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Trends of progression of student level of reasoning and generalization in numerical and figural reasoning approaches in pattern generalization

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Abstract This study explored progression of students' level of reasoning and generalization in numerical and figural reasoning approaches across grades and in different pattern generalization types. An instrument that included four figural patterns was administered to a sample of 1232 students from grades 4 to 11 from five private schools. The findings suggest that there was progressive development in the level of reasoning and generalization in each reasoning approach across clusters of grades. The level of reasoning and generalization in figural approach was higher than that for numerical approach in each grade. In addition, the level of reasoning and generalization for each approach and in each grade was not limited to one level but to several levels. The type of generalization influenced the progression of students' level of reasoning and generalization in each approach.

Keywords Numerical reasoning approach · Figural reasoning approach · Pattern generalization · Type of generalization · Level of generalization · Level of reasoning · SOLO taxonomy · Trends of progression

1 Introduction

Pattern generalization (PG) is a core area in mathematics that is characterized by more strategic and reasoning knowledge than mathematical content knowledge (El Mouhayar & Jurdak, 2015, 2016). This study addresses the progression of student ability and level of generalization in numerical and figural reasoning approaches across grades in the context of generalizing growing figural patterns and how the progression is influenced by different generalization types.

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This study was conducted in private schools in Lebanon where almost 60% of K-12 Lebanese students are served. The findings have implications for contexts in which PG may be implicit and/or may have a very limited place in the mathematics curriculum as well as for contexts in which PG is explicit in the curriculum. In Lebanon, like many countries, PG has a limited place in the mathematics curriculum. The Lebanese students' experience in generalizing figural patterns is moderate, since it does not stand on its own as a curricular topic in algebra. However, several processes that are closely related to PG, such as generalization, representation, and modeling span the algebra and arithmetic official curriculum for all grades, including the upper elementary school cycle (grades 4–6). Relations, functions, and their representations constitute a core area in the algebra curriculum. Many of these topics usually appear explicitly in other countries' high school algebra or pre-calculus official curricula or are considered by teachers as basic to cover the explicitly stated topics in algebra, relations and functions. Consequently, the findings may motivate researchers in other countries.

2 Theoretical perspectives

2.1 Generalizing strategies, reasoning approaches and types of generalization

One direction of research in math education was the study of students' reasoning and strategies in PG. Findings in previous studies (e.g., El Mouhayar & Jurdak, 2015, 2016; Healy & Hoyles, 1999; Radford, 2003, 2008, 2010b; Rivera & Becker, 2008; Rivera, 2010; Stacey, 1989) empirically demonstrate that students use different strategies and ways of reasoning to generalize patterns. In particular, El Mouhayar and Jurdak (2015) demonstrate that students in Lebanon frequently use the recursive strategy (pointing the common difference between pairs of consecutive terms and repetitively adding the constant from term to term to extend the pattern) and functional strategy (relating parts of the pattern to the figural step number) to generalize patterns.

We agree with Lannin et al.'s (2006) suggestion that strategies in PG often emerge from various ways of reasoning. For example, to extend the pattern in Fig. 1, an additive strategy may emerge from two different ways of reasoning: (1) noticing that the number of squares increases by two each time: 3, 5, 7, 9, 11, 13, 15, 17, 19; and (2) recognizing the structural growth of the pattern and that two squares increase in the top and bottom rows by each step.

We also distinguish between simple and more advanced strategies. We consider that strategies that enable finding a general term or a term in a specific figural step without the need to find the preceding figural steps (e.g., functional) are more advanced than strategies that involve determining the preceding steps (e.g., recursive).

We shall make a distinction between two classes of students' reasoning approaches, which was adopted in previous studies (El Mouhayar & Jurdak, 2016; Healy & Hoyles, 1999; Küchemann, 2010; Rivera & Becker, 2008). The figural reasoning approach (FRA) in which students generalize patterns by looking at the relational structures embedded in the figural steps of those patterns; a process that involves analyzing generic cases or finding relationships

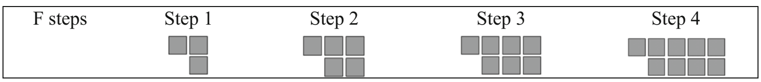

Fig. 1 A growing figural pattern

that students perceive between figures. The numerical reasoning approach (NRA), which involves generalizing patterns based on attributes that the students see in numbers that they generate from specific steps of the pattern. During this process, students neglect the figures that structure the pattern. El Mouhayar and Jurdak (2016) note that the NRA for the recursive strategy is more dominant than the FRA. However, the latter seems to be more dominant than the former for the functional strategy.

We note that the distinction between NRA and FRA is not universally accepted. For example, the Pythagoreans explored and established properties of natural numbers by using collections of figural dots to represent numbers: "●" represents "1"; "●●" represents "2"; etc.

Previous studies (e.g., El Mouhayar & Jurdak, 2015, 2016; Jurdak & El Mouhayar, 2014; Rivera, 2010; Stacey, 1989) distinguish between near and far generalizations. The former involves finding the value of a step that is near from a given step (e.g., step 9), whereas the latter consists of determining the value of a step that is distant from given figural steps (e.g., step 100).

2.2 Neo-Piagetian perspectives regarding stages of cognitive development

According to the Piagetian classical theory, a child's (or an adult's) cognitive development is perceived as a continuous process in stages. Neo-Piagetian theories (e.g., Biggs & Collis, 1982), however, refute this assumption. For example, McClelland (2010) points out that developmental progress that children make is not uniform and that there are stage-like progressions that exist as possible emergent consequences of a gradual learning process.

