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# Technology-based inquiry in geometry: semantic games through the lens of variation

Carlotta Soldano[1] · Yael Luz[2] · Ferdinando Arzarello[3] · Michal Yerushalmy[2]

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## Abstract

The paper describes two versions of an inquiry-based activity in geometry, designed as a game between two players. The game is inspired by Hintikka's semantic game, which is a familiar tool in the field of logic to define truth. The activity is designed in a dynamic geometry environment (DGE). The inquiry is initially guided by the game itself and later by a questionnaire that helps students discover the geometry theorem behind the game. The activity is emblematic of describing a geometry-based inquiry that can be implemented with various Euclidean geometry theorems. The analysis of the first "student vs. student" version associates the example space produced by the students with their dialogue, to identify the different functions of variation. Based on the results of this version, we designed a "student vs. computer" version and created filters for the automatic analysis of the players' moves. Our findings show that students who participated in the activity developed forms of strategic reasoning that helped them discover the winning configuration, formulate if-then statements, and validate or refute conjectures. Automation of the analysis creates new research opportunities for analyzing and assessing students' inquiry processes and makes possible extensive experimentation on inquiry-based knowledge acquisition.

**Keywords** Logic of inquiry · Strategic games · Automatic filtering · Example space · Variation

## 1 Introduction

Numerous studies (Balacheff, 1988, 1999; de Villiers, 2010) have emphasized the importance of experiencing an experimental and investigative phase as part of the proving process. Arbib (1990), who researched psychological processes of mathematicians producing proofs, noted that:

✉ Carlotta Soldano
carlotta.soldano@unito.it

[1] Department of Philosophy and Educational Sciences, University of Turin, Torino, Italy

[2] Department of Education, University of Haifa, 3498838 Haifa, Israel

[3] Department of Mathematics, University of Turin, Via Carlo Alberto 10, 10123 Torino, Italy

> The usual proof generated by a mathematician does not involve the careful application of a specifically formalised rule of inference, but rather involves a somewhat large jump from statement to statement based on formal technique and on intuitions about the subject matter at hand. (p. 55)

These processes have a non-algorithmic nature, and school books often ignore them and present the proof in an orderly manner. The need for inquiry in the geometry curriculum has long been recognized. According to Yerushalmy and Chazan (1992), students need to be competent in different aspects of the inquiry process and experience conjecturing, data collection, generalization, and proof. Weber (2001) argued that the knowledge of facts, concepts, and theorems of a mathematical domain, along with the knowledge of the logical way of reasoning, is insufficient for building proofs, and that students often fail to use their knowledge for proof construction because they lack the strategic knowledge required; therefore, it is important to teach strategic skills. Hintikka (1999) noted that "in concentrating their teaching on the so-called rules of inference, logic instructors are merely training their students in how to maintain their logical virtue, not how to reason well" (p. 37).

Inquiry, exploration, generalization, verification, and refutation of properties are core elements in the study and teaching of Euclidean geometry. Dynamic geometry environments (DGEs) are useful tools to support these functions. There are concerns, however, that the opportunity afforded by the software of testing a myriad of diagrams by using the "drag" function may reduce the understanding of the logic machinery and the perceived need behind deductive reasoning for verifying the truth (Chazan, 1993; Hoyles & Jones, 1998; Laborde, 1993; Yerushalmy, Chazan, & Gordon, 1990).

In this paper, we present a geometry game activity for middle and high school students, which develop a new approach to DGE inquiry. The game challenge requires to discover spatial relationship between the involved geometrical objects, paving the way to the understanding and the analysis of the logical relations between the statement of a theorem and its premises. These types of activities require careful design of both the game and the questionnaire used to trigger the geometric inquiry of the game. The game dynamic is inspired by Hintikka's game semantics, which provides a useful theoretical tool for designing activities within DGE. When performing this type of activity, students act as detectives: exploring, observing facts, asking, and answering questions to discover connections between them. These processes constitute what Hintikka (1999) called the *logic of inquiry*, which is a coherent enlargement of classical deductive logic. We provide a design method that has the potential to turn any geometry statement that can be formulated as a *for all-exists* sentence ($\forall\,\exists$ −sentence, e.g., in each triangle the bisectors meet in the same point) into an inquiry activity. We also present a method for analyzing these inquiry activities for teachers and researchers, which makes it possible to grasp a visual picture of the epistemology behind the student's cognitive processes.

## 2 Theoretical background

### 2.1 Inquiry learning and strategic games

The interrogative model of inquiry developed by Hintikka (1998) is based on the assumption that scientific inquiry and knowledge acquisition are question-answer processes. The model sees the inquiry as a process of asking questions and drawing logical inferences from the received answers. Hintikka described this set of questions and answers as a two-player strategic game. One player, the *inquirer*, asks the questions, and the other player, *nature* or *oracle*, answers. Hintikka (1998) referred to sources of information and answers as Nature or Oracle: "The oracle can be the database stored in the memory of a computer, a witness in a court of law, or one's tacit knowledge partly based on one's memory" (p. 34).

Hintikka described two aspects of games: one concerns knowledge construction, the other the establishment of the truth of statements. He called the games dealing with the first aspect *interrogative games*, and those dealing with the second aspect *semantic games*. The two types of games are two sides of the same coin: in a semantic game, the role of inquirer is replaced by the refuter, who tries to show that the statement is wrong, and the oracle is the verifier, who tries to show that it is true. Therefore, semantic games are always associated with a statement that must be verified or refuted ("falsified" in Hintikka's terminology). The statement is proven to be true or false only if a winning strategy can be found by one of the players.

