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Making sense of student mathematical thinking: the role of teacher mathematical thinking

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Abstract
In mathematical whole-class discussions, teachers can build on various student ideas and develop these ideas toward mathematical goals. This requires teachers to make sense of their students' mathematical thinking, which evidently involves mathematical thinking on the teacher's part. Teacher sense-making of student mathematical thinking has been studied and conceptualized as an aspect of teacher noticing and has also been conceptualized as a mathematical activity. We combine these perspectives to explore the role of teacher mathematical thinking in making sense of student mathematical thinking. In this study, we investigated that role using video-based teacher discussions in a teacher researcher collaboration in which five Dutch high school mathematics teachers and one researcher developed discourse based lessons in cycles of design, enactment, and evaluation. In video-based discussions, they collaboratively reflected on whole-class discussions from the teachers' own lessons. We analyzed these discussions to explore the mathematical thinking that teachers articulated during sense-making of students' mathematical thinking and how teachers' mathematical thinking affected their sense-making. We found five categories concerning the role of teacher mathematical thinking in their sense-making: flexibility, preoccupation, incomprehension, exemplification, and projection. These categories show how both the content and the process of teacher mathematical thinking can support or impede their sense-making. In addition, we found that the teachers often did not articulate explicit mathematical thinking. Our findings suggest that sense-making of students' mathematical thinking requires teachers to (re-)engage in reflective thinking with regard to the mathematical content as well as the process of their own mathematical thinking.

Keywords Teacher sense-making · Teacher mathematical thinking · Whole-class discussions · Teacher noticing · Video-based discussions

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1 Introduction

Developing students' mathematical thinking requires that mathematical thinking takes place and that the teacher has some access to students' thinking in order to make decisions about how to respond. Teachers can create these conditions in discourse-based lessons—lessons in which students work on a mathematical problem or question and reflect on each other's various solution methods and ideas in a whole-class discussion. Teachers' sense-making of students' mathematical thinking is crucial and has been investigated and conceptualized from multiple perspectives.

Several researchers who conceptualize the work of mathematics teaching as inherently mathematical describe teachers' sense-making of student thinking as a mathematical activity (e.g., Ball, 2017; Ball & Bass, 2000; Chick & Stacey, 2013). Making sense of a student's mathematical thinking requires the teacher to adopt the student's perspective, to imagine what the student's line of thinking could be, and to connect this to the teachers' own thinking as well as the other students' thinking. This is teacher mathematical thinking that has to happen *on the spot*, each time a teacher is making sense of student thinking, whether it is in the moment during whole-class discussion, before or after a lesson, or during a professional development setting.

Sense-making of student thinking, by which we mean interpreting student thinking, is closely connected to teacher noticing (for an overview, see Schack et al., 2017; Sherin et al., 2011). The widely used conceptualization of noticing by Jacobs et al. (2010) comprises (1) attending to student thinking, (2) interpreting student thinking, and (3) deciding how to respond. During whole-class discussions, these activities have to happen almost instantaneously (Jacobs et al., 2010). Professional development settings that focus on teacher noticing usually involve teachers in a situation apart from—but close to—their actual teaching practice, such as "video clubs" (Sherin & Han, 2004). Such professional development relates to Mason's (2002) view of noticing as a discipline: "a collection of practices which together can enhance sensitivity to notice opportunities to act freshly in the future" (p. 59). Studies such as those of Sherin and van Es (2009) and Teuscher et al. (2017) support the underlying premise that development of sense-making in video-based discussions goes hand in hand with improved sense-making during actual interactions with students.

Although research into teacher noticing has brought many insights into how teachers make sense of student thinking, the role of the teachers' mathematical thinking in such sense-making has not been thoroughly investigated, and more research is needed to "understand the mathematical reasoning that underlies the decisions and moves made in teaching" (Ball et al., 2008, p. 403).

In this study, we investigated five teachers' sense-making of student thinking during *video-based discussions* in which they collaboratively watched and reflected upon episodes from video recordings of whole-class discussions selected from the teachers' own lessons. We explored the *role of the teachers' own mathematical thinking in making sense of student mathematical thinking*: how teacher mathematical thinking occurs during their sense-making and how it affects their sense-making.

2 Theoretical background

2.1 Mathematical thinking

We conceptualize mathematics as a human activity, in accordance with Freudenthal (1973), Schoenfeld (1992), and many others, and this activity is what we call mathematical thinking. If, as Watson and Barton (2011) state, "Knowing mathematics means *being able to use* mathematical concepts mathematically" (p. 66, emphasis added), then mathematical thinking means the activity of actually *using* mathematical concepts mathematically. Mathematical thinking takes on different forms which are inherently intertwined, although scholars have decomposed mathematical thinking into categories or highlighted specific forms, such as conceptual understanding (Kilpatrick et al., 2001), problem solving (Schoenfeld, 1985), generalizing and specializing (Mason, 1985), or modeling (Blum et al., 2007). By mathematical thinking, we do indeed mean all those activities, not as practicably separable categories, but as an amalgam of overlapping and mutually interdependent activities.

Two activities are pivotal to the view of mathematics learning that underlies this study: *solving problems* and *reflection*. A problem is defined by Schoenfeld (1985) as a task for which a solution method is not obvious: it requires mathematical thinking. Furthermore, as argued by Wheatley (1992), it is not enough to solve problems; students should also reflect on *the way* they solve problems. Ongoing reflection is described by Skemp (1971) as a prerequisite for learning mathematics with understanding. Wittman (1981) discusses reflective thinking to be complementary to intuitive thinking: both are essential in learning mathematical thinking. To develop trustworthy intuitions requires experience with reflective thinking, so students should be supported in reflective thinking. Discourse-based mathematics teaching is aimed at providing rich opportunities for both problem solving and reflection, by encouraging students to think about problems, explain their thinking, and discuss each other's ideas in a whole-class discussion (Wheatley, 1992).

