Newcombe & Shipley, 2015; Sarama & Clements, 2009); and (3) the focus on minimalist artworks as a context for visual arts with explicit use of geometric shapes and forms.

2 Theoretical background

2.1 Mathematics and visual arts in education

Studies on mathematics and visual arts education have focused on a wide range of topics: theoretical discussions of the interdisciplinary education of visual arts and mathematics (Bickley-Green, 1995; Danesi, 2020; Hickman & Huckstep, 2003); the transfer of arts learning to geometry (Goldsmith et al., 2016; Walker et al., 2011); the integration of visual arts with mathematics in learning settings (Ernest & Nemirovsky, 2016; Portaankorva-Koivisto & Havinga, 2019; Schoevers et al., 2020; Shaffer, 1997); visualization in problem solving (Edens & Potter, 2007); aesthetical approaches to mathematics and visual arts (Eberle, 2014; Sinclair, 2006); and the effect of art-based instruction on motivation and performance in mathematics (Forseth, 1980).

This brief review of the literature highlights two important aspects explaining how the current study differs from the previous studies. First, very few studies have described the features of the tasks and the art and math-related learning environments (e.g., Ernest & Nemirovsky, 2016; Schoevers et al., 2020; Shaffer, 1997). These features include (1) a studio-like environment in which students have both physical and intellectual freedom (Shaffer, 1997) and (2) asking students to observe, create, and reflect on artworks (Schoevers et al., 2020). The questions "What kind of artworks were observed?," "Why were they observed?," and "What were the questions/tasks directed to students?" remain to be answered.

The second crucial issue concerns the nature of thinking and learning processes that emerge through the collaboration of visual arts and mathematics. Experimental studies have focused on measuring mathematics achievement scores and attitudes toward mathematics (Forseth, 1980); students' understanding of symmetry and use of visual thinking strategies in problem solving (Shaffer, 1997); and geometric ability (spatial sense and spatial visualization), geometrical creativity, use of geometric vocabulary, and visual arts perception (Schoevers et al., 2020). In these studies, students' performances were only measured with tests given before and after the experiments, highlighting methodological limitations (random assignment, concerns regarding the reliability and validity of the tests, lack of observational data) (e.g., Schoevers et al., 2020; Winner et al., 2013). There is a need to analyze students' spoken data and documents and explain the mathematical aspects of this discourse in such environments, which paints a fuller picture of what occurs during the experiments.

Some studies have suggested examining the commonalities between the visual arts and mathematics (Bickley-Green, 1995; Goldsmith et al., 2016; Portaankorva-Koivisto & Havinga, 2019). Visuospatial thinking could be a common element between these disciplines (Goldsmith et al., 2016; Walker et al., 2011). Interdisciplinary studies on mathematics and visual arts are rarely conducted in the field of mathematics education. This study contributes to this growing field by characterizing visuospatial thinking processes that emerged in the specific context of mathematics and visual arts and by providing detailed explanations of studio works, situating itself within the framework of Studio Thinking and studies on spatial thinking.

2.2 Visuospatial thinking

This study is grounded in research conducted in psychology, mathematics, and visual arts education. Visuospatial thinking is defined as "thinking about the shapes and arrangements of objects in space and about spatial processes, such as the deformation of objects, and the movement of objects and other entities through space" (Hegarty, 2010, p. 266). Studies have defined the components of spatial thinking and examined its underlying cognitive processes (e.g., Hegarty, 2010; Newcombe & Shipley, 2015; Tversky, 2005). Newcombe and Shipley (2015) proposed classifying spatial thinking into four categories: intrinsic-static, extrinsic-static, intrinsic-dynamic, and extrinsic-dynamic. Table 1 briefly describes and gives examples of each category, which provided the base for designing studio works and analyzing students' data.

Researchers in the arts, mathematics education, and psychology have discussed visual perception, which also serves as the foundation of the present study. Rudolf Arnheim (1969) described visual perception as visual thinking. Visual perception begins with encoding the remarkable arrangements of objects. As observers carefully view an object, their eyes become more equipped to see its details and explore its spatial features and relations. Perception is thus active in nature. Similarly, Noble et al. (2004) addressed seeing as an active process, especially when someone is not familiar with a visual display. They described three aspects of seeing: (1) *seeing-as* (multiple ways of seeing an object, e.g., the reversible seeing of a rabbit and a duck in a drawing), (2) *not-seeing a whole* (not seeing the object even though one knows it), and (3) *recognizing-in* (between seeing-as and not-seeing a whole). Additionally, they characterized students' mathematical learning as seeing a visual display (a velocity-time graph) in a new way (recognizing-in experiences). Researchers in mathematics education also highlighted the importance of processing visual information in geometric problems (e.g., Duval, 1999; Gal & Linchevski, 2010).

Table 1 Classification of visuospatial thinking (Newcombe & Shipley, 2015)

Category	Examples
Specifying objects and their parts, particularly in a distracting background (intrinsic-static)	Recognizing a shape or pattern (Battista et al., 2018) Disembedding (Newcombe & Shipley, 2015; Oltman et al., 2003; Sarama & Clements, 2009) Discriminating an object from another (Uttal et al., 2013)
Transforming objects (intrinsic-dynamic)	Rotation, cross-sectioning, folding, transformations in size, relating 2D and 3D representations (Battista et al., 2018; Cohen & Hegarty, 2014; Pittalis & Christou, 2010)
Encoding the relation between objects (extrinsic-static)	Determining the location of an object with respect to other shapes or a reference frame (scaling) (Frick & Newcombe, 2012; Uttal, 1996; Vasilyeva & Bowers, 2006)
Transforming the relation between objects (extrinsic-dynamic)	Navigation, perspective taking (Cohen & Hegarty, 2014; Newcombe & Shipley, 2015)

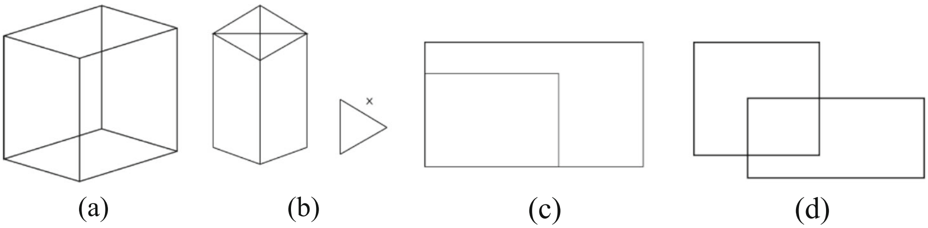


Fig. 1 Examples of shape perception. **a** Necker cube (Necker, 1832). **b** An example from embedded figures test (Oltman et al., 2003). **c–d** Examples of nested and overlapped figures

Perception of geometric shapes can be seen in studies on reversible figure perception (Attneave, 1971), embedded figures (Oltman et al., 2003; Sarama & Clements, 2009), and the artistic illusion of shapes and forms (Gombrich, 1984). For example, some pictures and geometrical forms can be seen differently after someone looks at them for a while (Attneave, 1971). The Necker cube is a prominent example of multi-stable perception (Fig. 1a). The embedded figures test (Oltman et al., 2003) highlights the process of finding a simple shape (e.g., the shape named "x" in Fig. 1b) in a complex figure. In mathematics education, recognizing simple geometric shapes in overlapped and nested structures has been described as disembedding and embedding for young children, an important skill for solving geometric problems (Sarama & Clements, 2009) (Fig. 1 c–d).