Consider, for example, the statement "for all x, there exists y such that S [x,y] is true." The role of the refuter is to find an instance of $x$ for which there is no instance of $y$ such that $S$ [$x$,$y$] is true; the role of the verifier is to find an instance of y for each instance of $x$ presented by the refuter. If the verifier succeeds in finding such an instance of $x$, the verifier wins a single match. Otherwise, the refuter wins the match. The statement is true if a strategy can be found for the verifier to always win the game. The statement is false if the refuter has a strategy to win the game.

Looking at mathematical statements through the lens of semantic games activate a reversed way of reasoning (Gómez Chacón, 1992): starting with the results of the single matches, the players should develop strategies and infer the winning one. To validate the truth of a conjectured winning strategy, the players can activate the "logic of not" (Arzarello & Sabena, 2011), i.e., validating by showing the impossibility to find a counterexample. Otherwise, they can activate what we refer to as the "logic of yes," and try to validate the winning strategy inductively, by showing its truth on different examples. Using the *logic of yes*, it becomes natural to check the truth of a claim by means of generic examples; using the *logic of not*, it becomes natural to think about how the situation may be if it were not like this. The former is particularly useful for grasping the geometric structure behind the game; the latter is useful for grasping the impossibility of the existence of different situations, in other words, the generality of geometric properties.

### 2.2 Inquiry into geometry through the lens of example variation

Dragging in a DGE links spatio-graphic with geometric aspects; such dragging is crucial in the dialectic of perceptual vs. theoretical aspects, because spatial invariants in the moving diagrams represent geometric invariants (Laborde, 2005). By experiencing what remains invariant during the dragging it is possible to discover geometric properties. Dragging is therefore a strong tool for purposes of inquiry.

Leung (2008) organized and interpreted known dragging practices (Arzarello, Olivero, Paola, & Robutti, 2002) using the lens of variation theory (Marton, Runesson, & Tsui, 2004), to show the students' progression of awareness. Arzarello et al. (2002) identified seven dragging practices (wandering, guided, bound, dummy locus, line, linked, and drag test) showing an evolution in students' cognitive modalities, from empirical to abstract ideas. Marton et al. (2004) observed that what people discern in the course of inquiry into a new mathematical knowledge is variation: the only way to experience aspects of a thing is to experience the way in which it varies. Learning consists of becoming aware, through variation,

of the new features that constitute an object. Marton et al. defined four categories *of patterns of variation*:
*Contrast*: to experience and understand an aspect of a thing, one must experience something else to compare it with. When inquiring about a mathematical property, one should experience examples of the property as well as non-examples of it (Antonini, 2003), that is, examples that do not have this property.
*Separation*: to experience something, and to separate this aspect from other aspects, a person must experience its variation while other aspects remain invariant. Within mathematical inquiry, one aspect of the property should vary through different examples, while other aspects should remain unvaried.
*Generalization*: to experience something, a person must experience varying appearances of it. The mathematical property should be exemplified through different examples.
*Fusion*: to experience different aspects, they must be present at the same moment. All the aspects and properties that are objects of the inquiry should be present simultaneously.

### 2.2.1 A new construct: *Actions|Logic*[1]

In the concrete application of the two lenses above to interpret students' behaviors during the activity, it is apparent the intertwining between their actions stimulated by the given situation and their logical thinking with respect to the geometric dynamic figures, upon which they operate through dragging. The two aspects, actions with the device within the environment, and logical ways of looking to what is happening because of their actions, continuously interact each other. This is not new at all: see the links between dragging actions and logical productions pointed out in the literature (Baccaglini & Mariotti, 2010). Here, the technology allows to point out another aspect. This deep interaction between actions and logic recalls an analogous construct, doing|mathematics, introduced by Maheux and Proulx (2015), where the Sheffer stroke serves to emphasize the dynamic relationship between the two terms:

> *Doing|mathematics* is both *doing* something (some thing) recognizable as *mathematics*, but also producing *mathematics* as this thing that we are *doing* when we do what we do. (p. 215)

We so introduce the construct *Actions|Logic* to indicate that students' actions produce logical thinking while they drag the geometric figures within the designed environments (or reflect on what they have done), and conversely.

Hintikka's notion of oracle and Marton's theory of variation are at the core of the design of the DGE-based inquiry activity, while the *Actions|Logic* constructs adds to support the analysis methods used in this paper.

## 3 Design of the activity

This study is a two-cycle design-based research (Prediger, Gravemeijer, & Confrey, 2015) on learning processes. The first cycle included a "student vs. student" version of the game, and the second cycle included an online "student vs. computer" version of the game. The activity

[1] We thank one of the reviewers for pointing out this interpretation.

is designed to engage students in exploring new knowledge by empirical inquiry, identifying patterns and conjecturing. Each activity is meant to introduce to students a new geometry theorem through a game-based guided inquiry. By playing the game, it is possible to grasp the universal validity of the theorem on which the game is based from both an empirical and a theoretical point of view. The design of the game exploits notions and concepts belonging to the field of logic and game theory. When students realize that there is no strategy that allows the refuter to produce a counterexample, they validate the truth of their discoveries. Successively, guided by the questionnaire, the students explain why this happens from a theoretical point of view. For this purpose, they investigate both the dynamic construction of the game and the goals of the players, creating logical links between them.