Another major difference between classical Piagetian and neo-Piagetian theories is that the former classifies an individual as belonging to a specific developmental stage of reasoning. Moreover, a child or an adult who demonstrates a specific level of reasoning in solving a problem will have the tendency to show the same level of reasoning on other problems. In contrast, neo-Piagetian theories report that individuals demonstrate different levels of reasoning on various problems. Individuals, regardless of their age, tend to develop their level of reasoning in correspondence to their expertise in a particular domain (algebra, geometry, etc.). Consequently, a novice in a particular domain (e.g., algebra) will exhibit a lower level of reasoning than that same individual will exhibit in a domain (e.g., arithmetic) by which s/he is more expert. Furthermore, neo-Piagetian models describe an individual's precise level of reasoning based on his/her work on a specific problem—as we will examine in this study.

2.3 The structure of the learned outcomes (SOLO) framework

The Structure of the Learned Outcomes (SOLO) is one of the neo-Piagetian theories that was developed by Biggs and Collis (1982). By investigating indicators of specific cognitive abilities, the SOLO taxonomy characterizes "the structure of any given response as a phenomenon in its own right, that is, without the response necessarily representing a particular stage of intellectual development" (Collis, Romberg, & Jurdak, 1986, p. 207).

A level of reasoning (LOR) (Table 1), in the SOLO taxonomy, refers to a cognitive ability of a specific response classified at a certain level reflected in the SOLO structure. An increased use of applicable elements, operations, and relationships related to a mathematical task produces a response of higher structural complexity. However, the structural complexities at each stage are identical. Moreover, each of the levels encompasses the preceding one, and they recur across developmental stages such that the highest level becomes the lowest at the following stage (Collis et al., 1986).

Table 1 A comparison between Radford and SOLO levels

Radford's levels of generalization	SOLO levels
	Prestructural (P level). Does not involve using important features
Abduction. Noticing a commonality	Unistructural (U level). Involves using one important feature
Arithmetic generalization. A generalization of the local commonality observed on some figures or numeric facts without the ability to use this information to provide an expression of whatever term of the sequence.	Multistructural (M level). Involves using a number of important, but disconnected, features
Factual generalization. Applies to objects at the same concrete level (e.g. numbers) and involves various types of semiotic means of objectification: spatial positional linguistic terms (e.g. "the next"); adverbs (e.g. always); rhythm and movement (e.g. 1+2, 2+3, 3+4, 4+5, …), etc.	Relational (R level). Involves using several important features that are connected together
Contextual generalization. Nonsymbolically based type of generalization that goes beyond the realm of specific figures and deals with generic objects (e.g. "the figure", "the top row").	Extended abstract (E level). Involves using important and connected features in an inclusive manner in relation to abstract principles
Symbolic generalization. Involves expressing the generalization through alphanumeric symbols and bypassing the positioning problem to produce nonspatially based symbolizations.	

2.4 Levels of generalization

Radford, Bardini, and Sabena (2007) describe algebraic generalization of patterns as a "shift of attention that leads one to see the general in and through the particular" (p. 525). This shift of attention involves *objectification*: "making something visible to the view" (Radford, 2003, p. 40) and to arrive at it, students rely on the use of various types of artifacts that Radford (2003) identifies as *semiotic means of objectification*.

In a series of research papers, Radford (2003, 2008, 2010a, 2010b) empirically distinguishes between different levels of generalizations and demonstrates the existence of three layers of algebraic generality: *factual*, *contextual*, and *symbolic* (Table 1). Furthermore, Radford (2010b) points out that, "just as not all symbolization is algebraic, not all patterning activity leads to algebraic thinking" (p. 40) and distinguishes between algebraic generalization, arithmetic (non-algebraic) generalization, and naïve inductions. According to him, generalizing a pattern algebraically involves (1) grasping a commonality, (2) generalizing this commonality to all the terms of the sequence, and (3) providing a rule that allows determining any term of the sequence. However, arithmetic generalization involves generalizing a local commonality observed on some figural steps without the ability to come up with a rule that determines any term of the pattern. In naïve inductions, generalizations are made by guessing rules.

2.5 Logical correspondence between levels of reasoning and generalization

Table 1 presents Radford's levels of generalization (column 1) with the suggested corresponding SOLO levels (column 2) as they appear in the literature. The levels in each taxonomy assume a hierarchy. Table 1 shows that the two taxonomies may be reasonably matched in terms of compatibility of respective hierarchies. Abduction, arithmetic, factual, contextual, and symbolic levels respectively correspond to U, M, R, and E levels. The unistructural level (U) involves the use of one relevant aspect of a task, which entails noticing a commonality—abduction—in the context of generalization. The arithmetic generalization level corresponds to the multistructural level (M) because both levels in the two taxonomies correspond to a step-by-step generalization. More specifically the arithmetic generalization involves extending a commonality to subsequent terms of a pattern without the ability to come up with a rule that determines any term of the pattern (Radford, 2010b). Similarly, the multistructural level involves the use of several relevant, but disjoint, aspects of a pattern (e.g., noticing a local commonality) to determine particular steps. Consequently, both levels are based on making use of a regularity to determine a near or far generalization but without the ability to provide a rule or to determine any term of the pattern.

Factual generalization level and the relational level (R) are also compatible. The former is classified as a presymbolic algebraic generalization, which applies to generalization of objects at the same concrete level. The latter involves generalization, which is based on using all relevant and connected features of a generalization task. Although factual generalization reveals a higher level of generalization in comparison with arithmetic generalization and the relational level is characterized at a higher SOLO level in comparison with the multistructural level, both levels do not go beyond the realm of specific steps of a pattern and do not deal with generic objects.

Table 1 also shows that contextual and symbolic generalizations correspond to the extended-abstract level (E). Both generalization levels go beyond concrete objects and deal with generic objects. Similarly, the extended-abstract level is characterized by using a generic case.