These studies explain why we selected particular artworks with a focus on minimalist artworks. In the present study, students were asked to observe minimalist artworks, including geometric shapes with embedded, hidden, overlapped, and reversible figures. Note that visual arts are not only about the perception of these shapes. Visual arts also involve a range of traditions, with artists creating artworks for a range of purposes: searching for beauty, creating wonder, expressing their fears, dreams, and fantasies, and communicating their states of mind and ideas (Hurwitz & Day, 2007). For example, Maurits Cornelis Escher did not only explore the ways of surface division but also wanted to create wonder, stating, "…I guess the thing I mainly strive after is wonder, so I try to awaken wonder in the minds of my viewers" (Ernst, 2012, p. 37). Furthermore, the interaction between visual arts and mathematics has been more evident at certain times in history (e.g., the Renaissance, geometric art after World War II, and computer graphics especially after the 1980s) (Emmer, 2005; Gamwell, 2016). This study stresses the importance of this interaction by focusing on mathematical aspects of minimalist artworks.

2.3 Studio thinking

The Studio Thinking Framework (STF) describes the nature of learning and teaching in visual art studios (Hetland et al., 2013). In the present study, it was used to design studio works specific to the context of the study and to make students' thinking visible within the art studio environment. The researchers identified four structures of a visual arts studio that teachers can use to organize their instruction and eight non-hierarchical studio habits of mind that teachers can use to teach visual arts courses (Table 2).

We focused on the first three structures of the studio and the habits of mind of observing, envisioning, exploring, and reflecting, considering their potential in making students' visuo-spatial thinking visible, a process that has common thinking dispositions (e.g., observing and

Table 2 Studio Thinking Framework proposed by Hetland et al. (2013)

Studio Thinking Framework	Descriptions
Structures of studio	
Demonstration	Introducing visual examples, artworks, tools, techniques, assignments, which was adapted to the context of study as introducing artworks of artists to observe individually and with peers
Students-at-work	Creating own artworks
The critique	Explaining and evaluating own and/or peers' artwork
Exhibition	Preparing artwork to present to the public
Thinking dispositions	
Observing	Looking at something closely, seeing what is seen and what is not seen
Envisioning	Mentally depicting something that is not seen directly and imagining possibilities
Reflecting	Explaining and evaluating own artwork, the art-making process, and the artwork of others
Expressing	Conveying meaning such as feelings, ideas, or thoughts within artworks
Exploring	Attempting to do something new and embracing the idea of making mistakes
Engaging and persisting	Working on a task over a period of time rather than giving up
Understanding art worlds	Learning the history of art and artworks, and learning to become part of the arts community
Developing craft	Utilizing tools effectively and carefully, and having a sense of which tools to use

describing, reasoning) with Artful Thinking (Tishman & Palmer, 2006). We chose this focus for three reasons: (1) observing and envisioning involves looking at something closely and mentally depicting something that is not seen directly (Hetland et al., 2013; Tishman & Palmer, 2006), which has a spatial nature (Goldsmith et al., 2016); (2) students' discussions about what they are doing, how they do it, and their judgments and written explanations enable them to reflect on their ideas (Hetland et al., 2013; Tishman & Palmer, 2006); and (3) students' every attempt during exploration might elicit different visuospatial thinking processes. Sketching is important for exploring spatial relations and detecting new perceptual cues (Tversky & Suwa, 2009), and it may support envisioning as a way of representing and communicating ideas (Goldsmith et al., 2014; Kantrowitz et al., 2017).

2.4 Mathematical thinking and learning in informal settings

Science centers, science and art museums, after-school programs, and other similar settings are considered informal learning settings, a term sometimes interchangeably used with out-of-school learning environments. There have been few studies on the evolving field of out-of-school mathematics education (e.g., Cooper, 2011; Guberman et al., 1999; Gyllenhaal, 2006; Mueller & Maher, 2009; Nemirovsky et al., 2013; Nemirovsky, 2018).

This literature review shows that researchers have investigated a variety of topics (e.g., emergent reasoning about fractions, the role of perceptuomotor integration in a mathematics museum exhibit, and visitors' and museum educators' experiences). However, very few studies have focused on the nature of mathematical thinking and learning in informal settings (Nemirovsky et al., 2013). One study examined the nature of learning in informal settings related to crafts and mathematics and suggested conducting case studies in this field (Nemirovsky, 2018). Students' learning of mathematics was explained from the perspective of emergent learning (the unpredicted flow of students' paths and freedom from predetermined outcomes) during basket weaving in an after-school club and in an art museum (Nemirovsky, 2018), highlighting the importance of perception and imagination (e.g., imagining a basket in half) and new images of mathematics. In line with these studies, we aim to contribute to this field by examining students' visuo-spatial thinking in the informal setting of the art studio.

3 Method

We employed the instrumental case study method since the studio works and environment we designed became tools for understanding students' thinking processes. In an instrumental study, the case plays a secondary role to better understand something else (Grandy, 2010). We first designed the studio works and environment since there was not a particular program for the interdisciplinary education of visual arts and mathematics. The present study involved three in-depth cases of studio works. Before this study, we conducted a preliminary study with different participants to design and revise the studio works.

3.1 Research context and participants

This study was conducted in the art studio of a public middle school in Ankara, Turkey. In the Turkish national education system, students are taught mathematics and visual arts as separate subjects in separate classes. None of the participants had received such interdisciplinary education. The current study was conducted outside of the scheduled classroom time for formal education. The participants had previously been taught the basic concepts of angles and length measurement, proportions, and geometric shapes and their properties.

The participants included six seventh-grade students (four females, two males). Two of them did not attend the final studio work due to personal reasons, and their thinking processes were analyzed due to their potential to affect the other students' thinking processes. Visual arts teachers recommended seven students, and mathematics teachers recommended six students (based on the students' interest, performance, and use of creative approaches to solving problems in mathematics and visual arts courses). Two students were recommended by both teachers. Students were asked if they were willing to participate in the study. Due to the mismatch between the time of this program and their schooling time, six students agreed to participate. Their appointed pseudonyms were Ali, Emre, Melek, Esra, Burcu, and Fatma. While Fatma and Melek had an interest in and were relatively successful at visual arts compared to their peers, Esra, Ali, and Emre were found to be more interested in and successful at mathematics. Burcu was successful and interested in both disciplines. Note that the purpose of the study was not to make comparisons among students. We assumed that the participation of students with different interests and performances in visual arts and mathematics would enable us to observe a variety of thinking processes.

3.2 Description of the studio environment and studio works

The studio environment, in which the participants were engaged in studio works, was described as "flexible," and the first author acted as their coach. The students were free to take breaks whenever they wished. During the studio works, the students each had the opportunity to explain their ideas to their peers by tracing and drawing the shapes of the artworks presented on the smartboard and the whiteboard. Table 3 briefly explains the studio works. Tables 4 and 5 describe the artworks used during each studio work, and present illustrative examples to give the reader a sense of the key feautures of interest in the artworks that students were presented with. References for the actual artworks used in the study are included.

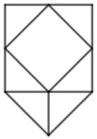
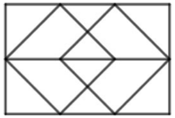
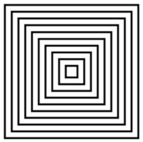
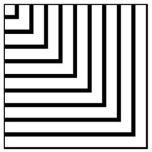
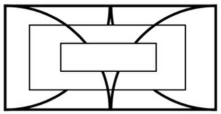
Students were engaged in different practices of arts education (creating their artworks and copying artists' artworks) based on art history. For example, van Gogh copied the drawings (see *The Daughter of Jacob Meyer*) in the manual of Charles Bargue many times during his career (Bakker, 2013). Through these different art practices, students mainly engaged with the use of geometric shapes. The use of color in the artworks was not the primary focus, although we appreciate its role in artistic engagements.