This type of activity was designed for various geometric properties and theorems; here we present it for the following theorem: a quadrilateral ABCD is a parallelogram if and only if its diagonals mutually bisect. To trigger the dynamics described above, the DGE construction must be a soft one (Healy, 2000). As shown in Fig. 1, ABCD is not a parallelogram but a general quadrilateral that the player can transform into a parallelogram. In quadrilateral ABCD, points C and D are two draggable vertices, whereas A and B are fixed. E and F are midpoints of diagonals, and G is their intersection point. The verifier controls point C and seeks to make E, F, and G coincide; the refuter controls point D and seeks to prevent the verifier from reaching the goal.

Following the game, students complete a questionnaire intended to help them shift their attention from visual to geometric properties, which does not always occur naturally.

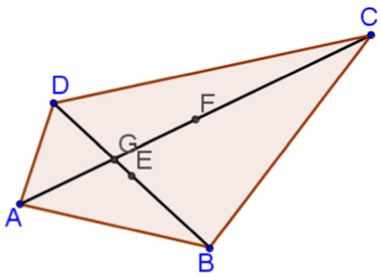
Figure 1 shows the first cycle of the activity, in which the game was played in a "student vs. student" setup.

Player X controls point C, and player Y controls point D. The goal of player X is to make points G, E, and F coincide (or only E and F if G disappears); the goal of player Y is to prevent X from making the three points coincide.

Players take turns dragging their respective points. Player Y begins.

If Y can position point D in such a way that prevents X from making the three points coincide, Y wins. If X succeeds in making the three points coincide, X wins.

After you finish one match, switch roles and play again.




Answer this question only if you believe that X can always win. In which way should the quadrilateral be modified for X to make points E, F, and G coincide? Why is it that points G, F, and E coincide if the quadrilateral is modified in the way you suggested?

Answer this question only if you believe that there exists position D that prevents X from making points E, F, and G coincide. In which way should the quadrilateral be modified for X not to be able to make points E, F, and G coincide? Why is it that points G, F, and E do not coincide if the quadrilateral is modified in the way you suggested?

**Fig. 1** The student vs. student version of the activity

Based on the analysis results of this version, we designed the student vs. computer online version of the game. This version included automatic analysis of students' submissions, which can support research with a large number of students. In the student vs. computer version, the computer generates an immediate win/lose feedback on each match and automatically makes the next move. First, the student plays as verifier; then, they switch roles and the student plays as refuter. As a refuter, the computer randomly chooses a location for point D. As a verifier, the computer automatically calculates the winning position of point C. In the online version, the point G was the intersection of the lines AC and BD, as opposed to the intersection of the diagonals AC and BD in the first version (the difference is due to the DGE used, GeoGebra in the first cycle and JSXGraph in the second cycle). The student vs. computer game is followed by an online questionnaire, which is part of an online assessment system (Luz & Yerushalmy, 2015).

## 4 Methodology

The data for the student vs. student version cycle were collected from games played by 31 students. The data include video recordings of two groups of students (a pair and a group of three, with the third student observing the pair playing and helping answer the questionnaire), and the completed questionnaires of all pairs. The students were at the beginning of 10th grade; they studied the diagonals' properties of parallelograms at the end of 6th grade but they did not prove the theorem. They played the game in GeoGebra: dynamic geometry software that they have already used with their teacher. For the student vs. computer version cycle, data were collected from games played by 20 students. The students were in 9th grade and had one-year experience in deductive proofs; however, they did not yet learn the theorem of parallelogram's diagonals. The data consist of the diagrams submitted by the students and by the computer during the game, the completed questionnaire of all students, and a single videotaped game played by a pair of students against the computer in think-aloud mode, which was used to triangulate databased data.

The analysis of the videotaped activity focuses on dragging motions and other moves of the students and on the function of their pronouncements regarding the game in the course of their conversation. Dragging behavior was analyzed according to the dragging modalities defined by Arzarello et al. (2002); the students' moves and the diagrams produced were analyzed using the theory of variation. Dialogues were analyzed according to the various game functions referred to during the activity: we distinguish between *played games*, aimed at defeating the opponent, and *reflective games*, aimed at formulating geometric conjectures and establishing their truth (Soldano & Arzarello, 2016). In the former, the students played against each other or against the computer and their aim was to win; in the latter, the game served as the oracle, helping students in the discovery and validation of the new knowledge. The students are no longer opponents and play the game cooperatively. In dialogue analysis, attention was paid to types of strategic thinking, including *anticipatory thinking* (Harel, 2001), in which one predicts the result of the opponent's move and *reversed type of reasoning* (Gómez Chacón, 1992), and the types of logic behind them, i.e., the logic of yes and logic of not.

Based on results of the first cycle, we added four filters to the automatic analysis system of the student vs. computer version. In our analysis, we sought to associate the example space produced by the students with their dialogue, to identify the different functions of variation (contrast, separation, generalization, and fusion), and to analyze how these functions led students to the insight concerning the parallelogram configuration.

## 5 Analysis

The analysis was organized in the following three phases: the game, the questionnaire, and the geometric interpretation of the game. We choose to present the findings through the videotaped activity of three Italian students (S1, S2, and S3) and through samples from the student vs. computer activity database, which was made in Israel. Each sheds light on different aspects of the learning process through the students' inquiry activities.