Even though there is a logical correspondence between the levels, the relationships between the two taxonomies are not as simple as they appear because of differences in the theoretical insertions of each taxonomy. Radford's (2003) taxonomy proposes a semiotic-cultural approach to study students' semiotic means of objectification, which includes an analysis of gestures, body, posture, and other embodied signs and resources (e.g., language writing, speech, tools, analogies) in presymbolic (factual and contextual) and symbolic generalizations. In contrast, the SOLO taxonomy is inserted in the cognitive realm by which the body (including gestures and body posture) and other material culture play a secondary role.

The high degree of correspondence between the two sets of levels suggests adopting the SOLO approach as a methodological simplifying move that does not detract from Radford's approach. Matching the two sets of levels suggests that classification of a response of a generalization task to a particular SOLO level falls within the corresponding layer of generality. Thus, when the SOLO taxonomy (which is of general nature and predominantly cognitive) is applied to PG tasks, the SOLO levels of reasoning are very similar to levels of generalization (which are specific to forms of algebraic generalization thinking and correspond to cultural semiotic perspective). However, there seem to be one exception: The pre-structural level has no matching level in Radford's model. This is comprehensible since this level indicates refusal or inability to engage in the task.

3 Rationale of the study

This study builds on previous research (El Mouhayar & Jurdak, 2015, 2016; Healy & Hoyles, 1999; Lannin, Barker, & Townsend, 2006; Radford, 2003, 2008, 2010a, 2010b; Rivera & Becker, 2008) regarding strategies, ways of reasoning and levels of generalization in PG. Little is known about the progression of student level of reasoning and generalization associated with the use of NRA and FRA across grades in the context of generalizing growing figural patterns and how these are influenced by PG type. Such knowledge is essential for researchers, teachers, and curriculum designers. If the progression is identified and how it is influenced by PG type, tasks could be designed to promote students' performance throughout different school cycles.

We agree with Radford's (2008) suggestion that one perspective (e.g., neo-Piagetian cognitive viewpoint) is not enough to offer a satisfactory scientific explanation of students' processes to generalize patterns due to the complexity of generalization. Thus, the present study uses two theoretical perspectives to analyze students' responses: the SOLO and Radford's taxonomies. As we previously noted, the former is of general nature and predominantly cognitive, whereas the latter is specific to forms of algebraic thinking and corresponds to cultural semiotic perspective.

The approach that we adopt in this study is first to establish a logical correlation between the SOLO taxonomy and Radford's taxonomy (see Table 1). Second, to present an analysis of illustrative examples in order to demonstrate empirically the results of the logical analysis. Such a logical and empirical correspondence between the two taxonomies does not mean that SOLO taxonomy may replace Radford's taxonomy; however, it will enable the researcher to infer students' level of generalization based on SOLO classification. Third, to systematically explore and describe the progression in the levels of reasoning and generalization across developmental stages and age and to study the influence of PG on the progression of those levels.

4 Research questions

The research questions were as follows:

1. Are there trends of progression in student level of reasoning and level of generalization in numerical and figural approaches across grades? How do those trends compare across grades (using SOLO and Radford taxonomies)?
2. How does the generalization type (immediate, near, and far generalizations) influence the progression of student level of reasoning and level of generalization across grades in numerical and figural approaches?

5 Method

5.1 Participants

A sample of 1232 students from grades 4 to 11 in five private schools, which are located in greater Beirut area in Lebanon, participated in the present study. The distribution of students by grade was as follows: 147 students in grade 4 (11.9%), 169 students in grade 5 (13.7%), 164 students in grade 6 (13.3%), 172 students in grade 7 (14%), 200 students in grade 8 (16.2%), 160 students in grade 9 (13%), 123 students in grade 10 (10%), and 96 students in grade 11 (7.8%).

5.2 Instrument

The instrument used in the present study contained four figural patterns formed of growing number of squares (Fig. 2). For each pattern, the first four steps were given and the students were requested to find the number of squares in steps 5, 9, 100, and/or n and to explain their responses.

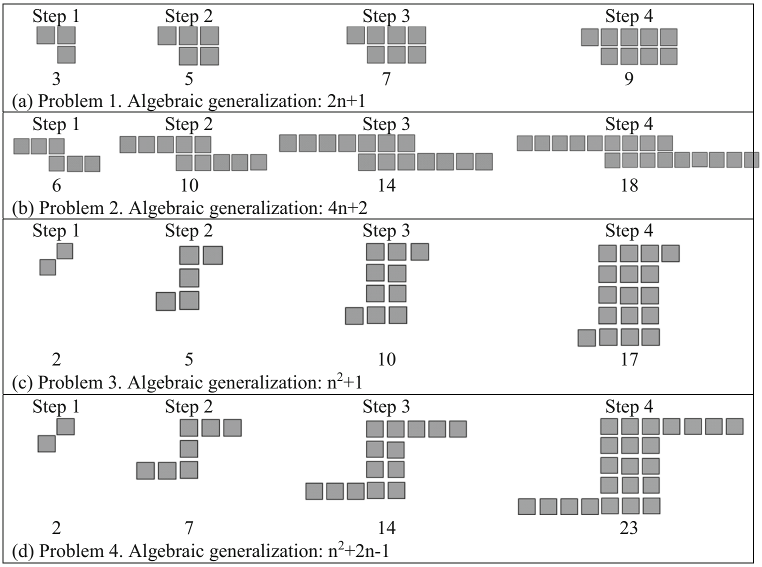

Fig. 2 The four PG problems in the instrument

5.3 Data collection

Data collection took place during classroom mathematics sessions, each of duration 55 min in the presence of the author. Students worked individually to solve at least two of the following problems: one linear pattern (problems 1 or 2) and one non-linear pattern (problems 3 or 4). Some students solved more than two problems. This occurred when students finished solving the two problems that were assigned to them and there was still sufficient time to solve other problems.