2.2 Teacher mathematical thinking

Most research into the special mathematical thinking involved with teaching mathematics has focused on the term "knowledge." Based on Shulman's work (1986), scholars have developed categorizations of knowledge for teaching mathematics, such as Pedagogical Content Knowledge (Magnusson et al., 1999; Shulman, 1986) and Mathematical Knowledge for Teaching (Ball et al., 2008). A widely acknowledged problem of such cognitive approaches is that knowledge is only relevant if it is available to the knower at the right time, and in the right manner. Hodgen (2011) describes how a teacher may demonstrate rich knowledge in a particular situation but fail to apply the same knowledge in a different situation. As Mason and Davis (2013, p. 187) state about teachers "having knowledge": "Possession is not an appropriate metaphor. Rather, 'knowing' is not so much about having as it is about doing." Likewise, Davis and Renert (2013) view mathematical knowledge for teaching as "a flexible, vibrant category of *knowing* that is distributed across a body of professionals" (p. 247, emphasis added). Research focusing on teachers' mathematical activity, or the "mathematical work of teaching" (Ball, 2017), builds on the idea that "teachers must act mathematically in order to enact mathematics with their students" (Ruthven, 2011, pp. 90–91). A focus on mathematical thinking in teaching combines aspects of cognitive and situated perspectives and moves beyond those in a search of not only "what teachers know, but how they know it, how they are aware of it, how they use it, and how they exemplify it" (Watson & Barton, 2011, p. 67).

2.3 Teachers' mathematical thinking involved in making sense of student thinking

What is especially unique in teacher mathematical thinking is that it concerns "students' mathematics" (Steffe & Thompson, 2000): it should be aimed at interpreting and developing the students' construction of mathematics. From a modeling perspective, teachers make mental models of their students' mathematics (Lesh & Doerr, 2003), which requires explicit recognition of essential connections and reasoning steps involved in mathematical ideas. This goes much further than conceptual understanding of the mathematics (Ball et al., 2008). In fact, a teacher's personal understanding of certain mathematics is not always useful for interpreting student thinking. For example, Thompson and Thompson (1994) argue that some teachers experience such strong mathematical connections that when a student utters calculation steps, they hear or "see" the underlying reasoning steps. Making sense of a mathematical idea from a student's perspective challenges the teacher to imagine a situation in which they[1] do not understand the idea yet. This is not to say that teachers should "forget" their previous thinking. Rather, they should try to place the student's understanding within the frame of their own understanding (and the understanding they aim for) and be able to move back and forth between ideas and reasoning steps. This requires teachers' deconstruction of their "own mathematical knowledge, into less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98).

Working with "decompressed" or "unpacked" mathematics is explained by Ball et al. (2008) as part of specialized content knowledge: mathematical skill and knowledge unique to teaching. Unpacking mathematical ideas requires explicit, coherent reasoning, as opposed to compressed mathematical thinking, "where no explicit display of understanding (the reason employed) is required" (Adler & Davis, 2006, p. 289). Given our focus on teacher mathematical thinking instead of (static) knowledge, our interpretation of "unpacking" seems to be in line with what Davis and Renert (2013) call "substructuring": "re-collecting and re-membering" what infuses the meaning of a concept, and re-presenting in various ways to "compel new integrative structures and novel interpretations" (p. 252). We agree with Wasserman (2015) that "unpacking is required to help students recognize the complexity of the concepts they are working with—the teacher is using his/her knowledge as a means to explicitly point toward, scaffold, and unpack mathematical complexity so that students develop a more complete understanding of the concept at hand" (p. 78). However, we would like to add that unpacking is also required for the teacher to recognize the complexity in students' thinking and is used as a means to adopt the students' perspective and make sense of their thinking.

[1] We use "they" and "their" as gender-neutral singular pronouns.

2.4 Prior research and research question

Prior research has examined the development of mathematics teachers' sense-making in studies that included video-based discussions. Researchers strongly tie growth in noticing and sense-making to both an increased focus on student thinking and deeper discussions of the mathematics in student statements (e.g., Borko et al., 2008; Sherin & Han, 2004; Sherin & van Es, 2009; Stockero et al., 2017; van Es, 2011). For example, Sherin and van Es (2009) discern how teachers describe, evaluate, or interpret students' thinking, and they show how the teachers shift from mainly describing and evaluating at the beginning of the project, toward mainly interpreting at the end. These studies put much emphasis on teachers' sense-making of student mathematical thinking, but do not synthesize on the teachers' mathematical thinking or knowledge involved in this sense-making. In the words of Philipp et al. (2017):

> [W]e still have much to learn about how the mathematics must be understood for a teacher to effectively engage in professional noticing of students' mathematical thinking or how a focus on students' mathematical thinking leads to teachers' deeper learning of the mathematics. (p. 119)

Theoretical connections between the concepts noticing and mathematical knowledge for teaching are described by Thomas et al. (2017), and several studies have empirically investigated the role of teachers' mathematical knowledge in their noticing. Dick (2017) shows how specialized content knowledge and interpretation of student thinking develop hand-in-hand. Sánchez-Matamoros et al. (2019) show how awareness of connections between different representations—which we regard as an example of unpacked knowledge—is necessary for teachers' sense-making of their students' thinking. Likewise, Cengiz et al. (2011) found links between teachers' own mathematical knowledge regarding a problem and their abilities to build on student thinking in a whole-class discussion about the problem. These studies work with the concept of mathematical knowledge for teaching and speak of "having" and "drawing on" knowledge. In describing situations of contingency—involving in-the-moment noticing with regard to unanticipated student reactions—Rowland and Zazkis (2013) also use the term "knowledge" but add that teachers need to have a "mathematical disposition" and take an "inquiry stance" in order to fruitfully build on students' unanticipated reactions. Their three cases "exemplify mathematical ways of being" (p. 144). This is more in line with our own conceptualization of mathematics as an activity.