### 5.1 The game

The three players of the student vs. student version, after having read the rules of the game and chosen their roles (S1 is the verifier (player X), S2 the refuter (player Y), and S3 the observer), start the game. Figure 2 shows the diagrams created during the first three matches. Each column shows one match, which includes the refuter's move (first line) and the verifier's move (second line). Each cell also indicates the duration of the move.

The refuter's moves are long and explorative, lasting 72 and 42 seconds. Before making Fig. 2(a–c), S2 examines different positions where he could leave point D, and chooses the most convenient one by activating a *strategic type of thinking*. He says: "If I were player X, I would make them [E, F and G] coincide immediately because I already knew [where to move the point]!" This sentence reveals the activation of *anticipatory thinking* (Harel, 2001), which helps him to predict the result of the opponent's move. Unlike the refuter, the verifier activates a *reversed type of reasoning* (Gómez Chacón, 1992) and takes some steps backward in his reasoning to plan the move, as demonstrated by the question he asks: "What did you do?"

Figure 2(d) poses a problem at the perceptual level as far as the actual coincidence of the three points on the screen is concerned, because the drawing is very small. S3, the observer, is convinced that the three points coincide; he takes control of the mouse and demonstrates his claim making Fig. 2(e, f). S3 uses the game to provide a generic example (Fig. 2(e, f)) of his claim ("The points coincide"). This way of validation follows the *logic of yes* and results in the production of a generic example (Mason & Pimm, 1984), obtained by eliminating the conditions that make it specific—in this case, the small dimension of the diagram. It is not a proof, but it can help students produce one because it shifts their attention from the particular to the

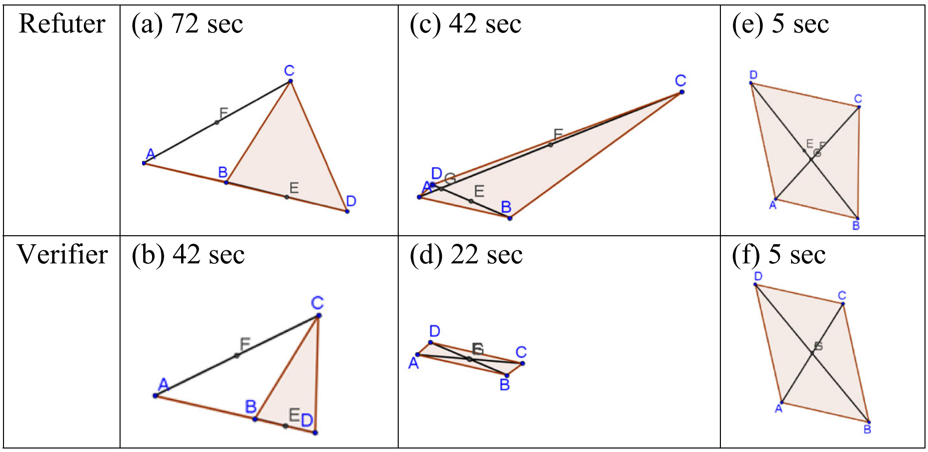



**Fig. 2** Example spaces of the first three matches

general, in other words, to the mathematical structure. S3 solved the perceptual problem of the coincidence of the points activating a theoretical/logical frame: he uses a configuration in which the three points clearly coincide (Fig. 2(f)) to validate the coincidence in the case of empirical ambiguity (Fig. 2(d)). Differently from S3, S2 and S1, instead, activated an empirical/visual frame which did not allow them to easily solve the perceptual problem: they need to more deeply explore the situation. These situations cannot appear in the students vs. computer version because the winning of the verifier is established according to an implemented acceptable error. This makes evident that the software design affects students' actions and considerations.

The example space contains three diagrams in which points E, F, and G coincide, and three diagrams in which they do not. Their juxtaposition creates the *contrast*: each diagram submitted by a player constitutes a non-instance of the possible diagrams produced by the opponent, and consequently "something else" to compare with. At this time, the students were not explicitly aware of the parallelogram and non-parallelogram configuration, but the game was preparing the ground for this discernment.

The videotapes of the played games show that in the first matches, the verifier drags point C randomly, without a plan, until finding the direction that makes the E, F, and G points come closer to each other. This type of dragging is characterized as *wandering dragging*, and its duration is generally long. In subsequent matches, when the verifier discovers the existence of the winning configuration (the parallelogram), the wandering dragging turns into faster *guided dragging*, guided by the location of C, which makes ABCD into a parallelogram. By designing a filter on move duration to capture this phenomenon, we found that diagrams such as the one shown in Fig. 2(d) causes longer move duration, because the diagram becomes smaller and dragging more sensitive, making the move more complicated to perform.

**Contrast filter** Normalized duration of a move. Rather than filtering by the absolute duration of a move, we found that the move duration normalized by the number of zoom-in and zoom-out actions represents the analyzed patterns. The student vs. computer version automatically calculates the ratio between move duration and the number of zooms required for it (Fig. 3). The filter criterion is the decreasing moving average of this ratio in a series of verifier moves. We interpret a decreasing ratio as an indication of awareness of the contrast aspect of variation.