5.4 Data analysis

The content validity of the instrument was previously confirmed by the author and other researchers (Judak & El Mouhayar, 2014; El Mouhayar & Jurdak, 2015, 2016) based on literature review and a pilot study, which was conducted on a sample of 50 students in grades 4 to 11.

The data were subjected to a series of analyses. First, each student's response for each task of the instrument was assigned a SOLO level using the rubric (Table 2). Our coding scheme began with classification of different levels of SOLO as found in the literature (Biggs & Collis, 1982; Collis et al., 1986), which provided us with initial categories for coding (Strauss & Corbin, 1998). A similar approach was adopted for assigning the reasoning approach and strategy use, by which the coding categories were created in previous studies (El Mouhayar & Jurdak, 2015; 2016). For example, Table 3 shows the coding categories of the functional and recursive strategies in NRA and FRA approaches. Then, a constant comparative method (Glaser & Strauss, 1967) was applied to test and revise the coding of SOLO, reasoning approach and strategy use in pattern generalization.

All students' responses were scored independently by three raters (including the author). Several meetings occurred among the raters and disagreements in coding were discussed until a consensus was reached.

Table 2 Rubric for scoring student LOR in PG

LOR	SOLO-based classification associated with NRA	SOLO-based classification associated with FRA
U level	Uses one numerical relevant aspect of the pattern.	Uses one relevant structural aspect of the figural pattern.
M level	Uses several numerical disjoint facts or relations with a focus on adding or counting the numbers in the consecutive numerical steps.	Uses several relevant structural facts or relationships of the figural pattern with a focus on adding or counting the growing components in the consecutive figural steps.
R level	Uses several numerical relationships or facts beyond counting consecutive numerical steps.	Connects the components of a particular figural step with the step number or uses several figural relationships beyond counting the consecutive figural steps.
E level	Relates the numerical growth of pattern with the *n*th step number.	Uses a generic case in the pattern and relates the figural growth of pattern with the nth figural step number.

Cross tabulations and error bars were used for the quantitative analysis. First, a cross tabulation of SOLO level in each reasoning approach by grade was done to explore potential differences of significant values across grades. For this purpose, the researcher explored the chi-squared and adjusted residual values. Second, error bars showing 95% confidence intervals (CIs) around the average in each grade were used to represent the variation in LOR in each approach. While the author used the averages to visualize and explain the growth in student LOR across grades, the CIs were used to represent variations in each grade. In a similar manner, the author analyzed the impact of PG type on the progression of student LOR across grades.

Students' responses were analyzed qualitatively in order to understand differences in their SOLO levels of reasoning and their levels of generalization in numerical and figural reasoning approaches. In particular, typical examples of students' responses were selected for each SOLO and generalization level in order to support deeper interpretation of the developmental trends that were identified in the quantitative analysis and to show the influence of PG on those trends.

6 Findings

6.1 General trends of progression of student LOR in NRA and FRA across grades

Cross tabulation of student NRA and FRA by LOR by grade is shown in Table 4. The findings show that chi-squared was significant ($p = 0.00$) for NRA (χ^2 (21) = 410.741) and for FRA (χ^2

Table 3 Rubric for coding student strategy use within NRA and FRA in PG

Strategy	NRA	FRA
Functional	Recognizes a rule based on the numeric pattern.	Identifies the growing components of the pattern as well as the constant components and relates them with each other and with the figural step number.
Recursive	Recognizes a numeric pattern by finding the difference between consecutive terms and then adds it to a numeric value in given step to reach next step.	Recognizes the structural growth of the pattern between consecutive figural steps and then adds the value of the growth to a given term in order to reach the next figural step.

Table 4 Cross tabulation of student SOLO by NRA and FRA by grade

Grade		LOR				Total (%)
		U level (%)	M level (%)	R level (%)	E level (%)	
		NRA				
4		58.8*	37.8*	3.4*	0.0*	100.0
5		55.4*	42.7	1.9*	0.0*	100.0
6		40.5*	54.1	5.5*	0.0*	100.0
7		39.8	52.5	7.1	0.7	100.0
8		33.1*	54.6	11.3*	1.0	100.0
9		27.1*	62.8*	8.1	2.0	100.0
10		17.9*	65.0*	13.9*	3.2*	100.0
11		16.0*	68.8*	9.2	6.0*	100.0
Total	Frequency	1471	2192	302	55	4024
	%	36.6	54.5	7.5	1.4	100.0
		FRA				
4		58.7*	30.3*	7.3	3.7*	100.0
5		54.2*	40.8*	4.2	0.8*	100.0
6		41.1*	49.1	6.5	3.3*	100.0
7		29.6	56.5	7.0	7.0*	100.0
8		21.1*	63.1*	10.4*	5.3*	100.0
9		26.7	56.0	4.3	13.0	100.0
10		13.8*	58.6*	6.7	20.9*	100.0
11		16.7*	47.9*	5.5	29.9*	100.0
Total	Frequency	562	1101	139	253	2055
	%	27.3	53.6	6.8	12.3	100.0

*Contributor to the significance of chi-squared

10.8% of the responses were classified at the P level (empty response), and 10% were classified as "unscored response" because student's response was unclear

(21) = 317.342) suggesting progression in student LOR within each approach. Table 4 shows that the majority of the cells contributed significantly to $\chi 2$ because the adjusted residual was greater than |2| in those cells.

Figure 3 represents trends of progression in the LOR associated with NRA and FRA. Figure 3 shows that the former was lower than the latter in each grade. The error bars show that the students used different SOLO levels for each of the NRA and FRA in each grade. Moreover, the length of the 95% confidence interval for the FRA was longer than that for the NRA indicating a larger variation in LOR.