The following STF-derived principles (Kus & Cakiroglu, 2019) explain the features of the art studio environment that make students' thinking visible: (1) introducing sixteen minimalist artworks with different characteristics (overlapping and hidden geometric shapes, reversible figures from different points of views, Tables 4 and 5); (2) asking questions that prompt the students to observe artworks ("What shapes do you see?," "What else do you see?"); (3) asking students to create their own artwork and copy artworks (through hiding and embedding shapes, continuing an existing artwork as if creating a new artwork, and copying an artwork); (4) asking questions that prompt students to envision ("How would you look at this pyramid?," "Imagine embedding one shape into the other"); (5) encouraging students to sketch and explore new possibilities (asking students to imagine and sketch first, then experiment and revise their ideas and sketches); and (6) encouraging students to reflect on their own and others' artworks

Table 3 Description of studio works

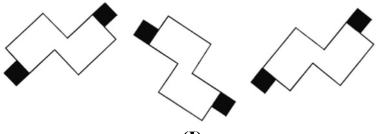
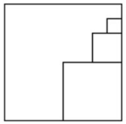
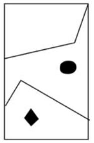
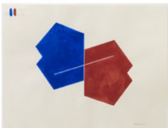
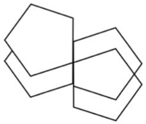
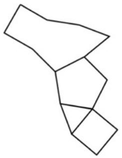
Studio Works	Descriptions
Studio Work 1	(1) Observing artworks (see Table 4) and finding hidden and embedded geometric shapes, including reversible figures, perceived as 2D and 3D; (2) creating own artwork with an aim of hiding or embedding shapes; (3) describing and evaluating their own and their peers' artworks.
Studio Work 2	(1) Observing a series of artworks to determine identical artworks (see Table 5); (2) creating their own artwork through completing an artwork from a series by Frank Stella (based on the task, "What if this artwork was the beginning of another artwork, how would you continue it?"); (3) describing and critiquing their own and their peers' artworks.
Studio Work 3	(1) Students' opinions regarding difficulties experienced in copying artworks; (2) copying or restructuring artworks with different characteristics, without the use of a measurement tool, to a 1:4 scale; and, (3) describing and critiquing restructured or copied artworks. The properties of each artwork were described in Table 5.

Table 4 Description of artworks of Studio Work 1 and their illustrative examples

Illustrative examples	Description of artworks
 (I)  (II)	(a) (Sol LeWitt, 2004a). Original artwork can be interpreted as a cube embedded in a circle, while it can also be interpreted as a hexagon consisting of three rhombi. Figure (I) alongside presents an exemplar of this situation of different interpretations (2D and 3D). Here, a cube is embedded in a hexagon. It can be perceived as 2D (a hexagon) and 3D (a cube). Other 2D shapes such as triangles, squares, and parallelograms also feature in the images (b) (Sol LeWitt, 2003). Original artwork can be perceived as 2D (e.g., an equilateral triangle consisting of three isosceles triangles) or 3D (a triangular pyramid embedded in a rectangle). The triangular pyramid can be seen from the top or front. Figure (II) illustrates an example of reversible perspective (depth reversal). Here, the cube embedded in a hexagon can be perceived as if looking from left or right. (c) (Sol LeWitt, 2004b). Original artwork can be perceived as 2D (e.g., two congruent parallelograms and three right triangles) and as 3D (e.g., a half of rectangular prism or a triangular prism). Figure (II) shows an example of a 2D representation of a triangular prism and reversible perspectives. In this figure, four triangular prisms can be perceived from different perspectives (front face to the right or to the left). The image can also be interpreted as overlapped trapezoids, squares, parallelograms, and triangles.
 (III)  (IV)	(d) (Frank Stella, 1967). Original artwork can be perceived both as 2D (nested rectangles) and 3D (a truncated rectangular pyramid seen from the top or bottom). Concentric rectangles are repeatedly used. Figure (III) shows an exemplar of using growing or shrinking concentric squares repeatedly for depth illusion. Here, 3D perception of the figure shows a truncated square pyramid seen from the bottom or top, while a 2D perception might show nested squares. (e) (Frank Stella, 1961). Original artwork can be interpreted as nested squares whose sizes change proportionally and also as a square pyramid as if looking from the top or the top and left side, depending on the observer's position and obliquity of the pyramid. It represents only two lateral faces of a pyramid, shown in the Figure (III), to fit into a square. Figure (IV) presents an example of a 2D representation (with two faces) of a pyramid.
 Mel Bochner. Four shapes, 1973-1976. Courtesy of the artist.  (V)	(f) (Mel Bochner, 1973-1976). This artwork has four figures that consist of hidden geometric shapes. Each shape involves a combination of a regular pentagon, triangle, and square so that there is no space between them (see also Table 5e). (g) (Frank Stella, 1971). Original artwork consists of embedded and hidden geometric figures (different-sized squares and sectors). The sizes of nested squares and sectors are getting smaller or bigger. Figure (V) alongside illustrates an example of this situation of superimposing shapes and hiding some parts of them. The figure shows superimposed rectangles and sectors of circles.

In this table some illustrative examples of embedded, overlapped, and hidden figures are presented. References for original artworks used in the study are included

Table 5 Description of artworks of Studio Works 2- 3 and their illustrative examples

Illustrative examples		Description of artworks
 (I)		Artworks of Studio Work 2 (a) (Frank Stella, 1964; 1968a; 1968b; 1968c). *V series* of Frank Stella were shown to students together. It involves rotating analogous series of the artworks. One of them (Frank Stella, 1968c) is not identical to the others. Figure (I) alongside presents an example of this situation of rotating series of shapes. The last one differs from the others.
 (II)	 (III)	Artworks of Studio Work 3 (b) (Robert Mangold, 1974). In the original, there are four squares located at the corners. The sizes of each square differ proportionally. The spaces between squares are also located proportionally. Original artwork has an ordered layout with decomposed shapes. Figure (II) shows an exemplar of this ordered layout. In this figure, squares are clearly seen (not hidden) and arranged in a way that their sizes getting smaller proportionally and one is placed on the top of another. (c) (Agnes Martin, 1957). Original artwork has an asymmetrical configuration of non-regular geometrical shapes. It has a scattered layout with decomposed shapes. In other words, they are not placed in a geometric pattern and geometric shapes are not hidden. Figure (III) shows an exemplar of a scattered layout of irregular geometric shapes.
 Mel Bochner. First fulcrum, 1975. Courtesy of the artist	 (IV)	(d) (Mel Bochner, 1975). Original artwork has a symmetrical-like configuration formed by a line segment located at the center, and involves two identical shapes. Two overlapping regular pentagons were hidden inside each shape. It has an ordered layout with composite figures. Figure (IV) shows a sample analysis of original artwork. (e) (Mel Bochner, 1976). Original artwork has an asymmetrical configuration of regular polygons that are hidden inside two-colored irregular shapes. The hidden shapes might be square, pentagons, and equilateral triangles that are connected with a common contour. Each side of regular polygons is equal. It has a scattered layout with composite figures. Figure (V) on the left shows a sample analysis of original artwork.
 Mel Bochner. Two shapes, 1976. Courtesy of the artist	 (V)	

In this table some illustrative examples of embedded, overlapped and hidden figures are presented. References for original artworks used in the study are included

("How did you do that?," "What made you say that?," "Did you experience any difficulty?").