## 5.2 The questionnaire

In the questionnaire, we asked students to generalize the invariants of the game: the geometric nature of points E, F, and G, and the geometric shape of the winning configuration. Students can discover them observing the variants and invariants of the game and can check their

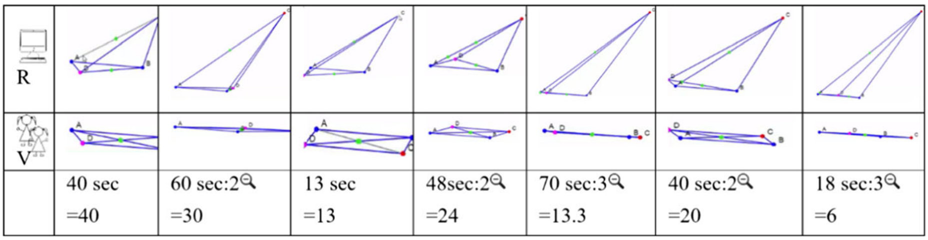



**Fig. 3** Contrast filter in student vs. computer version

conjectures using the GeoGebra toolkit, but the task did not require them to make it. We noted that students chose to address the questionnaire after succeeding in making their moves faster. We interpreted decreasing duration of the matches as the students having reached some discernment regarding a winning strategy, possibly only a visual discernment that did not necessarily translate into geometric language.

The examples in Fig. 4 were created while the students played the game, after they read the first question.

## 5.3 Excerpt 2

After reading the questionnaire, students started playing the reflective game to investigate its geometric aspects. It is clear from the dialogue that the students are not opponents, but cooperate to discover the geometric invariants. To do so, they use the game as an oracle, as a sort of virtual laboratory, as suggested by the expressions “Have a look” (line 2), “Look” (line 3), and “Try to make … ” (line 6). The students are conducting controlled experiments that allow them to formulate geometric conjectures: “I believe … ” (line 1), “in order to … you should form …” (line 3).

The students discern a conditional link between the actions of making the points E, F, and G coincide and of creating a parallelogram (lines 3–4). With this knowledge, their moves become faster, and they begin guided dragging, which is guided by the discovered winning shape. S2, who carries out both the verifier’s and the refuter’s moves, creates 5 examples and 4 non-examples in 25 seconds, which is less than the average time it took to make one move in the first match. The dialogue between S2 and S1 reveals that the discovered conditional link and invariants are visual and not logical.

At this stage of the game, the students’ moves are guided by the *logic of yes*, as they are checking the validity of the sentences “it is always a rectangle” or “it is always a

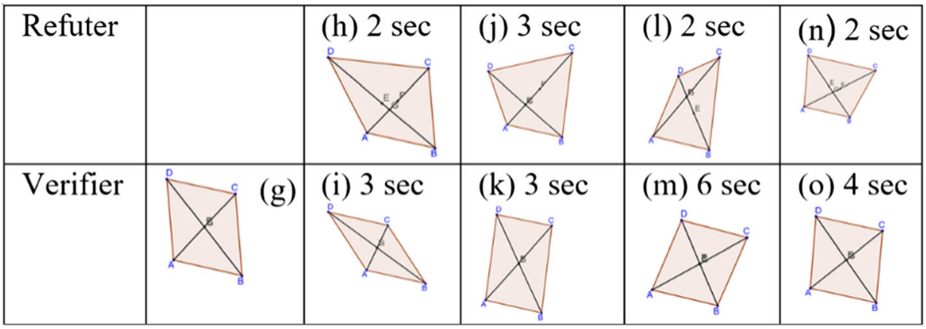



**Fig. 4** Example space of matches 4–7

| | | | |
|---|---|---|---|
| 1 | 19:05 | S2 | I believe they [points E, F, and G] always meet in the midpoint [of ABCD] (looking at Fig. 4(g)) |
| 2 | 19:08 | S1 | It’s true! Have a look! (rapidly creates Fig. 4(h–j)) |
| 3 | 19:20 | S2 | It’s always a rectangle! Because in order to make them meet you should form a rectangle! Look (making Fig. 4(k, l)) |
| 4 | 19:29 | S1 | It’s not a rectangle, it’s a parallelogram! (looking at Fig. 4(m)) |
| 5 | 19:31 | S2 | It’s a particular rectangle. |
| 6 | 19:32 | S1 | Make it. Try to make them coincide again (making Fig. 4(n, o)) |
| 7 | 19:37 | S2 | Rectangle (looking at Fig. 4(o)). Maybe a square … |

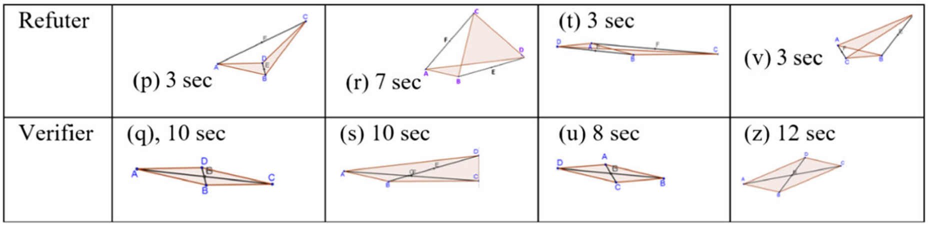



**Fig. 5** Example space of matches 8–11

parallelogram" (line 3, 4, 7) on standard examples. The generalization resulting from experiencing varying appearances of the same object helps students establish the truth of the conjecture. In this example space, it is possible to notice a *separation* in the refuter's moves: Fig. 4(h, j, l, n) is all produced by varying the position of point D and keeping fixed the convexity of the quadrilateral. All these figures are standard non-examples.