Figure 3 also shows that the progression for each approach had several stages across clusters of grades. For NRA, the qualitative analysis suggests that students' responses that were classified at the U level in stages 1 (grades 4–7) and 2 (grades 8 and 9) used one numerical aspect of the pattern. Students' responses that exhibited characteristics of the M level, in stages 2 (grades 8 and 9) and 3 (grades 10 and 11) used several elements based on numbers that they generated from particular steps of the pattern. However, the students did not relate the elements together. This suggests that students' responses associated with NRA were classified, on the average, at two generalization levels: (1) abduction level (matches U level) and (2) arithmetic generalization level (matches M level). Thus, those students were capable of (1) grasping a local commonality observed on some numeric facts and (2) extending the commonality to subsequent numerical terms of the pattern; however, they were not able to use the commonality to provide a direct expression of whatever term of the pattern (see Table 5 illustrative examples).

For FRA, students generalized patterns based on relational structures that the students perceived in the figures forming the pattern. Students' responses that were classified at the U level in stage 1

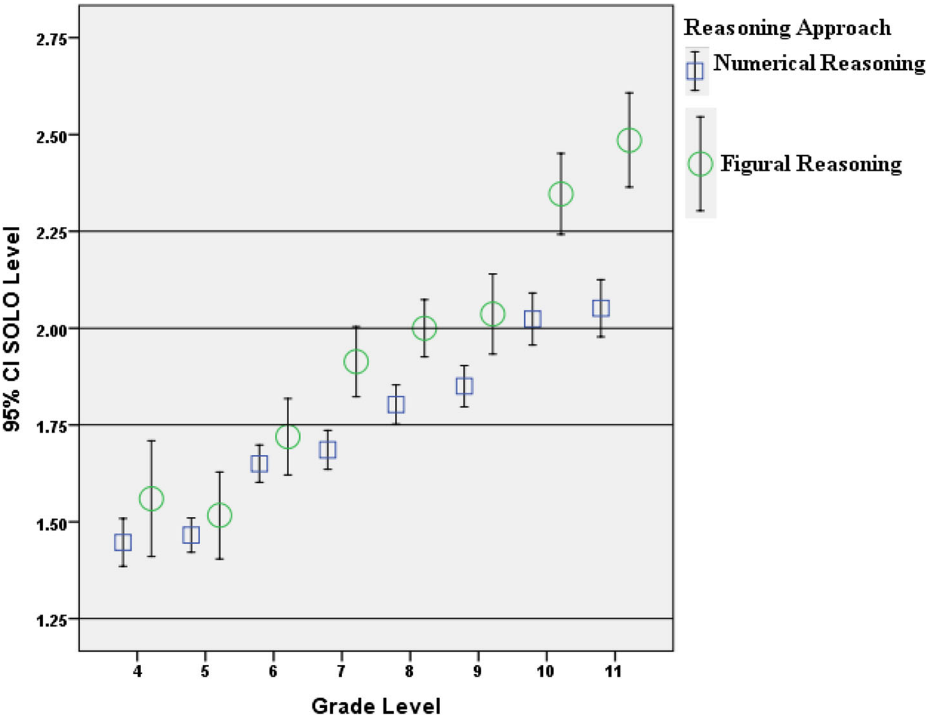

Fig. 3 Developmental trends of student LOR across grades associated with NRA and FRA. Student LOR for a grade was defined as close to SOLO level L if the mean (M) is between L < M < L + 0.25; within the average of L and L + 1 if L + 0.25 < M < L + 0.75; approached L if L—0.25 < M < L

(grades 4–6) used one aspect of the pattern by looking at relationships that they saw between the figures. However, students' responses that were classified at the M level made use of various elements of the pattern by looking at the relational structures of the figures forming the pattern. The focus of those responses was to add or count the squares in consecutive figural steps of the pattern. Students' responses that were classified at the R level in stage 4 (grades 10 and 11) focused on either finding relationships between the figures and the step number or finding relationships beyond counting consecutive steps of the pattern. This suggests that students' responses associated with FRA were classified, on the average, at three levels of generalization: (1) abduction level (matches U level); (2) arithmetic generalization level (matches M level); and (3) factual generalization (matches R level). Thus, students in stages 1 (grades 4–6), 2 (grades 7 and 8), 3(grade 9), and 4 (grades 10 and 11) were capable of (1) grasping a local commonality observed on some figural facts and (2) extending the commonality to subsequent figural terms of the pattern. Only students in stage 4 were able to use the commonality to provide a direct expression of whatever term of the pattern; however, their generalizations applied to objects at the same concrete level (see Table 5 illustrative examples).

6.2 Effect of PG type on the trends of progression of student LOR and level of generalization associated with NRA and FRA

The cross tabulation of student LOR, within NRA and FRA, by grade by type of generalization (immediate, near, and far) were done. The findings suggest that the trends of progression of student LOR, across grades, in each approach was moderated by generalization type and by grade. For

Table 5 Illustrative examples of students' SOLO level and level of generalization

Problem; Question	SOLO level	Level of Generalization	Reasoning approach	Illustrative example
P1, Q1	U level	Abduction	NRA	11 squares; I counted by 2
P1, Q2	M level	Arithmetic generalization	NRA	19; since we have to add 2 then we keep going with the pattern till we reach $9 \rightarrow 9+2=11$, $11+2=13$, $13+2=15$, $15+2=17$, $17+2=19$
P1, Q1	U level	Abduction	FRA	9 squares; I obtained by counting Figure 4 and added another square to up and down
P1, Q2	M level	Arithmetic generalization	FRA	17 squares; I kept adding one square on both sides till I reached fig. 9, which is 9 on the top and 8 on the bottom
P2, Q2	R level	Factual generalization	FRA	38 squares; number of squares under each other is increasing one square $\rightarrow 5+4=9$. In each figure, 4 squares are added $\rightarrow$ 9–5 = 4/4 × 4 = 16 $\rightarrow 22+16=38$
P4, Q3	R level	Factual generalization	FRA	10,199; on each side there is 100. So 200. In the middle, row is (100–1) column is (100 + 1). So there is 99 × 101 = 9999 boxes in the middle. Thus, 200 + 9999