3.3 Data collection

The study's primary data sources included (1) video recordings of each students' individual working process and the whole studio environment; (2) student documentation (artworks, written notes, and sketches); and (3) non-video stimulated recall interviews. Non-video stimulated recall interviews (De Smet et al., 2010) were conducted as it was not feasible to ask students to watch their actions from the recorded videos shortly after the studio works. During these interviews, one of the researchers asked the students to recall their experiences while working on their drawings and artwork creations and explain why they took certain critical actions. These were audio- and video-recorded and lasted between twenty and thirty minutes. During the interviews, the students were asked questions about their works (sketches, artworks, and notes in the sketchbook or observation sheets) depending on the nature of the studio works and their performances (e.g., "Where did you start from?," "What was your first idea in doing this?," "Why did you make these changes?," and "Why did you give up?"). Each of the students' working processes was video recorded. The duration of each video-recorded studio work for each student ranged from three to five hours, including breaks. If a studio session was not completed in a single day, it was continued on the following day. The students' mathematics and visual arts teachers were invited to attend the critiques of each studio work to help motivate and encourage their students and to allow us to seek experts' opinions on their students' artworks.

3.4 Data analysis

For data analysis, each studio work's video records and documents were imported into the qualitative data analysis software. All interviews with students and video records were transcribed by one of the authors. Timestamps were imported into the transcripts of video recordings, which enabled us to make connections between the video and the transcripts. Each timestamp showed a narrative that involved critical events, illustrating students' visuospatial thinking processes. The unit of analysis was students' simple sentences and extended utterances. The researchers identified a total of 460 critical events concerning visuospatial thinking from the video records (approximately thirty hours excluding breaks). Video recordings and documents were analyzed for each student during each studio work. The data were first examined holistically to make sense of the overall process. We then watched videos by looking back and forth repetitively and wrote memos describing categories based on a discussion between ourselves and studies on visuospatial thinking.

The data were analyzed for students' visuospatial thinking and studio thinking. At first, we did not use a particular framework or predetermine a coding list regarding spatial thinking. We employed the constant comparative method to analyze the visuospatial thinking data (Glaser & Strauss, 2006). This method was used for building an understanding of visuospatial thinking processes in a special case combining visual arts and mathematics rather than simply for data processing. There was not a predetermined coding scheme for data analysis. It is worth noting that our interpretation of students' thinking processes was not independent of the studies on visuospatial thinking and visual perception (Table 6). We also admit that the data analysis is inevitably influenced by our experiences and backgrounds. The first author had experience in

Table 6 Indicators of visuospatial thinking

Codes and descriptions	Terms from the relevant literature
Recognizing geometric shapes Identifying and naming 2D and 3D geometric shapes in the artworks	
Identifying shapes as real-life objects Relating geometric shapes with real-life objects based on visual similarities (e.g., likening a geometric shape to a path or a shoe)	*Recognizing-in experiences* (Noble et al., 2004)
Naming shapes and their properties Identifying geometric shapes based on their geometric properties (e.g., naming a geometric shape and explaining its length and angular relations, number of faces, vertices, and edges)	*Property-based visualization* (Battista et al., 2018); *recognition of 3D shapes' properties* (Pittalis & Christou, 2010)
Identifying shapes through disembedding and embedding shapes Picking out a shape that is embedded or hidden within other shapes (disembedding) or creating nested or overlapped shapes (embedding) (e.g., perceiving a reversible figure as 2D and 3D; drawing overlapping 2D and 3D shapes)	*Reversible figures* (Attneave, 1971); *disembedding* (Newcombe & Shipley, 2015); *embedded figures* (Oltman et al., 2003); *disembedding and embedding* (Sarama & Clements, 2009); *figure-ground perception of geometric shapes* (Gal & Linchevski, 2010)
Identifying shapes from different viewpoints Identifying and imagining shapes from a particular point of view (e.g., imagining the view of 3D shapes represented in the artwork and the view when the perspective is changed)	*Perspective-taking* (Cohen & Hegarty, 2014; Newcombe & Shipley, 2015)
Decomposing and composing shapes Placing shapes together to produce new shapes, or reducing shapes into smaller shapes	
Decomposing shapes Partitioning a whole shape into a smaller shape (e.g., slicing a shape into same-sized units, decomposing an irregular shape into regular polygons)	*Spatial structuring* (Battista et al., 2018; Sarama & Clements, 2009); *decomposition of 2D shapes* (Gal & Linchevski, 2010; Sarama & Clements, 2009)
Composing shapes Producing a new whole shape by combining individual units or units of units repeatedly, or combining different geometric shapes so as to make a coherent whole (e.g., rotating equilateral triangles to make a coherent whole such as hexagon)	*Composition of 2D shapes* (Sarama & Clements, 2009)
Spatial patterning Identifying and creating repeating and growing geometric patterns	
Recognizing a visual pattern Identifying the regularity in the arrangement of shapes, units of pattern, or the rule of a pattern in artwork (e.g., identifying a growing pattern in an artwork in which the sizes of squares are proportional to each other)	*Recognizing patterns and units* (Sarama & Clements, 2009); *Spatial pattern analysis* (identification of part of patterns) (Akshoomoff & Stiles, 1995)
Creating a visual pattern Locating/drawing shapes in a regular or predictable manner through informal or formal strategies (e.g., creating a pattern through rotation and reflection)	*Spatial pattern analysis* (integrating part to make a whole) (Akshoomoff & Stiles, 1995); *construction of geometric patterns in copying tasks* (Feeney & Stiles, 1996)
Transforming geometric shapes Identifying and imagining manipulations of shapes rigidly or non-rigidly that preserve their properties	
Scaling Identifying transformations in sizes of shapes and changing the sizes mentally by preserving the	*Scaling* (Frick & Newcombe, 2012; Hodgkiss et al., 2018; Möhring et al., 2018) *mapping* (Vasilyeva & Bowers, 2006); *reconstruction and scaling of spatial*

Table 6 (continued)

Codes and descriptions	Terms from the relevant literature
relations within the shape or between shapes (e.g., copying an artwork to a particular scaling factor)	*configurations* (Uttal, 1996)
Rigid transformations Identifying transformations in the orientation of shapes and changing the orientation of shapes mentally so that the size and properties of shapes do not change. (e.g., identifying identical shapes within an artwork, drawing rotated images of triangles to create artwork)	*Mental rotation* (Bruce & Hawes, 2015; Newcombe & Shipley, 2015); *mental rotation of 3D objects (identifying identical objects)* (Shepard & Metzler, 1971); *mental transformations of shapes* (Gal & Linchevski, 2010)

both mathematics education and visual arts and studied in art studios. Additionally, we analyzed the same data by using a predetermined coding scheme regarding studio thinking to describe at which of the studio structures critical events happen (observing artworks individually and with peers, students at work, critique) and the thinking dispositions related to a critical event (observe, envision, reflect, and explore).

To establish inter-rater reliability, a second coder, who is a researcher in mathematics education, analyzed one participant's data throughout three studio works. The participant was randomly selected, and the data concerning this participant comprised almost 20 percent of the total coded events. The percent agreements between two coders on the visuospatial thinking and studio thinking were above 80 percent (84 and 82.9 percent respectively), a reasonable degree of reliability (Miles & Huberman, 1994). The differences in coding were discussed until a full consensus was reached and the coding scheme was revised (see the final version in Table 6).