Although there is a steadily growing interest in mathematics teacher noticing, much remains to be investigated. First, research in teacher noticing is primarily done with primary school teachers, and secondary school teachers' noticing brings about its own challenges, as described by Nickerson et al. (2017). Second, most research into noticing considers pre-service teachers or teachers who are highly skilled, whereas Jacobs et al. (2010) found that experienced teachers who are beginning to develop their noticing skills exhibit significantly different noticing than pre-service or highly skilled teachers. Furthermore, the field has much attention for students' mathematical thinking, but lacks investigation and conceptualization of teachers' mathematical thinking involved in noticing and supporting students' thinking. In this study, we combine the theory of noticing with the view that making sense of student thinking involves mathematical thinking on the spot. We analyzed video-based discussions involving teachers who were experienced teachers but novices with regard to noticing and discourse-based teaching, and we address the following research question:

> What role does teacher mathematical thinking play in making sense of student mathematical thinking?

In line with previous research (e.g., Borko et al., 2008; Sherin & Han, 2004), we investigated teachers' sense-making during video-based discussions in a teacher-researcher collaboration. As teachers discuss student thinking, they articulate their own thinking, which gives insight into their sense-making and the mathematical thinking inherent in their sense-making. An underlying premise of this study is that insight into teachers' sense-making during video-based discussion regarding their own and their peers' lessons can lead to insight into their in-the-moment noticing during teaching.

3 Method

To answer our research question, we adopted an explorative qualitative research design involving content analysis of excerpts from video-based discussions in which teachers make sense of student mathematical thinking.

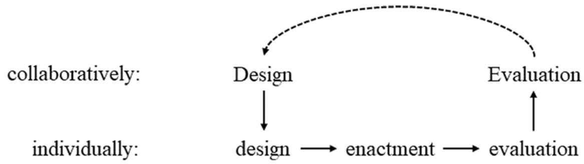
3.1 Context and design features of the collaboration[2]

Five Dutch high school mathematics teachers and one researcher (the first author) collaborated during the 2018–2019 school year to develop discourse-based mathematics lessons in which students (grade 10 or 11, age 15–17) work on a problem[3] and then share and reflect on their different ideas during a whole-class discussion. Each month (nine times in total), the group met for a 4-h group meeting at the researcher's university. The setup of the project was based on design research (Cobb et al., 2003); in the sense that between two group meetings, each teacher designed, enacted, and evaluated one discourse-based lesson (see Fig. 1), and the group iteratively developed guidelines for discourse-based teaching. During each meeting, the teachers chose one discourse-based lesson to design for their own classrooms and collectively began designing those lessons (Design in Fig. 1). In between meetings, each teacher continued the design of their own lesson, enacted the lesson and recorded it on video, and individually evaluated the lesson using the video. During the subsequent group meeting, lessons were evaluated collectively (Evaluation in Fig. 1) by focusing on particular parts of the video recordings (selected by the teachers or the researcher) for a video-based discussion.

[2] For a more elaborate description of the design, goals, and outcomes of the project, see Kooloos et al. (2020).

[3] See Appendix Table 2 for three examples of problems from the teachers' lessons.

Fig. 1 One cycle of development (figure from Kooloos et al., 2020)



3.1.1 Setup of video-based discussion

In the first four meetings, the setup structure for video-based discussion was based on the protocol "Case Stories" (Hughes et al., 2008), primarily to establish a safe climate for reflection. In these discussions, student mathematical thinking was rarely mentioned, which suggested that the teachers attended to other things rather than student mathematical thinking. In the fifth meeting, the researcher, in consultation with the teachers, introduced a new structure for video-based discussion. The purpose of this new setup was to target teachers' attention to student mathematical thinking and then foster sense-making. From the fifth meeting on, the setup of each video-based discussion comprised two phases:

1. The presenting teacher shares the story of the lesson's design and enactment.
2. The group watches a segment of the video recording of the lesson's classroom discourse.
3. At certain times, immediately following a student or teacher utterance, the researcher pauses the video and asks the group to make sense of the classroom situation.

Examples of researcher questions were as follows: "What are possible scenarios for what could happen now, what scenario would you prefer, and why?" or "What does the student mean?" In this setup, direct attention was paid both to what a student said and the teacher's possibilities to work with the student utterance. In our analysis, we investigated the teachers' sense-making during these video-based discussions.

3.2 Data collection

The primary data for this study were audio recordings of the video-based discussions. A total of 21 video-based discussions from nine group meetings took place over the course of one school year. Each group meeting included between one and four video-based discussions, each typically lasting between 20 and 30 min. Secondary data included the video recordings of lessons, photos taken during the lessons, and the teachers' personal lesson plans.

3.3 Data analysis

Data analysis consisted of four consecutive steps, as visualized in Fig. 2. In the first step of data analysis, we explored:

1) The appearance of sense-making of student mathematical thinking in the video-based discussions

The first step resulted in the selection of five video-based discussions that included significant discussion of student thinking. Within this selection, we explored:

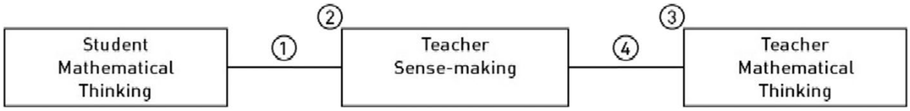


Fig. 2 Steps of data analysis

2) The nature of teacher sense-making regarding student mathematical thinking
3) The occurrence of teacher mathematical thinking in this sense-making

Steps two and three prepared us to explore:

4) The role of teacher mathematical thinking in their sense-making of student mathematical thinking

We adopted a form of content analysis in which each step combined certain aspects of conventional and directed content analysis (Hsieh & Shannon, 2005). The first author was the principal investigator, and the analysis was discussed with the other authors during frequent discussions. In these discussions, we aimed for "agreement through a rational discourse and reciprocal criticism between those interpreting a phenomenon," described by Kvale and Brinkmann (2015, p. 279) as "dialogical intersubjectivity."