P refers to problem, Q refers to question

NRA, the findings show that chi-squared was significant ($p = 0.00$) for the immediate ($\chi2$ (21) = 134.033), near ($\chi2$ (21) = 204.033) and far generalizations ($\chi2$ (21) = 137.394). For FRA, the findings show that chi-squared was also significant ($p = 0.00$) for the immediate ($\chi2$ (21) = 222.722), near ($\chi2$ (21) = 81.65), and far generalizations ($\chi2$ (21) = 72.155).

Tables 6 and 7 show that the adjusted residual was greater than |2| for several cells indicating that these cells contributed in a significant manner to $\chi2$. For NRA, most of the cells that contributed significantly to $\chi2$ were at the U and/or M levels. However, for FRA, most of the cells were at the E level (except for immediate generalization where the adjusted residual was larger than |2| in most of the cells at the U, M and E levels).

For NRA, the developmental trends of student LOR for the immediate, near, and far generalization types were distinct (Fig. 4a). SOLO for near generalization was higher than that for far generalization and the latter was higher than that for immediate generalization in each grades 4–11. For FRA, the trends of progression of student LOR for the immediate, near and far generalizations were distinct (Fig. 4b). SOLO for far generalizations was higher than that for near generalizations and the latter was higher than that for immediate generalizations in each grades 4–11. The three trends of progression in each of the NRA and FRA showed different developmental stages across clusters of grades.

Below are two typical examples of students' responses that adopted the NRA in near and far generalizations. We also discuss each example using the theoretical perspectives (SOLO and Radford's taxonomies). The first typical example (Fig. 5) represents a response in a near generalization task. The excerpt is from grade 7:

From a SOLO perspective, the response was classified at the M level. The student used more than one relevant aspect of the pattern: (1) noticed that the terms of the pattern increase by two, (2) identified the number of squares in step 5 and (3) repetitively added the common increment between pairs of consecutive steps four times from step 5 to reach step 9. From Radford's perspective, the student's response corresponds to the arithmetic generalization level, where the student generalized a local commonality observed on some steps of the pattern. The response

Table 6 Cross tabulation of student LOR associated with NRA across grades by PG type

Grade	% within LOR				Total (%)
	U level (%)	M level (%)	R level (%)	E level (%)	
	Immediate generalization				
4	73.2*	26.8*	0.0	0.0	100.0
5	65.9*	34.1*	0.0	0.0	100.0
6	50.6	49.4	0.0	0.0	100.0
7	46.9	52.5	0.6	0.0	100.0
8	36.2*	63.8*	0.0	0.0	100.0
9	30.5*	68.7*	0.0	0.8	100.0
10	31.6*	68.4*	0.0	0.0	100.0
11	14.5*	81.8*	1.8*	1.8*	100.0
Total (%)	48.5	51.1	0.2	0.2	100.0
	Near generalization				
4	51.3*	40.0*	8.7	0.0	100.0
5	46.5*	50.2*	3.3*	0.0	100.0
6	29.0	65.2	5.8*	0.0	100.0
7	30.4	60.7	8.2	0.7	100.0
8	21.1*	66.4	12.2	0.3	100.0
9	22.3	64.2	12.8	0.8	100.0
10	11.5*	63.5	23.5*	1.5	100.0
11	11.6*	69.1*	14.4	5.0*	100.0
Total (%)	26.7	61.3	11.0	1.0	100.0
	Far generalization				
4	47.6	51.2	1.2*	0.0	100.0
5	52.6*	44.5	2.9*	0.0*	100.0
6	45.1	43.8	11.1	0.0*	100.0
7	48.8*	38.7*	11.3	1.2	100.0
8	47.2*	33.2*	17.0*	2.6	100.0
9	32.3*	55.7*	7.0	5.1	100.0
10	22.0*	65.9*	4.9	7.3*	100.0
11	23.9*	61.9*	4.4	9.7*	100.0
Total (%)	40.9	47.3	8.8	3.1	100.0

*Contributor to the significance of chi-squared

provides us with an obvious suggestion that, for this student, the common increase refers not only to the terms that were explicitly cited but also to all the terms of the pattern. One clear indicator is the statement: "We are always skipping 2." However, the student was not able to use this information to provide a response to the far generalization. Thus, the student showed inability to make use of the already-noticed regularity: "We are always skipping 2" to provide an exact value for the number of squares in step 100.

Qualitative analysis revealed that students who used NRA to establish far generalizations in steps 100 or n were more likely to determine differences between successive steps of the pattern (recursive strategy) or did not distinguish between the number of squares in step n and the step number. The following excerpt (Fig. 6) from grade 5 illustrates the recursive strategy:

In the above excerpt, we do not find adverbs, such as "always" that appeared in the previous example. Actually, the student's response does not reach a verbal description. From a SOLO perspective, the response was classified at the M level since it involved the use of several relevant disjoint aspects of the task. From Radford's (2003, 2010b) perspective, the student relied on a rhythm during the course of the numeric actions in order to investigate the pattern. Moreover, this rhythm played the role of the adverb "always." Radford (2003, 2010b)

Table 7 Cross tabulation of student LOR associated with FRA across grades by PG type