Table 7 The number of occurrences of visuospatial thinking processes at different structures of the studio

	Studio Work 1			Studio Work 2			Studio Work 3	
	OA	SW	C	OA	SW	C	SW	C
Recognizing shapes								
Identifying as real-life objects	7	9	1	7	7	5	8	6
Disembedding and embedding	48	34	8	10	3	4	1	3
Identifying from different viewpoints	7	6	2	0	1	4	0	1
Naming shapes and their properties	12	12	1	2	9	1	17	10
Total	**74**	**61**	**12**	**19**	**20**	**14**	**26**	**20**
Decomposing and composing								
Decomposing	9	3	0	16	7	2	5	5
Composing	2	6	0	1	17	3	0	0
Total	**11**	**9**	**0**	**17**	**24**	**5**	**5**	**5**
Spatial patterning								
Recognizing patterns	5	4	2	1	6	5	4	10
Creating patterns	0	23	0	0	11	0	0	0
Total	**5**	**27**	**2**	**1**	**17**	**5**	**4**	**10**
Transforming shapes								
Scaling	2	13	0	3	5	0	27	6
Rigid transformation	2	4	2	23	39	9	10	1
Total	**4**	**17**	**2**	**25**	**44**	**9**	**37**	**7**

OA observing artworks (individually and with friends), *SW* students-at-work, *C* critique

4 Findings

Four interconnected major visuospatial thinking processes emerged during the studio works: (1) recognizing geometric shapes, (2) decomposing and composing, (3) patterning, and (4) transforming shapes. Table 7 summarizes the number of occurrences of each thinking process that emerged during different structures of three studio works. Recognizing shapes was mainly elicited during Studio 1, decomposing and composing during Studio 2, patterning during Studios 1 and 2, and transforming shapes during Studios 2 and 3.

4.1 Recognizing geometric shapes

The students identified geometric shapes as real-life objects, both when observing artworks themselves and with their peers (e.g., a wall and path in the artwork of Frank Stella, *Hampton roads*, 1961; Table 4e) and also when creating and copying artists' artworks. During Studio Work 3, for example, students were asked to copy artworks on a 1:4 scale. This process required students to observe the original artwork, envision the relationships within and between shapes, and explore the position of the shapes on larger paper. Ali and Fatma respectively imagined a dragon and the shoe of a cartoon character in the artwork of Mel Bochner (*Two shapes*, 1976; Table 5e) that included hidden geometric shapes (Fig. 2). Fatma, for example, stated, "I couldn't adjust the size; I could not do it before I imagined it as the shoes. I mean, I had a hard time. It was difficult for me before I could not think this way; then I likened it to something, and it was easy."

Students also recognized geometric shapes through disembedding and embedding shapes. During their individual observations of artworks, they identified more 2D shapes than 3D shapes. All identified the cube shape in the artwork of Sol LeWitt (Table 4a). Burcu was the only student to identify the pyramid from the artwork of Frank Stella (*Hampton roads*, 1961; Table 4e). While making observations with their peers, they discovered new shapes and reflected on their properties. Students also identified geometric shapes (e.g., two square pyramids embedded inside of a square prism, nested cubes, nested triangular prisms [Fig. 3a]) when they were asked to embed or hide shapes to create their artworks and when they examined their peers' artwork during the critique. During the critique of Studio Work 1, Emre

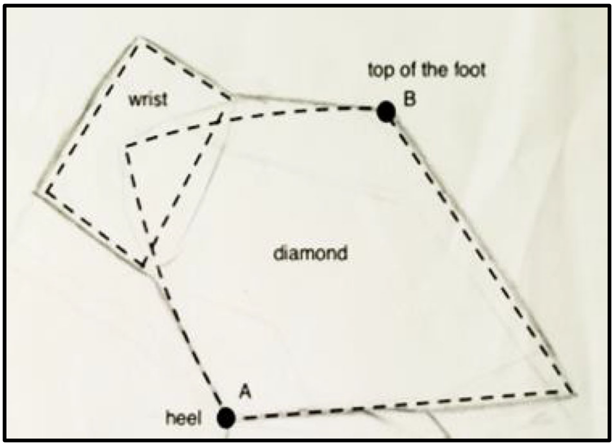
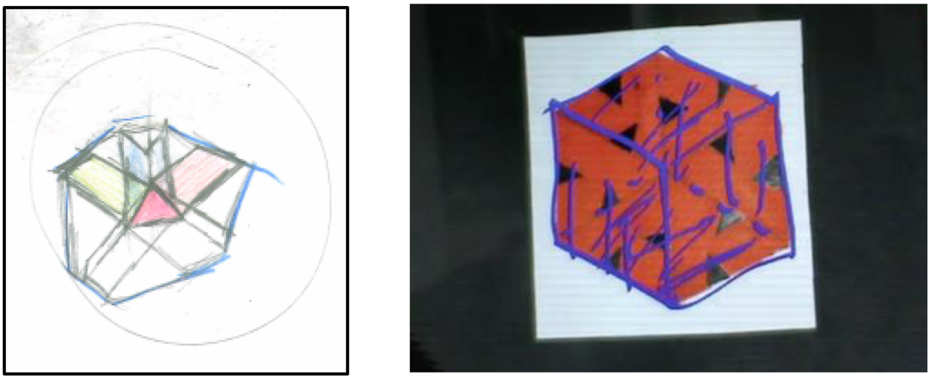


Fig. 2 Identification and representation of a shoe of a cartoon character (Fatma)



(a) (b)

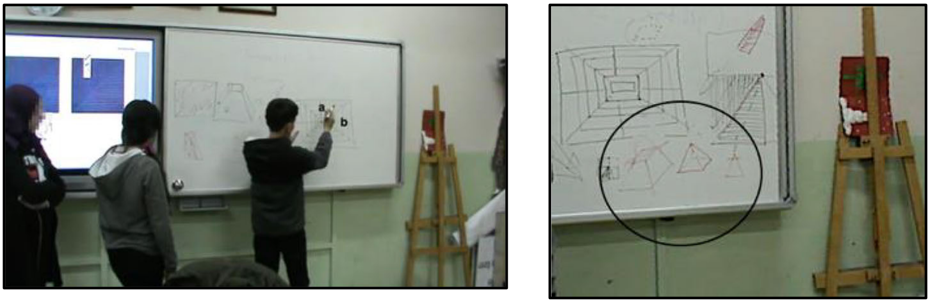
Fig. 3 Embedding and disembedding shapes

described his ideas and expressed how he noticed the cube shape in Ali's artwork, stating, "He is not seeing it, but there is a cube here… Teacher, here is the base and the top, and here are the side faces, I see it!" (Fig. 3b)

The third finding was that students often named 2D geometric shapes based on the number of their sides and 3D geometric shapes based on the number of side faces or their bases. During individual observations of the artwork in Studio Work 1 (Frank Stella, *Tomlinson Court Park*, 1967; Table 4d), the students identified rectangles. When the researcher asked them to observe it again along with their peers, they recognized other shapes and reflected on the shapes' properties to justify their identification, particularly for a pyramid and a prism. The following conversation between the students exemplifies the confusion that some of them experienced with pyramids during Studio Work 1. It also shows how students were encouraged to reflect on ideas by answering the question, "What makes you say that?" Emre perceived the artwork (Frank Stella, *Hampton roads*, 1961; Table 4e) as a pattern of squares rather than a pyramid since he seemed to conceptualize a pyramid with only four side faces:

Emre	"I don't think this can be a pyramid anyway. It is a painting created by growing in certain dimensions only in a certain order."
Researcher	"What makes you say that?"
Emre	"This is not a complete pyramid. There are just two sides of the pyramid, and two sides are missing."
Melek	"A pyramid should have three sides. It was a pyramid from the top (Table 4d), but not this (Table 4e). These are squares."