3.3.1 First step of data analysis

All transcripts were coded in vivo (Saldaña, 2016)**,** with "student mathematical thinking" as a sensitizing concept. Analysis revealed that what the teachers attended to in the first four meetings rarely concerned students' specific mathematical thinking, while the remaining meetings displayed—as we had intended—more student mathematical thinking in the discussions. For a further, more refined analysis regarding teachers' sense-making of student mathematical thinking, we selected the five video-based discussions from meetings five through nine that contained the highest incidence of teacher utterances regarding student mathematical thinking.

3.3.2 Second and third steps of data analysis

In preparation for the final step of analysis, we separately coded sense-making and teacher mathematical thinking. Teacher utterances were coded individually, taking into account the previous part of the group discussion—individual utterances were analyzed in light of their contribution to the group discussion. To analyze teacher sense-making, we adopted a hybrid form of inductive and deductive coding (Saldaña, 2016), based partly upon a framework created by Sherin and van Es (2009), and partly upon our emerging data. We coded the stance that teachers took in sense-making—*(re)state*, *evaluate*, or *investigate*—as well as what aspect of students' articulated thinking teacher sense-making attended to: the *formulation*, *meaning*, *intention*, or *reasoning* (see Appendix Table 3 for our categorization). Teacher mathematical thinking was coded in vivo (Saldaña, 2016). We coded all mathematical activity in teachers' sense-making: for example, when teachers reasoned to establish the mathematical truth of a statement, or when they explicitly connected notation to mathematical meaning, or when they reflected on the steps of a solution method.

3.3.3 Fourth step of data analysis

For each utterance that was coded with regard to both sense-making and teacher mathematical thinking, we described the role of the teachers' mathematical thinking within their sense-making. Subsequently, we searched for patterns, identifying two dimensions of variation in the role of teacher mathematical thinking. First, we distinguished different levels on which their thinking played a role: namely, the mathematical *content* of their thinking and the *process* of their mathematical thinking. Second, we distinguished whether the role of teacher mathematical thinking was supporting or impeding teacher sense-making. Our analysis resulted in five categories, as displayed in Table 1, that depict how the teachers' mathematical thinking affects their sense-making of student mathematical thinking. We selected examples from three video-based discussions to illustrate the five categories and report our findings in the following section.

4 Findings

In this section, we present our findings. First, we point out an overall finding with regard to the explicitness of teacher articulated mathematical thinking. Second, in the body of the section, we present our categorization of the role that teacher mathematical thinking plays in making sense of student thinking and provide descriptions and examples from the data. Appendix Table 2 presents the mathematical problems that are relevant to the examples in our findings. We invite readers to look into those problems before continuing to read and to think of different solution methods or lines of thought that might be expected from students.

4.1 Explicitness of teacher mathematical thinking

Overall, the role of the teachers' mathematical thinking in their sense-making often remained unclear, because they often did not articulate mathematical thinking. The multiple steps of data analysis entailed repeated reduction of data, with a focus on sense-making about student thinking: The first step reduced the data from twenty-one video-based discussions to the five that contained the most discussion of student thinking. In the second step, we coded sense-making, reducing the data to 14–50 utterances per discussion. In the third step, we coded teacher mathematical thinking, reducing the data to 2–20 coded utterances per discussion. Thus, in the majority of teacher sense-making, teachers' mathematical thinking was not articulated. Evidently, if teacher mathematical thinking is not made explicit, the role it plays in sense-making remains unclear, as the following example illustrates:

Table 1 Role of teacher mathematical thinking in making sense of student mathematical thinking

	Supporting		Impeding
Content	Flexibility	Incomprehension	Preoccupation
Process	Exemplification		Projection

In[4] Anna's lesson,[5] the students had worked on finding a trajectory for a fiberglass connection between a coastal city and an island, where the costs for the cable differ between land and water. In a whole-class discussion concerning various student solution methods, Anna asked the class for similarities or differences between solution methods.

Nika: Between the method on the left and the one on the right of the whiteboard: left has calculated the number of kilometers horizontally that the cable must extend through the water, and right has calculated how many kilometers horizontally the cable must extend across land.

The two solution methods that this student referred to were similar and involved similar pictures. In both methods, a variable x is introduced to vary the point where the fiberglass reaches the coast. The difference is in what x stands for: the distance that the fiberglass spans over land or the part of the 20 km over land that the fiberglass does not span. Hence, what is x in the one method is $20 - x$ in the other method. At first glance, it seems Nika misspoke, because she said "horizontally" twice, and in the solution methods, the cable did not go horizontally over the water. However, it might be that this student decomposed the distance traveled over water into a horizontal and a vertical component. In that case, the horizontal component and the distance traveled horizontally across land must add up to 20 km. This lesson episode was watched and discussed in the group meeting:

> The teachers discussed possible lines of continuation for the whole-class discussion.[6] During this discussion, the teachers repeatedly evaluated Nika's formulation as wrong, but her intention as right. For example: 'The student said it wrong, but she meant it right,' and '[her] intention is correct, but the way [she said] it is not completely [correct].' However, the teachers did not make explicit what exactly the student had said incorrectly, nor what she 'meant' correctly.

The teachers were making sense of the student's articulated thinking by evaluating her formulation and her intention, but the discussion does not show evidence of the teachers' mathematical thinking. Therefore, although their mathematical thinking supposedly played a role in evaluating Nika's formulation as incorrect and her intention as correct, this role remained unclear.

4.2 Role of teacher mathematical thinking in sense-making

The following subsections present the descriptive categories for the role of teacher mathematical thinking in making sense of student thinking. These are structured as follows: (1) the category is described generally; (2) an example is given, and, if relevant, our interpretation is explained; and (3) the example is analyzed for the role of teacher mathematical thinking in sense-making.

[4] To distinguish descriptions of classroom situations and classroom discourse from the situations and discussions in the group meetings, descriptions of classroom episodes and classroom discourse are in italics.

[5] See Appendix Table 2 for the problem that the students were given.