Grade	% within LOR				Total (%)
	U level (%)	M level (%)	R level (%)	E level (%)	
	Immediate generalization				
4	68.4*	27.6*	2.6	1.3	100.0
5	58.2*	41.8*	0.0	0.0*	100.0
6	45.4*	53.2*	1.4	0.0*	100.0
7	28.3	67.0	1.4	3.3	100.0
8	21.4*	75.0*	2.5	1.1*	100.0
9	24.7	69.3	1.9	4.2	100.0
10	11.1*	78.7*	3.2	6.9*	100.0
11	17.6*	68.8	1.0	12.6*	100.0
Total (%)	28.1	65.8	1.9	4.2	100.0
	Near generalization				
4	27.3	50.0	18.2	4.5*	100.0
5	47.4*	42.1	5.3	5.3	100.0
6	32.0	50.0*	12.0	6.0*	100.0
7	36.8	33.3	17.5	12.3	100.0
8	23.9	35.8	29.9*	10.4*	100.0
9	26.4	32.1	11.3	30.2	100.0
10	16.1	29.0	19.4	35.5*	100.0
11	17.2	17.2*	12.1	53.4*	100.0
Total (%)	26.3	34.0	17.0	22.7	100.0
	Far generalization				
4	54.5*	9.1	18.2	18.2	100.0
5	30.0	30.0	40.0	0.0*	100.0
6	34.8	21.7	26.1	17.4*	100.0
7	25.0	28.1*	25.0	21.9*	100.0
8	12.9	16.1	38.7*	32.3	100.0
9	40.6*	6.3	9.4	43.8	100.0
10	22.9	6.3	6.3*	64.6*	100.0
11	13.0*	3.7*	14.8	68.5*	100.0
Total (%)	24.9	12.4	19.1	43.6	100.0

*Contributor to the significance of chi-squared

suggests that the rhythm forms a fundamental semiotic means of objectification that helps to make awareness of a particular order and to go beyond the specific steps of a pattern. However, this example shows that the student remained at the arithmetic generalization level and did not reach the factual generalization. The student's response did not lead to a schema through which the student was able to determine the number of squares in any particular step of the pattern.

The following excerpt (Fig. 7) from grade 7 illustrates a student's response that used the FRA to deal with immediate, near, and far generalizations:

From a SOLO perspective, the response was classified at the M level in the immediate and near generalizations. The response was classified at the E level in far generalization. From Radford's (2003, 2010b) perspective, the excerpt shows that the student used arithmetic generalization for immediate and near generalizations, whereas he/she used contextual generalization for the far generalization task. In immediate generalization, the student noticed that the terms increased by two squares. The common increase between the consecutive steps applies not only to the terms that were addressed in the questions but also to any term of the pattern: "Since in each figure one square is being added on the top side and on the bottom side." Moreover, the student did make use of the regularity that he/she noticed in the

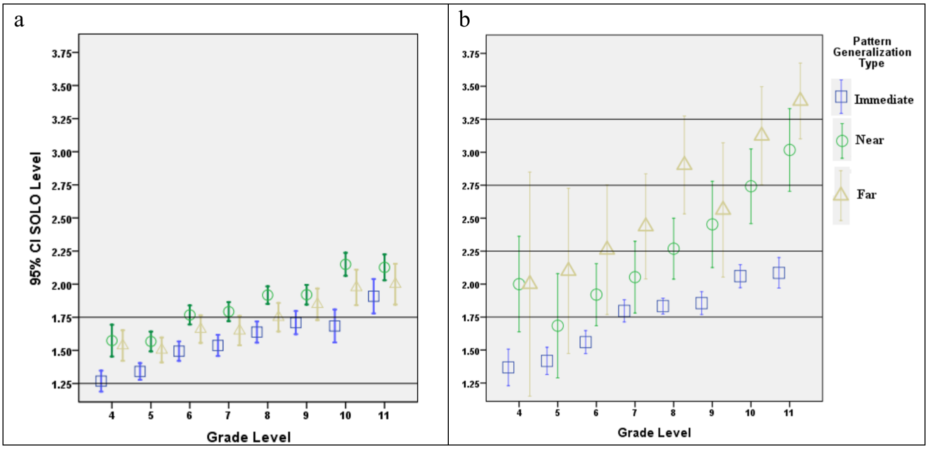

Fig. 4 Developmental trends of student LOR associated with NRA (a) and FRA (b) across grades for the immediate, near and far generalizations. Student LOR for a grade level was defined as: close to SOLO level L if the mean (M) is between L ≤ M < L + 0.25; within the average of L and L + 1 if L + 0.25 ≤ M < L + 0.75; approached L if L—0.25 ≤ M < L

immediate generalization in order to provide the number of squares in the near generalization task. In fact, the student kept on adding one square in each side until reaching figural step 9 of the pattern: "I kept on adding one square on both sides till I reached fig. 9 which is 9 on the top and 8 on the bottom" (Fig. 7). According to Radford's (2010b) perspective, the student has not yet used algebraic generalization for the near generalization task. Actually, the student generalized the local commonality that he/she observed on some figural steps without showing the ability to use this commonality to provide an expression of whatever term of the pattern. In far generalization, the student realized that there was another commonality that links the number of the squares in the bottom and top sides and the step number. The statement "Since the number of each figure is the number of squares in the bottom side, and the top side more than one square" indicates the student's awareness that this commonality holds to all the terms of the pattern. Drawing on this noticed regularity, the student was then able to provide an