The students also tried to envision shapes from different views. During their individual observations, except for Burcu, the students perceived the artwork as nested rectangles at first glance (Table 4e). When consulting with peers, the students discussed whether two of the artworks (Table 4d and e) could be perceived as pyramids. Then, four of the students (Ali, Fatma, Burcu, and Melek) attempted to imagine how the pyramids could be seen and might be represented from a particular point of view when the researcher asked them to envision the image as a pyramid. During this process, they compared the different views of the pyramids, and Burcu



(a) (b)

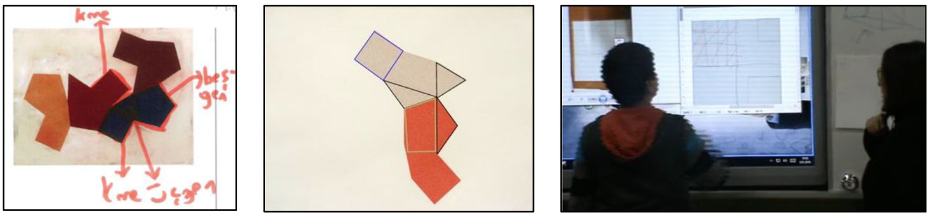
Fig. 4 Identification of the pyramids from different viewpoints in the artworks of Frank Stella

and Ali physically moved around to explain the view of the pyramid (Fig. 4a). Further in the process, they discussed how a pyramid is seen from the top when it is cut (Fig. 4b):

Researcher	"How do you imagine it as a pyramid?" (Table 4e)
Fatma	"Pyramid seen from side-view."
Researcher	"How would you look from the side?"
Fatma	"From the side and from the top."
Ali	"We stand here and are not seeing the other parts." (Fig. 4a, Ali indicates points "a" and "b" and demonstrates with body movements)

4.2 Decomposing and composing shapes

To decompose shapes, the students partitioned irregular shapes into geometric shapes with which they were familiar and then partitioned the shapes into equal parts. During individual observations of Mel Bochner's artwork (Fig. 5a), most students tried to decompose the irregular shapes. Ali and Melek decomposed an irregular shape into geometric shapes (square, pentagon, and triangle) (Fig. 5a). During Studio Work 3, the students were asked to copy another work by Mel Bochner (Fig. 5b). Throughout this process, they did not decompose the irregular shapes. While critiquing their drawings, the researcher asked the students to observe the artworks again and find the geometric shapes within them. They identified triangles and squares and named a trapezoid as a rectangle due to its visual similarity (Fig. 5b). They also



(a) (b) (c)

Fig. 5 Decomposing shapes in the artworks of Mel Bochner and Robert Mangold. **a** Ali pointed to the square, pentagon, and triangle. **b** Students pointed to the square (purple), triangles (black), and rectangle (green), traced and colored by the researchers. **c** Structuring the square into units (Emre)

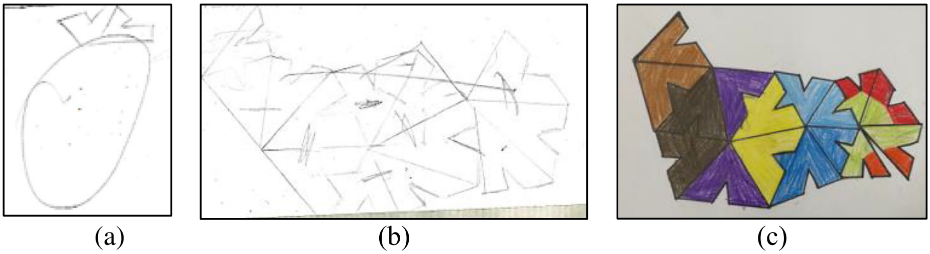


Fig. 6 Process of shape composition (Fatma). **a** A sketch of a head. **b** A sketch of a bird. **c** The final version of the bird

structured squares into the units of squares in Robert Mangold's artwork, in another example of decomposing (Fig. 5c).

To compose shapes, Fatma, Ali, and Emre tried to make a new whole shape by rotation, particularly during Studio Work 2, while others did not construct a coherent whole. Students were asked to choose one of Frank Stella's *V Series* (Table 5a; see also Fig. 10) and create their artwork by rotation. Fatma, for example, drew freehand sketches of rotated paired triangles and sketched a head and a bird (Fig. 6). She first created the image of the head by rotating triangles around a certain point. However, she thought it was too difficult: "The first thing that came to my mind was to fill a circle with triangles, make the hair out of triangles, but then I gave up as finding the exact angles was very difficult." While drawing the final version of the bird (Fig. 6c), Fatma used both intuitive and formal strategies to compose the shapes. She tried to measure the angles of triangles and construct equilateral triangles, using freehand sketches in areas where she had experienced difficulty in drawing the rotated images of triangles using a protractor.

Ali, on the other hand, combined paired triangles by employing a mathematically valid strategy to create his artwork. After finishing (Fig. 7a), he realized the whole shape (Fig. 7b) and divided the hexagon into identical triangles by comparing it with his artwork (Fig. 7c). This process is an example of simultaneously using both the decomposition and composition processes.

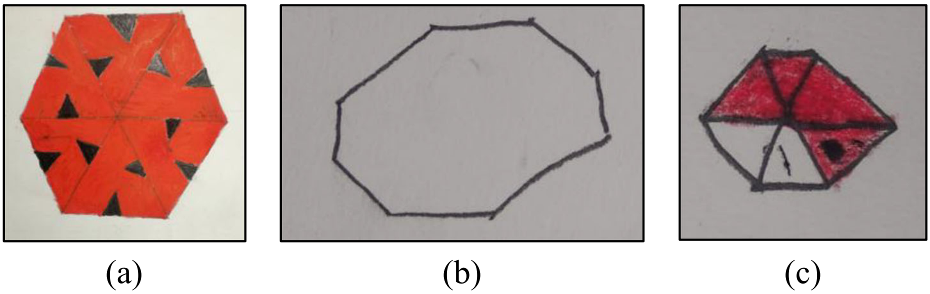
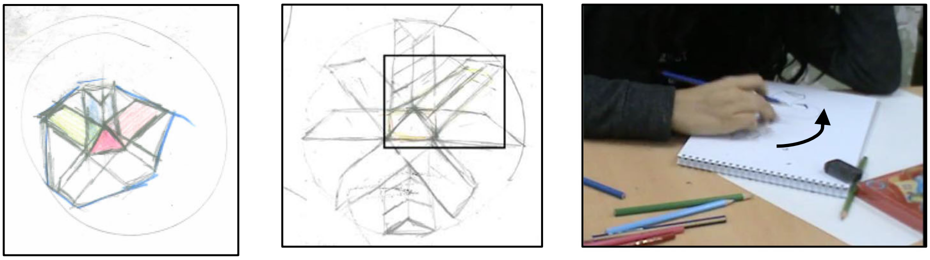


Fig. 7 Decomposition and composition of Frank Stella's *V Series*. **a** A composition of V series by Ali. **b** The recognition of the composite shape. **c** The decomposition of the hexagonal shape into six parts



(a) (b) (c)

Fig. 8 Sketches of a repeating pattern of triangular prisms (Melek)

4.3 Spatial patterning

The artworks involved growing patterns of squares, rectangles, or triangles. During their individual observations, one student (Ali) identified patterns and noted, for example, Frank Stella's artwork "continues and grows as a rhythm" (*Itata from the V series*, 1968; Fig. 10). In the subsequent process (observing artworks with peers and discussing artworks during critique), other students not only identified the patterns but also tried to discover the rule of the patterns. They structured the shapes into the units of shapes in the works of Frank Stella and Robert Mangold (Studio Work 3) to identify the rule of the pattern. During the critique of Studio Work 3, Melek described a pattern and explained its rule by segmenting the largest square into smaller, equal squares (see findings regarding scaling).