[6] General teacher responses that were highly regarded during the video-based discussions were (1) request the student to give further explanation in front of the class and (2) ask the class or a specific student for a reaction or a reformulation.

4.2.1 Flexibility

If the content of teachers' mathematical thinking includes varying and connected mathematical perspectives on a problem, this enables them to flexibly consider different reasoning paths with regard to a student's articulated thinking.

In her lesson about radical functions, Sigrid had asked the students to study the square root function $f(x) = \sqrt{x}$. One of her goals was to encourage the students to articulate connections between the different representations: formula, graph, and table. During a whole-class discussion, she had drawn a table on the whiteboard and plotted a graph with GeoGebra software (see Fig. 3).

Sigrid: I can see in the table that we cannot insert negative x-values. Ferdinand, can I see that in the graph as well?
Ferdinand: Yes, because the graph does not go beneath the x-axis.

Note that Ferdinand's utterance is not a correct answer to Sigrid's question. "Yes" is a correct answer, and "the graph does not go beneath the x-axis" is a correct statement. However, the fact that the graph does not go beneath the x-axis is not an argument for the fact that negative x-values cannot be inserted. In a group meeting, the video was paused at this point, and the group discussed possible continuation scenarios for the interaction:

First, some teachers assumed that the student misspoke and "actually meant 'not left from the y-axis'," but then the teachers tried to adopt the student's perspective and came up with possible lines of reasoning that the student might have followed to lead him to say that the graph does not go beneath the x-axis.

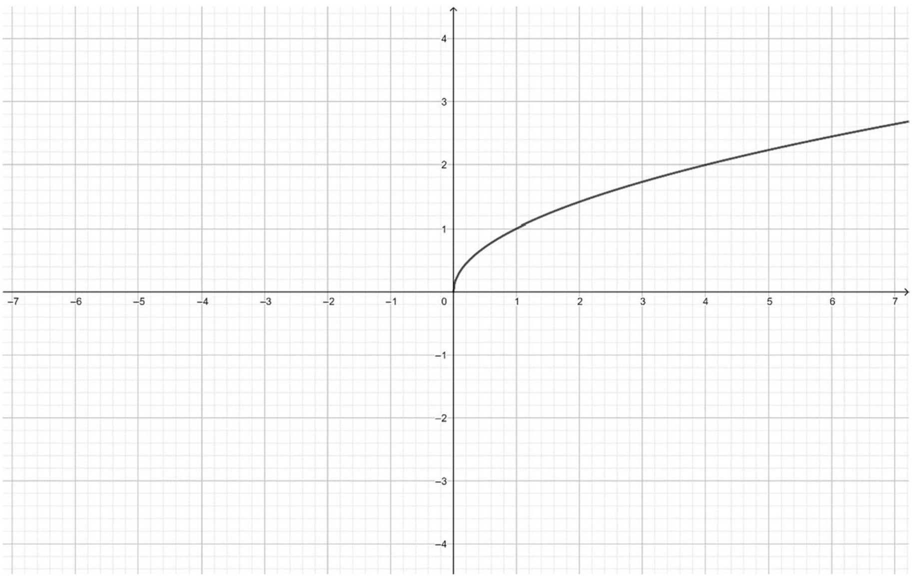


Fig. 3 The graph as plotted in GeoGebra (image created with GeoGebra software)

Jesse: Because what this student – that's what you have to figure out – the student is reasoning based on the shape of the graph. He uses that to predict the rest. I think.

The teachers discussed the possibility that the student was thinking dynamically, following the graph from right to left until the origin, where it stops and "does not go beneath the *x*-axis." Anna contributed by noting a different possible interpretation of not going "beneath the *x*-axis," because students could also think of going "beneath the *x*-axis" at the right "end" of the graph:

Anna: But the strange thing is, there are two points [endpoints of the part of the graph that is shown]. Because the graph is drawn like that, and does it not go beneath the *x*-axis like this [at the left side of the shown graph], or like this [at the right side of the shown graph]?

Additionally, Jesse mentioned a possible line of reasoning involving rotation symmetry:

Jesse: It could also be that he thinks this function arrives at the origin and then it smoothly continues like this [rotation of 180° around the origin], and arrives in the third quadrant.

These are examples of the teachers thinking mathematically in support of adopting the student's perspective. In Jesse's first statement, his mathematical thinking is not made explicit, but in the other statements, the teachers make their thinking explicit by connecting their speech to the graph drawing and the student's statement. The teachers are trying to investigate Ferdinand's reasoning behind his statement, and they go further than just assuming that he misspoke. Their ability to diverge from the "mathematically correct" answer and flexibly take different mathematical perspectives supports them in investigating the student's reasoning.

4.2.2 Preoccupation

If the content of teachers' mathematical thinking is limited to one solution method or a particular expected answer, this inflexibility can impede them in adopting a student's perspective that doesn't match their own thinking.

In his geometry lesson, Jesse had presented the students with a problem involving a circle and a tangent line in a Cartesian system of axes (see Appendix Table 2). Five solution methods were displayed across the classroom: four by students, and one by Jesse himself. He had included his own solution method because he wanted to show the students an elegant geometrical solution method that makes use of similar triangles (see Fig. 4). At a certain point during the whole-class discussion, Jesse invited his students to take a look at his solution method and asked if anyone could explain the method:

Ted: It's with ratios, right?
Jesse: Ted, please go ahead.
Ted: No, but I just know that it's with those ratios… with a triangle that has an angle of ninety degrees then you could work with… if one side is twice the hypotenuse