The investigation continues after the students read the first question again. Figure 5 shows the generated example space:

Comparing this example space with the previous one, we noticed another occurrence of *separation*: all the refuter's diagrams are non-prototypic non-examples, in particular, the concave and self-intersecting quadrilaterals (Fig. 5(r, t, v)). They are created by varying the position of D while keeping fixed the non-convexity of the quadrilateral. Students repeated the reflective game played previously with standard non-examples, this time with non-standard non-examples. Using the lens of variation theory, this *separation* results in the *generalization* of the parallelogram conjecture not only as a winning configuration for standard non-examples but also as a winning configuration for non-standard non-examples.

The students who generalized the idea "if we make a rectangle (parallelogram) we win," now check whether it is always possible to make a parallelogram. The refuter's moves are guided by the logic of reasoning by cases and the *logic of not*: students verify their hypothesis by trying to refute cases of potential counterexamples. S1's words provide evidence for this claim: he is looking for "weak points" and "most difficult positions" to "see if you can [make a parallelogram]," until he is convinced that "they always coincide." S1's dialogue starts with vague descriptions of the objects and their relations ("weak points" "most difficult positions") and progressively establishes a clear and understandable statement about the objects and their relationships: "they always coincide."

The non-prototypic non-examples are used as potential counterexamples of the conjecture. This way of playing leads students to check empirically that there is no situation that can refute

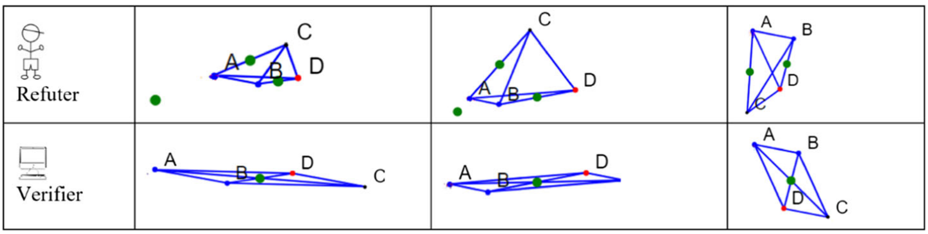



**Fig. 6** Separation filter in the refuter's inquiry: Various self-intersecting quadrilaterals

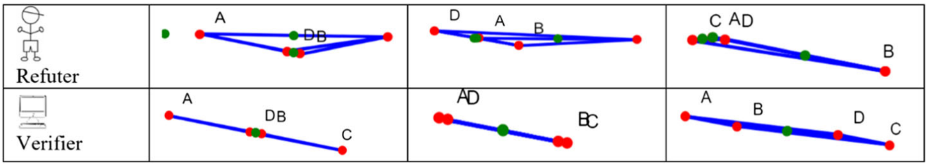



**Fig. 7** Separation filter in the refuter's inquiry: Various locations of point D along the line AB

the conjecture, and to conclude that "they always coincide, so the quadrilateral should be modified into a rectangle."

The separation aspect of the variation theory (such as the one demonstrated in Fig. 5) was revealed by a few strategies that comprise the separation filter.

**Separation filter** Point D location. Special positions of point D are interpreted as attempts to find a winning strategy for the refuter. We identified the following special locations:

- On the (invariant) points A or B (used by 57% of students)
- On the (variant) point C (used by 15% of students)
- On the (invariant) segment/line AB (59% of students, see Fig. 7)
- Far beyond the visible area (31% of students)
- To create a self-intersecting ABCD. In this configuration, G disappears from the visible area (59% of students, see Fig. 6)

Students generally used each strategy more than once, suggesting the activation of the separation aspect of variation. For example, Fig. 6 shows three different self-intersecting quadrilaterals created by one student. The self-intersecting aspect is the invariant, and the location of C is the variant.

Separation can also be identified when students systematically vary a single aspect, leaving other aspects the same, as can be seen in the filtered drawings of a student in Fig. 7. D is systematically placed in different positions along the line AB, first, close to B, then left of A, then on A itself. The computer responds to each of these moves with a degenerated quadrilateral, in the form of a line.

**Generalization filter** The role of points E, F, and G. In the questionnaire, students are asked to state the role of points E, F, and G. The answers are used to check whether students correctly generalized the invariants of the diagram. Different phrases are accepted as correct answers, such as (for E) midpoint of BD and the middle between points B and D.

### 5.4 Geometric interpretation of the game

It is interesting to observe a fusion phenomenon: when S1 said "they always coincide … so the quadrilateral should be modified into a rectangle," S1 was looking at a parallelogram, but he mixed up the terms and called it a rectangle. This utterance shows that S1 discerned the *fusion* between the two dimensions of variation explored by the verifier's moves: the coincidence of the three points and the parallelogram figure. The first dimension is induced by the rules of the game, but the second one is discovered through the game experience and the questionnaire. The students discuss the game from a theoretical point of view to validate their conjectures. Their reasoning is often insufficient (“it is only a parallelogram because in this way points G, E, and F coincide”), incorrect (“in the parallelogram the opposite sides are equal so of course they coincide in the midpoint”), and clumsy/wrong (“in a parallelogram/rectangle, the opposite sides are equal and the diagonals are equal. Since the diagonals are equal, these two [E and F] are in the middle while G is in the middle because it is in the midpoint”) but reveals that they start detaching from the visual/empirical aspects, validating the discovered visual invariants from a logical/theoretical point of view (S2: “Point F is in the middle of diagonal CA and point E is in the middle of BD. Point G is always the meeting point of the diagonals. Any position you place the quadrilateral in, it is always like this. So let’s write, it is a [parallelogram] for these reasons.”).