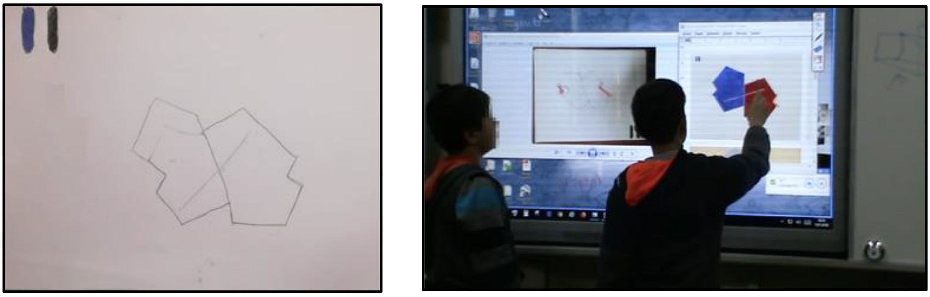
The students also created visually repeating and growing patterns by employing informal strategies: using circles to make the lengths of a shape equal (Melek), leaving equal distances between shapes when increasing the sizes of shapes (Ali), and rotating shapes in a particular manner without paying attention to their precise measurement (Melek). During Studio Work 1, students sketched shapes, embedded them into each other, and explored what happened. For example, Melek created a pattern by rotating triangular prisms in a circular movement and informally used a symmetry line to draw each triangular prism (Fig. 8a). After drawing two triangular prisms, she rotated her hand as a sign of rotation and drew a circle to make the prisms' sizes equal (Fig. 8b). The pattern she used combined triangular prisms on a common face, rotating them around a particular point in a circular motion so that they had symmetrical positions within the circle. During this exploratory process, she observed what she drew and envisioned what would happen if she rotated the triangular prism.

4.4 Transforming geometric shapes

4.4.1 Scaling

When the students were asked to copy artworks to a scale factor of 1:4, they attempted not only to multiply the lengths of the shapes to the scaling factor visually but also to encode geometric cues in the artworks (length/angular relations, shapes and properties, geometric properties of the layout).

Ali and Melek reflected that they mentally envisioned the enlarged sizes of shapes to make them four times larger. Ali said, "I looked at the distance between the shape and a finger like this; if it is four times, 1-2-3-4, there is a gap between it. When I look at that distance, it is similar." He multiplied the length of a shape or distance between the shapes by four mentally. Besides preserving the proportional relation between the original artwork and the enlarged



(a) (b)

Fig. 9 Copying the artworks of Mel Bochner at a 1:4 scale (Emre)

reproduction, Melek managed to identify the proportional relation between squares in Robert Mangold's artwork (*Four squares within a square*, 1974; Table 5b). She reflected on this process during the critique, in which the researcher asked students to propose a strategy to solve a problem in their peer's (Fatma's) drawing, stating, "We could use something like this: this is 1x, and this is 2x, which is 3x, 4x….For example, if we take its edge as two, then this is four times."

The students also relied on geometric cues in the artworks. For example, the researcher asked the students to observe and critique their friend's drawings, then explain how they could fix them during Studio Work 3. Ali critiqued Emre's and Fatma's drawings and analyzed the length relations between congruent shapes, stating, "The shape is not that symmetrical; for example, here is longer. The mouths of both shapes are at an equal distance. Here, these have the same lengths, but these are not equal" (Fig. 9).

4.4.2 Rigid transformations

Students identified and created rotational and flipped transformations, especially during Studio Work 2. First, the students were asked to observe and identify similarities and differences between Frank Stella's artwork series (Table 5a, see also Fig. 10). In the following conversation, the researcher asked them to explain how the directions of the shapes were different. This question elicited students' identification of rotation (Burcu

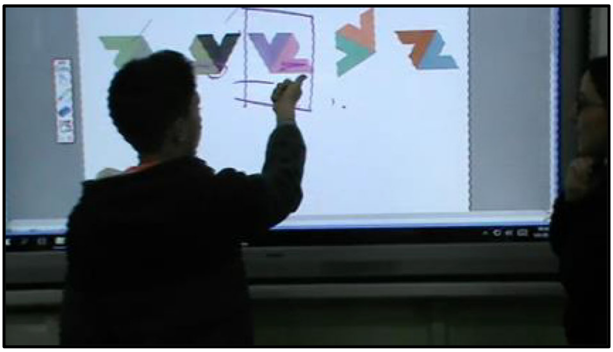


Fig. 10 Identifying the transformations of shapes on the smartboard (Ali)

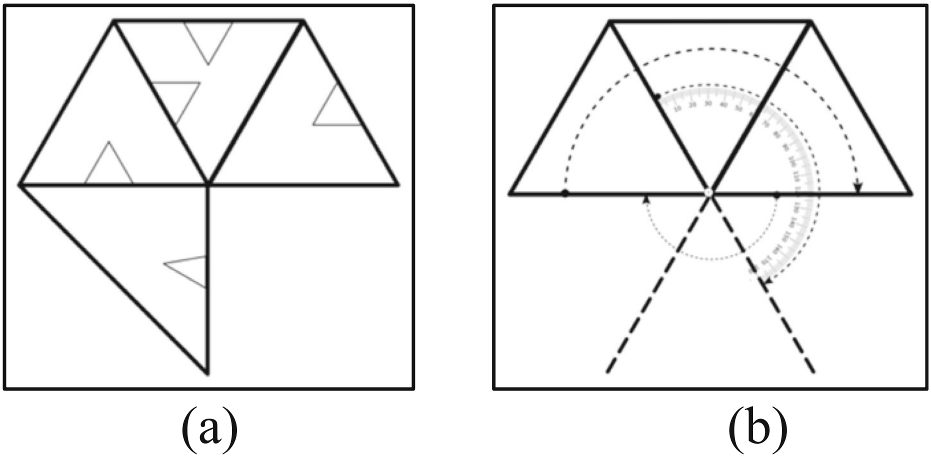
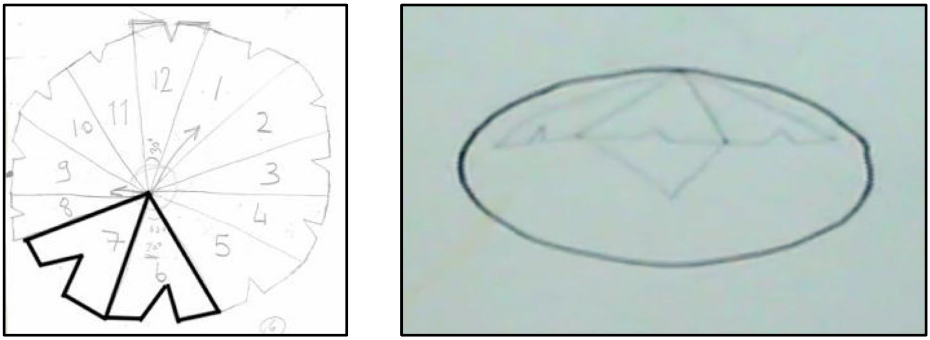


Fig. 11 Drawing nested *V-series* with a protractor (Ali). **a** The researchers represented Ali's drawing. **b** Ali's use of a protractor to draw rotated images of a triangle illustrated by the researchers with dashed lines

and Melek), reflection (Fatma), flip (Esra), and combinations of different transformations (Emre and Ali). The discussion also showed how the students envisioned these transformations. They explained holistically, referring to the change in direction (Emre and Esra) and also pointing more analytically to specific parts of shapes, such as the sides or vertices (Ali, Fatma, Burcu, and Melek):

Researcher "Now, you've taken note that their directions are different. How are they different?"
Esra "It's flipped. It's turned over, it's flipped." (uses her hands)
Researcher "Well, what else can it be apart from flipping?"
Fatma "Can I say something? Teacher, it is already a reflection in the mirror here; this is what is seen here."
Researcher "Do you agree with Fatma?"
Ali "Here, both of them are happening; if it is flipped to the side (vertically to the right), it becomes like this one (pink-colored shape). If we turn like this (uses hand gesture for rotation), it becomes like this one again. This part comes to the ground." (matches the parts of shapes) (Fig. 10)

While creating artwork, students observed their sketches and what did or did not work, envisioned how they would rotate the V-shape to create their artworks, and reflected on the problems in their drawings. For example, Ali drew a rotated image of a triangle for his first attempt. Then, the researcher asked him to rotate triangles to make a coherent whole. During this process, he seemed to experience difficulty in drawing rotated yet connecting images of triangles and used a paper triangle to imagine these images. However, he was not satisfied with the result, stating, "Teacher, it is not an equilateral triangle; it became a right-angled triangle with 90 degrees" (Fig. 11a). In the subsequent process, he discovered the angle of rotation by dividing the protractor visually into three equal parts and then turned it a few times to locate each triangle (Fig. 11b).