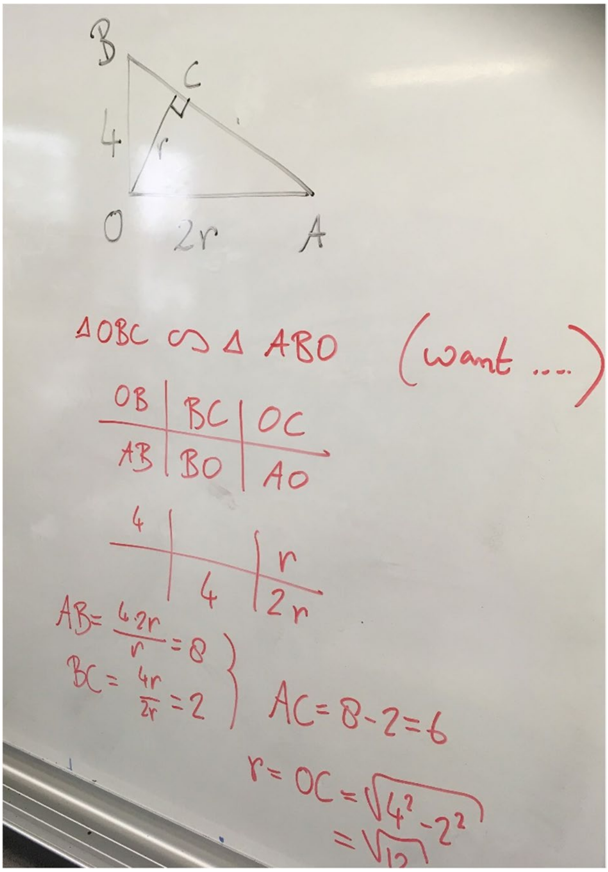


Fig. 4 Jesse's solution method

and then you could work with a ratio… but I don't completely remember how… but there's something there.

Ted articulated neither a refined solution method nor clear reasoning. It seems that he may have had a hunch about the use of ratios $1 : \sqrt{3} : 2$ of the "standard" right triangle with angles 30°, 60°, and 90°, which is a different solution method than the one Jesse was asking about. Hunches like these can be utilized by the teacher as a starting point for getting students to explore the hunch (which could have led to a solution method that had not yet been shared by any student), or they can be put aside if other matters are more urgent. During a group meeting, the teachers watched the episode and investigated the student's meaning and reasoning:

Some teachers tried to investigate a possible line of reasoning that involved the ratios of the standard triangle. Before watching the video, possible solution methods involving those ratios had already been discussed. During sense-making of Ted's thinking, Jesse repeatedly brought the discussion back to a solution method involving similar triangles: for example, by saying "No, this is about similarity" or by taking terms from Ted and supplementing them to construct a possible line of reasoning involving similarity:

Jesse: Yes, there is a right angle in triangle OAC and also in triangle AOB, which has a right angle, and angle A is a shared angle, so you can rotate that small one and then you have similarity.
[…]

Jesse: I think he sees that if you go from OC to OA, that's two times as large and that must be the same as from, let's see… from smallest leg to hypotenuse times two. So, OB is four, then AB is eight. That's what he's saying, I think.

The group re-watched the lesson episode and everyone, including Jesse, agreed that it was very possible that Ted was trying to reason about the "standard" triangle and the corresponding ratios, even though he said "if one side is twice the hypotenuse," instead of vice versa.

In this lesson episode, student Ted shared an idea that was not a clearly delineated solution method. To notice and extend such an idea during a whole-class discussion is very demanding for teachers. Our point here is that even in this situation, where the teachers were not subject to the immediate needs of teaching, and where they had time to make sense of Ted's thinking, Jesse remained focused on the solution method he had wanted to hear about. Jesse articulated his mathematical thinking quite explicitly, but it was inflexible, in the sense that he remained focused on his own idea of utilizing similarity. This impeded him in investigating possible lines of reasoning that did not match his own thinking. Eventually, he opened up to the possibility that Ted's reasoning may not have involved similarity.

4.2.3 Incomprehension

An additional role of the content of teacher mathematical thinking relates to incomprehension. If a teacher does not mathematically understand a student's solution method or idea, this can impede sense-making: if the incomprehension does not get resolved, the teacher is not able to fully investigate the student's meaning and reasoning. However, such incomprehension can also foster adopting the perspective of other students in the classroom: "If we don't understand, how will the students be able to understand?" Additionally, incomprehension can spur teachers to use the group discussions to explore the mathematics involved in a student's solution method.

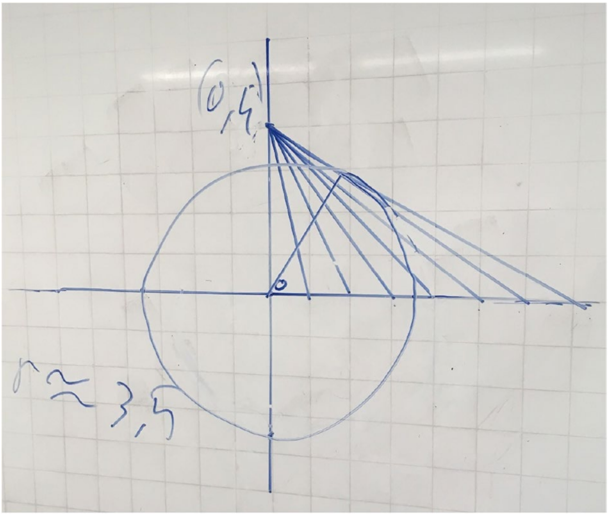
In Jesse's geometry lesson, one student had solved the problem by a trial-and-error method involving drawing lines from $(0, 4)$ *to several points on the x-axis (see Fig. 5).*

Jesper: Yeah, I actually just measured, because you knew that point $(0, 4)$ *and you knew that the other point should be right on the x-axis. So, I just drew several lines and then measured when the radius was twice as small as the point on the x-axis.*

During the group meeting, his method was discussed. At a certain point, Ward shared his incomprehension:

Ward: Could you please show me again what it is that he does, because I still don't understand.
Jesse: I think that he, he drew a coordinate system and point $(0, 4)$. Subsequently, he drew several lines like this […] and then I think he took half of this distance and

Fig. 5 Jesper's solution method



drew a circle to see whether that fits or something. Or with your protractor triangle, perpendicular to the line through the origin.