**Fusion filter** We wanted to assess the geometrical interpretation that students ascribed to the game. After they were asked to identify the invariants of the diagram (the role of points E, F, and G), the students were asked to generalize the winning shape and to formulate a geometrical statement, as shown in Fig. 8.

The fusion occurs with the realization that the parallelogram shape is formed simultaneously with the bisection of the diagonals.

We found that 53% of students selected the if-and-only-if option, 11% selected if option, and the others did not answer the question. We also found that 48% of students selected the parallelogram, 5% the rectangle, and the others did not answer the question.

Table 1 shows the filtering results of the student vs. computer game. The right column describes the percentage of students who passed a given filter from the population of those who passed the previous filter. Although some of the students failed to complete the task, the method seemed to provide a way to assess the students’ progression within the process of inquiry.

The table shows the evolution of the *Actions|Logic* construct through the mediation of the technological device: students’ patterns of actions within the geometric environment are linked to and express logical patterns. They are like the two sides of a same coin, which interact each other progressing in the disclosure of the logical links within the different geometrical dynamic figures that their actions produce. The contrast aspect is linked to the activation of the logic of yes; the separation reveals a way of reasoning by cases adopting the logic of not. These logics used while doing contribute to generalizing and fusing the doing in mathematics knowing.

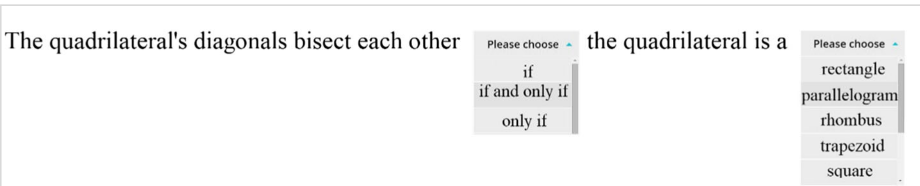



**Fig. 8** Online questionnaire question: formulate a geometrical statement

**Table 1** Summary of filtered automatic results in the student vs. computer version

| Aspect | Filter | Criteria | % passed | % that passed the previous filter |
|---|---|---|---|---|
| Contrast | $\frac{\text{Duration}}{\text{zoom}}$ | Decreasing by > 50% | 100% | – |
| Separation | Location of D | At least 1 strategy | 79% | 100% passed contrast |
| Generalization | Role of E, F, G | Midpoints or intersection | 65% | 85% passed separation |
| Fusion | Formulate geometric theorem | If and only if parallelogram | 53% | 100% passed generalization |

## 6 Summary

Each phase of the analysis (the game, the questionnaire, and the geometric interpretation of the game) is characterized by different types of student behaviors and triggers distinct types of reasoning and patterns of variation.

### 6.1 The game

When playing the first matches of the game as opponents, the players explore the rules of the game. The *contrast* aspect of variation is created by the juxtaposition of the examples generated by the verifier with the non-examples generated by the refuter. The verifier's moves, at first characterized by long wandering dragging, become faster as the more experienced player's dragging becomes guided dragging. The verifier's moves and reversed type of reasoning trigger the *logic of yes*.

The refuter's example space includes non-prototypical and extreme cases of non-examples. The *separation* aspect of variation is created by the natural desire to investigate in depth the situations that can spell trouble for the verifier. By anticipating how the verifier will respond to refuter moves, the refuter activates *anticipatory thinking* ahead of the verifier's response. Reversed reasoning and anticipatory thinking are manifestations of strategic thinking, indicating that the students consider and evaluate different move options and strategically choose the best move to make in a given situation.

### 6.2 The questionnaire

When the students feel that they grasped the idea of the game, they proceed to answering the questionnaire. Grasping the idea means producing some visual generalization of the game. Students know what they need to do, but they cannot yet articulate it. They keep playing the game, but their aim is different: they are playing the reflective game. They are investigating the game, trying to formulate its variants and invariants.

Strategic thinking is manifested through cooperation, such as the interaction between the *cooperative refuter* and the *cooperative verifier*. The motivation to defeat the opponent triggers the students' strategic thinking; but repeated failures produce a shift in their thinking about what is possible and impossible. The methodical positioning repeated in the refuters' moves demonstrate the students' intention to generate worst situations for the verifier. This type of behavior shows the activation of the *logic of not* for checking purposes. The main difference between the two versions is that when a student plays the refuter's role in the student vs. computer game, the verifier's role is played by the always correct computer, whereas in the student vs. student game, the verifier's role is played by the "possibly correct" student. The moves' acceptable error should be decided by the students and not by the computer.

### 6.3 Geometric interpretation of the game

In the last question, the students are asked to phrase the mathematical property on which the game is based. To answer this question, students must experience the simultaneity of the previous generalizations: "E and F are the midpoints of the diagonals," "if E, F, and G coincide we win," and "if we win the shape is a parallelogram."

Generally, the natural desire to investigate in depth a situation or conflicting opinions between the players induces students to start playing again the reflective game. The students' aim is not to defeat the opponent but to formulate geometric conjectures and to validate or refute them. At this stage, in the pair game, students do not respect their roles and turns, but cooperate with the goal of achieving new knowledge and checking it. They ascribe interrogative and semantic functions to their moves.