(a) (b)

Fig. 12 Representation of the rotation of shapes during artwork creation (Emre). **a** The freehand sketch of a clock by rotating the unit of the *V series*. **b** Reconstruction of the sketch of the clock using a protractor

Emre, on the other hand, drew rotated images of a triangle through freehand sketching by making use of a paper triangle. Initially, he sketched a clock (Fig. 12a), but then, it was seemingly difficult for Emre to draw each triangle and their rotated images with due consideration of their angles and sizes using a protractor (Fig. 12b).

5 Conclusion and discussion

Through conducting an in-depth case study, this study described students' visuospatial thinking processes that emerged in the informal setting of an art studio by identifying four major interrelated spatial thinking processes: recognizing geometric shapes, decomposing and composing shapes, patterning, and transforming geometric shapes. In addition, several visuospatial thinking sub-processes were involved, such as identifying shapes with their properties, relating geometric shapes with real-world objects, disembedding and embedding shapes, scaling, rotating shapes mentally, and perspective taking.

Visuospatial thinking has been suggested as a common point between visual arts and geometry (Goldsmith et al., 2016). This study shows a case supporting this argument. We examined and described students' emerging visuospatial thinking processes in an art studio environment specifically designed to elicit students' thinking processes. In contrast, previous studies have focused on measuring students' performances before and after the experiments (e.g., Schoevers et al., 2020; Shaffer, 1997) or transfer of learning in arts courses (e.g., Goldsmith et al., 2016; Walker et al., 2011) and have not described the students' thinking processes that have emerged through this collaboration. In addition, this study provided detailed information about the characteristics of the studio works and artworks by extending the previous studies on the interdisciplinary education of visual arts and mathematics.

The present study concludes that this specific studio environment has the potential to make students' visuospatial thinking processes visible. Observing different artworks (e.g., artworks with hidden shapes, shapes from different points of view, and artworks with symmetrical/asymmetrical layouts) elicited a variety of students' thinking processes. While observing artworks with their friends seemed to foster the participants' discussion of mathematical ideas (e.g., the rule of the pattern of growing squares, properties of a pyramid, recognition of new shapes), the individual observations helped us understand the students' initial perceptions.

Artwork creation and critique also have significant potential in making students' thinking visible. This study not only focused on students' finished artworks but also considered the process of creating artwork. During artwork creation, students were often in the processes of observation, envisioning, and exploration, in which students made a variety of changes (adding a new feature or deleting some parts) to their sketches and explored the relations between geometric shapes, a finding that is consistent with the unanticipated nature of the pedagogy of emergent learning (Nemirovsky, 2018) and with the idea that the research process in art leads students to new places (Marshal & D'Adamo, 2011).

The unique contribution of the current study is threefold. First, this study provided insight into students' visuospatial thinking processes in the informal setting of an art studio in which students were engaged in minimalist artworks. Making students' thinking visible is crucial to diagnosing how students think, learn to think, and learn specific concepts of disciplines (Tishman & Palmer, 2006). In attempting to trace students' thinking processes by using a variety of artworks and asking them to create and critique (explain and evaluate) their own and friends' artworks during three in-depth studio work activities, we hope to propose a useful categorization of students' visuospatial thinking processes by providing examples in this particular context. Still, a need remains for additional studies to elaborate on the visuospatial thinking processes in such informal settings.

The second major contribution relates to the study's theoretical contributions. The current study shows how the STF, designed in the field of arts education, and studies on spatial thinking were adapted to the interdisciplinary context of mathematics and visual arts. We attempted to interpret these studies from arts and psychology perspectives so that they are more sensitive to the nature of mathematics education. This study was enriched by the fields of psychology, arts education, and mathematics education. It also contributes to these disciplines by providing new interpretations of visuospatial thinking processes. Additionally, the minimalist artworks were interpreted in the context of mathematics education, demonstrating how a particular artwork could be used for educational purposes, especially in out-of-school settings (e.g., an art museum).

This study also offers new opportunities for mathematical experiences in informal settings. In this art studio, students had sensorimotor experiences (e.g., sketching and the use of bodily gestures and tools) in analyzing, creating, and copying artworks. The following are interesting instances arising from the study: Ali's and Fatma's application of the protractor in a new way, with the use of holistic, imaginary and property-based analytic strategies (Battista et al., 2018), Melek's use of hand gestures, including sketching, to imagine how triangular prisms are embedded in a circular way, and Ali's and Burcu's use of bodily gestures to experience rotation or a change in perspective. These examples primarily highlight the bodily and imagistic aspects of mathematics (Abrahamson & Lindgren, 2014; Nemirovsky et al., 2013). In the classrooms, students are not given opportunities to experience these sensations (Abrahamson & Lindgren, 2014; Sinclair, 2009). Students' bodily experiences, supporting perceptomotor integration as when an artist draws a figure fluently, are important in enacting the mathematical imaginary (e.g., Nemirovsky et al., 2013). Students also created their artworks based on their aesthetic sensibilities, resulting in a variety of mathematical explorations (e.g., circular compositions, including how to decompose a circle into twelve equal units and how to embed triangular prisms in a circular way so that they share a face). Such aesthetic sensibilities could evoke students' mathematical inquiries and develop a personal and intimate connection with the object or idea under study (Sinclair, 2006). In these regards, informal settings such as art studios could offer new opportunities for sensorial, bodily, and aesthetically rich engagements in mathematics education.

5.1 Future research and limitations

This study contributes to the comprehensive literature on mathematics education in a modest way. The emerging categorization of visuospatial thinking in the particular context of visual arts and mathematics raises new avenues for future research in the fields of visual arts and mathematics and for learning mathematics in informal settings. These avenues include (1) the validation and refinement of this emergent categorization in similar and different contexts and (2) the investigation of mathematics learning in such informal settings. These issues might be investigated from the perspectives of (1) learning to see a visual display in mathematics, (2) learning from and through sensorimotor experiences, and (3) identifying the features of tasks for interdisciplinary collaborations.

The findings of the current study should be interpreted within its limitations. We describe an intensive study with six seventh-grade students through three studio works that focused on two-dimensional minimalist artworks explicitly involving geometric shapes. This initial categorization may contribute to building a theoretical framework in this field by examining other art movements without explicitly using geometric shapes and considering 3D artworks as well. Additional studies with more participants could provide more support for the data and consider students' gestures and eye tracking. This study sheds light on research on visual arts and mathematics education and on mathematical thinking and learning in informal learning settings.

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Data availability and material Not applicable.

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Declarations

Ethics approval Ethical approval was obtained from the Human Research Ethics Committee of the researchers' institution and the Ministry of Education. Parental consent was obtained for all participants.

Competing interests The authors declare no competing interests.

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