Ward was aware of his incomprehension and made it public. He problematized the mathematics involved in Jesper's solution method and used the group discussion to investigate the student's reasoning, make the mathematics explicit, and resolve his incomprehension. Thus, Ward's incomprehension contributed to teacher sense-making of the student's mathematical thinking.

4.2.4 Exemplification

The process of teachers' mathematical thinking can serve as an example for the mathematical thinking they would like to develop in their students. It can support teachers in articulating what kind of thinking they aim to develop in their students as well as in recognizing to what extent students already articulate such thinking.

During Jesse's geometry lesson, Sarah presented a solution method that involved making formulas for the circle and the line, substituting, and calculating the radius for which the discriminant equals zero (see Fig. 6 for part of their drawing from the whiteboard):

Sarah: We made a formula for line l. Next, we substituted that formula in the formula of the circle. Then we calculated the discriminant and then discriminant equals zero.

The student said that she used "discriminant equals zero" in her calculation, but she did not say why this can be used here, and why this leads to an answer. During the group meeting, the teachers noticed this lack of justification.

The teachers investigated the reasoning that led Sarah to use the discriminant in her solution method. They discussed the possibility that for the students, using "discriminant equals zero" had become an automatic routine.[7] During this discussion, Sigrid manifested caution:

[7] To clarify: In the Dutch textbooks these teachers use students are presented with many exercises involving circles and tangent lines in which they use the procedure of setting up a quadratic formula and the fact that the "discriminant equals zero" if the line is tangent to the circle. Making these exercises does not always require understanding of why the method works.

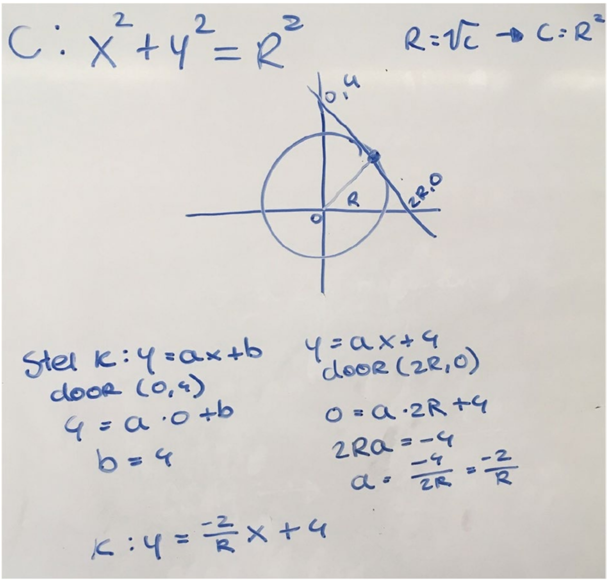


Fig. 6 Part of Sarah's solution method

> Sigrid: Yes, I think [the teacher's role includes asking 'why'], because there is a line of reasoning that is behind the possibility to use [discriminant equals zero] and if the situation is different, you want them to be able to reason in the same way, even if it's not about a circle and a tangent line. Then [their use of the procedure] must follow from the reasoning and not from an automatism, discriminant equals zero, right?

In this example, the mathematical content of Sigrid's thinking is not made explicit (she does not articulate the reasoning that she talks about). However, Sigrid articulates an important aspect of the process of her own mathematical thinking, which stands as an example for the thinking she wants to develop in students. This mathematical thinking includes basing solution methods on logical argumentation and being flexible in transferring lines of reasoning to new situations.

4.2.5 Projection

If teacher project their own ways of mathematical thinking onto their students, without taking into account the differences between their own expert thinking and the thinking of learners, this can impede them in adopting the students' perspective.

> During the group discussion about "discriminant equals zero," already mentioned in above in subsection Exemplification, the teachers investigated students' reasoning and to what extent this reasoning could and should be automatized:

> Ward: That's a good question: why must the discriminant be zero?
> Sigrid: Yeah, because perhaps they just learn, 'Okay, I need to apply that, so I'll just do it.'

Jesse: But that seems logical to me. If you have done it very often then it becomes 'discriminant equals zero, yeah, that's just the way we do this.' And it's not necessary to always think about why it works. It's just like riding a bicycle. Cycling is something you just do. You've learned that once.

During the discussion, Jesse repeatedly came back to his statement that it is logical that the process of equating the discriminant with zero becomes automatic. The mathematical argumentation behind these steps in the solution method was not made explicit by the teachers. In this example, it appears that Jesse considers his students' ways of reasoning with regard to "discriminant equals zero" as similar to his own reasoning, even though his own reasoning is based upon considerably more experience. Jesse's own thinking involved compressed ideas: for him, equating the discriminant with zero has become a standard process to which he has access in solving certain problems. Students, on the other hand, are still in the process of making sustainable connections and finding out which procedures are worth automatizing, and they need the teacher's support in articulating and scrutinizing underlying mathematical argumentation.

5 Discussion

Our investigation into the role of teacher mathematical thinking in sense-making about student mathematical thinking resulted in two findings. First, we found that in many instances of sense-making the role of teacher mathematical thinking remained unclear because the teachers did not articulate mathematical thinking. Second, our analysis of parts of the video-based discussions in which teachers did share explicit mathematical thinking resulted in our main findings, namely, five categories regarding the role of teacher mathematical thinking in sense-making about student thinking: *flexibility*, *preoccupation*, *incomprehension*, *exemplification*, and *projection*. These categories show how both the content and the process of teachers' mathematical thinking can either support or impede sense-making.

The first three categories concern the content of teacher mathematical thinking. (1) If the content of teachers' thinking involves multiple, well-connected perspectives, this enables them to be *flexible* in adopting students' perspective. This is in line with other studies into teacher noticing, such as Sánchez-Matamoros et al. (2019), who found that recognizing the importance of links between multiple representations is essential for interpreting student thinking. (2) If a teacher's thinking involves a single line of reasoning, this can result in *preoccupation* and impede the teacher in making sense of a deviant student idea. This relates to what Wallach and Even (2005) call "non-hearing": sometimes a teacher does not hear a student idea because it does not match her own thinking. (3) In case of teacher *incomprehension* of a student idea, it can go both ways: if incomprehension causes the teacher to stop engaging with the student idea, it impedes sense-making, but if the teacher takes their incomprehension as start of a mathematical inquiry, this can support them in making sense of student thinking.