Using the reflective game, the students discovered the invariants of the game. The uncertainty that students' work expresses on whether a rectangle or a parallelogram is produced by the action of making the three points coincide reveals that they are empirically looking to the discovered invariants and did not activate a logical control on them. Through the guided and the strategic dragging, the students make experience of the sufficient and necessary conditions of the parallelogram configuration for making points E, F, and G coincide. The cognitive step that allows students to switch visual invariants into logical ones is ready to be made by S2 and S1 but requires the mediation of an expert. In this process, the *Actions|Logic* is strongly active.

### 6.4 Automatic filtering in the student vs. computer version

Based on the results of the first cycle analysis of videotapes and dialogues, we designed filters for submitted diagrams of the players, and a questionnaire with associated filters. For the verifier, we designed a duration-based filter to assess the players' discernment of the contrast aspect and their progress toward the generalization aspect of the game; for the refuter, we designed a filter based on the location of point D, to assess the students' discernment of the separation aspect of the game.

We formulated the questionnaire question on the role of points E, F, and G for assessing the generalization of game diagram invariants, and their translation into geometric properties. Assessing how students grasp the logical connection between the winning shape and the bisection of the diagonal highlights the fusion aspect of the inquiry process.

Leung (2008) found a process of progression between the four aspects of variation when students engage in DGE-based inquiry. According to Leung's model, discernment starts with contrast and separation and develops into generalization, and finally fusion connects all the elements together as the property or theorem is discerned. Our automatic filtering system shows similar results of evolving discernment: this can be also described as an evolution of the *Actions|Logic* construct.

## 7 Conclusions and implications

Our two-cycle design-based research shows that students who participated in the activity developed certain forms of strategic reasoning (reversed type of reasoning, anticipatory thinking, reasoning by cases and use of the logic of error, of the *logic of yes* and of *the logic of not*), which helped them discover the winning configuration, formulate if-then statements, and validate or refute conjectures. The game offers to students the possibility to develop a mathematical correct exploratory behavior. This must be attributed to the students' participation in the design experiment, as we found a correlation between these forms of reasoning and participation in the game and questionnaire; it appears, therefore, that the design of the activity triggered these forms of reasoning and created the environment needed to facilitate this process. According to Cobb et al. (Cobb, Jackson, & Dunlap, 2015), these are the requirements that must be met in a design research.

Note that these dynamics are triggered and developed by the particular use of the game as an oracle. In these game situations, students naturally assume the role of inquirer and ascribe to the game the role of oracle or source of knowledge: they play what Hintikka called an interrogative game (Hintikka, 1998). The game answers implicitly through the matches that the verifier can always win, and that the winning configuration is a parallelogram. By visually inquiring the variants and invariants of the geometric configuration of the game, students conjecture that points E, F, and G are the midpoints and intersection point of diagonals, and that the diagonals bisect each other only if ABCD is a parallelogram. The moves prompt the discovery of visual conditional link between the point of intersection of the diagonals and the type of quadrilateral produced. The visual DGE invariants can help students to appreciate the dual logical nature of invariants. This activity can be a powerful instrument for teacher to understand how students manage logical/visual aspects in geometry and how the *Actions|Logic* interactions evolve. The shift from visual to logical invariants is a delicate step; hence, we recommend the mediation of an expert, who would follow the activity by discussions with the students.

A possible criticism of the role of the oracle in the construction of mathematics knowledge is that the failed attempt to find a counterexample in these activities does not prove the truth of students' claims or conjectures from a theoretical point of view, because it does not guarantee its non-existence; similarly, the use of generic examples does not guarantee the generality of a mathematical property. These represented a valid logical way of reasoning only if all possible configurations were explored, which is not possible to do empirically in a DGE. This approach to geometry conjectures is a powerful way of reasoning: the search for counterexamples creates the basis for actions that can promote understanding the way in which mathematicians think when they construct proofs by refutation: the production of generic examples helps students grasp the generality of a geometric property within the dialectic of *Actions|Logic*. Naturally, the teacher plays an important role in guiding students in this awareness.

A further comment on the notions of *oracle* and of *Actions|Logic* seems relevant to be underlined. These constructs appear close to some concepts of Brousseau's theory of didactical situations (Brousseau, 1997): for instance, nature or oracle is close to Brousseau's idea of milieu, and the distinction between interrogative and semantic games is close to the distinction between action/validation situations. This affinity can give further depth to the didactical meaning of our frame: however, we have not developed this issue for reasons of space and mainly since theories are not the focus of the paper. In any case, it is a point worthwhile of deepening.

Concluding, this type of activity helps to formulate and to check conjectures and systematize them according to a visual type of logic. In order to prove the theorem, students have to develop their reasoning on what is known as axioms or theorems. We believe that if students develop visual logical skills, they would be able to better understand the logic underpinning theorems' proof. In other words, conditional links based on visualization and DGE actions can prompt the understanding of logical links based on the results of proved theorems. The only difference between these two types of logical links is the source that allows students to derive them. If students are aware of the different (visual vs. theoretical) bases through which logical links can be made, the DGE experience not only would not prevent students' necessity of proving the visual results but also would help their understanding.

Finally, there is a possibility of generalizing the approach described here, because potentially any geometry theorem or property expressible in the ∀∃-form can be expressed as a DGE inquiry activity. This would require developing a functional approach to geometry, in which by means of verifier and refuter moves, the students visually grasp the generality of the conditional links between the objects involved in the theorem and formulate them through if-then statements.

The option to configure the relevant filters for various geometry statements creates new research opportunities for analyzing and assessing students' inquiry processes. Automation of the analysis makes possible extensive experimentation on inquiry-based knowledge acquisition.

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