The final two categories concern the process of teacher mathematical thinking. (4) Teachers can use their own way of mathematical thinking as *exemplification* of the kind of thinking they aim to recognize and develop in their students, which supports their sense-making. This brings to mind the chapter of Watson and Barton (2011) in which they investigate their own modes of mathematical enquiry, under the premises that these modes help teachers in supporting students in their own mathematical enquiries. (5) Teachers may also *project* their own way of thinking on their students, disregarding the differences between student and teacher thinking, which hinders sense-making. This is in line with claims by others such as Thompson and Thompson (1994), who describe how a teacher's own strong understanding makes him interpret "appropriate reasoning any time [his student] employed an appropriate calculation" (p. 299).

A particularly striking finding was that the teachers' mathematical thinking was often not made explicit in the video-based discussions of our group. This resulted in limited evidence of unpacking mathematics which comprises explicit recognition of essential elements and connections that underlie a specific idea. A possible explanation is that the teachers' mathematical thinking during sense-making frequently remained on a level of profoundness that can be expected from students—namely, thinking that is focused on finding the right procedures to arrive at an answer or in Richards' words: "School Math" (1991, p. 16). Another way of framing is that the teachers depend largely on intuitive thinking rather than on reflective thinking. As mathematics teachers, they are experienced in reflective reasoning about the mathematical content that plays a role in their lessons. As elaborated by Tall (2002), experience with reflective reasoning supports the development of refined and trustworthy intuitions. So, the teachers can rely on their own intuitions regarding the relevant mathematics, but not on their students' more initial intuitions.

Making sense of student thinking is pivotal in supporting it. Our findings—the five categories as well as the paucity of explicit teacher mathematical thinking—suggest that such sense-making not only requires teachers to think reflectively about the students' mathematical thinking, but on two additional levels: they need to "reexamine" their intuitions and unpack the mathematical content anew in order to be flexible in making sense of students' mathematical ideas and prevent preoccupation and they need to reflect on their own process of mathematical thinking so that they can use this as exemplification for student thinking. So, teachers not only need the capability to think mathematically, they also need the inclination to think mathematically, and they need awareness of their mathematical thinking. Awareness of mathematical thinking seems to be an aspect of mathematical thinking that is specific to teachers and is described by Mason and Davis (2013) as closely connected to teacher noticing.

A limitation of this study is that it provides insights into teacher sense-making during video-based discussions with colleagues and not during actual teaching. However, the categories that we found can be conceived to play a role during in-the-moment noticing during teaching as well. Also, our findings are based on Dutch secondary school teachers, and interpretation regarding other teachers should take into account local contextual characteristics. Furthermore, these five teachers may not have been exemplary for Dutch teachers as they were enthusiastic about, and participated in, a teacher-researcher collaboration regarding whole-class discussions.

6 Conclusion

Our structural synthesis of ways that teacher thinking plays a role in their sense-making contributes to research into the mathematical work entailed in teaching—and teacher noticing specifically—by combining insights from previous research and making a humble step toward a, much needed, conceptualization of teacher mathematical thinking. Our findings show the importance and variation of the role that teacher mathematical thinking plays in making sense of student thinking and suggest that teachers may benefit from explicit awareness of their own mathematical thinking. Future studies could hopefully confirm and expand our categorization in other contexts and develop a conceptualization of teacher mathematical thinking. Teachers— possibly supported by teacher educators or professional development facilitators—may build on our categorization and investigate their own mathematical thinking and its role in their sense-making.

Appendix

The problems

Table 2 The problems

Teacher	Anna	Sigrid	Jesse
Subject	Modeling and Optimization	Radical functions	Algebraic geometry
Problem	An island 15 km off the coast needs a fiberglass connection Two points are known on the coast: The island is 15 km from the coast (perpendicularly) and from that coastal point it's 20 km to the other coastal point The price for installation over water is 1000€ per km, and over land it's 400€ per km Search for a trajectory with the lowest possible cost	Investigate the square root function $f(x) = \sqrt{x}$ What do you notice? Can you describe the properties of this function?	The circle with center $O(0, 0)$ and radius r is tangent to line l, which intersects the axes at points $(2r, 0)$ and $(0, 4)$ Calculate r

Note that the problems were presented to the students without pictures. This was first advised by the researcher and later became a common practice. As a picture often already contains fragments of certain lines of thought, the absence of a picture broadens the possibilities of student lines of thinking. In addition, constructing a picture is an important step in solving problems, and it involves thinking that can lead to possible solution methods

Whether a mathematical question can be regarded as a problem depends on the students you give it to. These examples were all regarded as problems, because the students did not have an easy procedure at their disposal to find an answer

Categories of sense-making

Our classification of teacher sense-making was based upon the categorization of Sherin and van Es (2009) and adapted according to our data analysis. Whereas Sherin and van Es (2009) coded "idea units" and "segments in which a particular idea was discussed," we coded individual contributions by teachers, resulting in a more detailed variation of sense-making. In addition to the stance that teachers take with regard to student thinking, we found that their sense-making focused on different aspects of students' articulated thinking: namely, the formulation; the meaning of the utterance; the intention (what the student actually meant); or the reasoning that underpins a statement.

Table 3 Teacher sense-making of student mathematical thinking

Stance	Aspect of student thinking	Examples	
(Re)state Evaluate Investigate	Formulation Meaning Intention Reasoning	"The student said it wrong, but she meant it right" is *evaluating* the *formulation* and the *intention*	"The student is reasoning based on the shape of the graph. He uses that to predict the rest. I think" is *investigating* the *reasoning*

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