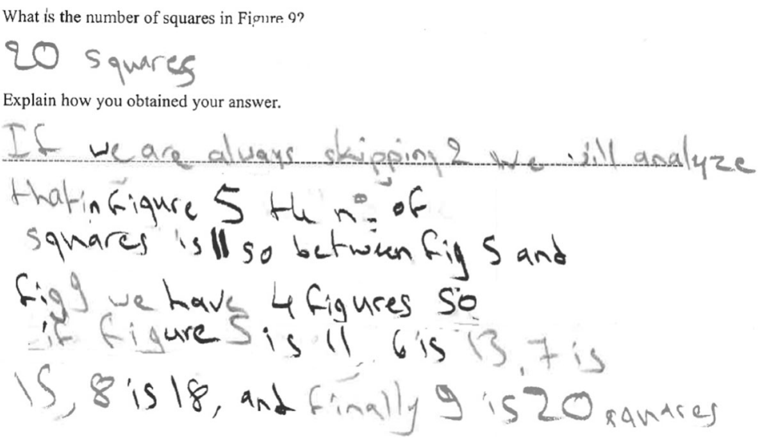
What is the number of squares in Figure 9?
20 squares
Explain how you obtained your answer.
If we are always skipping 2 we will analyze
that in figure 5 the n° of
squares is 11 so between fig 5 and
fig 9 we have 4 figures so
if figure 5 is 11 6 is 13, 7 is
15, 8 is 18, and finally 9 is 20 squares

Fig. 5 A student's response to a near generalization task in problem 1

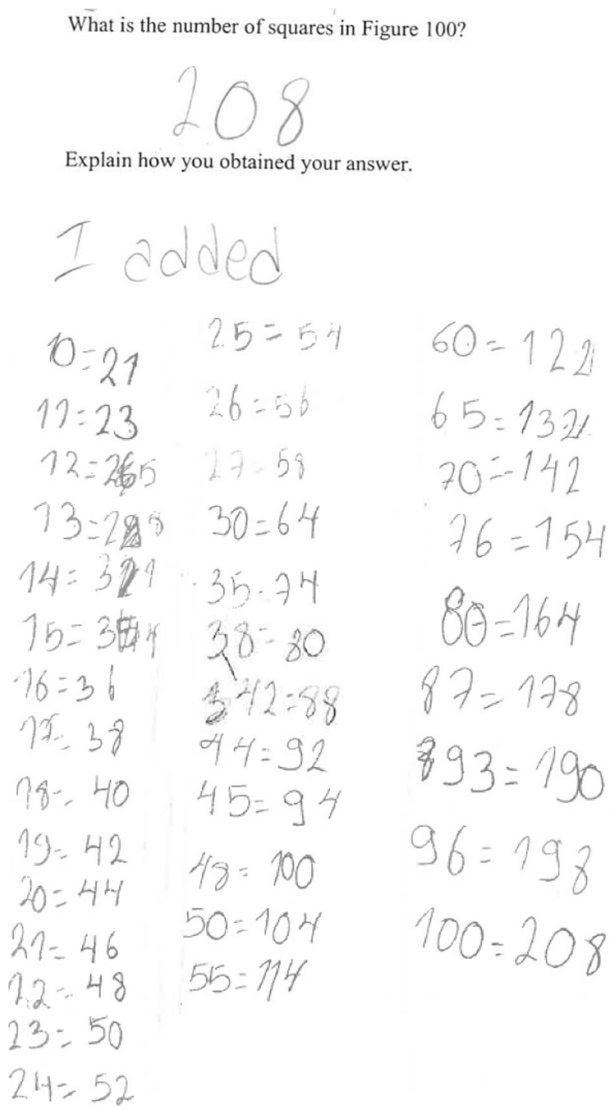

Fig. 6 A student's response (S10504) to a far generalization in problem 1

expression for the value of figural step 100. Thus, the student here made an algebraic generalization, one that Radford (2003) called contextual generalization.

7 Discussion

In this study, we distinguish between numerical and figural approaches although the dichotomy between the two approaches is not universal. We also use two theoretical perspectives to

What is the number of squares in Figure 5?

The number is : 11 squares

Explain how you obtained your answer.

Since in each figure one square is being added on the top side and on the bottom side.

What is the number of squares in Figure 9?

The number of squares: 17 squares

Explain how you obtained your answer.

I Kept adding one square on both sides till I reached fig 9. which is (9 on the top, and 8 on the bottom).

What is the number of squares in Figure 100?

The number of squares: 201 squares.

Explain how you obtained your answer.

Since ~~in each fig:~~ the number of each fig. is the number of squares in the bottom side,[3] and the top side is more than one square. So fig. 100 the bottom part is 100 and the top one is 101 the 100 + 101 = 201 square.

Fig. 7 A student's response to immediate, near, and far generalizations in problem 1

analyze students' responses: the SOLO taxonomy and Radford's taxonomy. The analysis allowed us to show that there seems to be both empirical and theoretical support for matching the SOLO and Radford's levels. Thus, students' responses to pattern generalization tasks can be characterized in a way that is compatible with Radford's levels without necessarily deriving students' behavior indicative of Radford's levels for every pattern generalization task. We note that the restriction to SOLO levels in the quantitative analysis was adopted as a methodological simplifying move that does not detract from Radford's semiotic-cultural approach.

One of the findings regarding the first research question shows that students demonstrated a progressive development in the LOR associated to NRA and FRA across clusters of grades, rather than from one grade to another. Drawing on the SOLO perspective, this finding suggests that students' responses in the different generalization tasks exhibited a significant increase in the usage of elements and relationships associated to NRA and FRA across clusters of grades, rather than across grades. In addition, the LOR, in each grade, was not limited to one SOLO level but to several levels. For instance, the students used the U, M, and R levels in each grade. The hypothesis of the existence of various levels of reasoning in each developmental stage (Biggs & Collis, 1991) may explain this finding.

The occurrence of different SOLO levels of reasoning within each developmental stage, with the dominance of one level, possibly explains the existence of various levels of generalizations observed in Radford's taxonomy in each stage with the dominance of one generalization level. From Radford's (2003, 2008, 2010b) perspective, this finding suggests that the variation of levels of generalization followed identifiable progressive trends across clusters of grades and the one associated with FRA was higher than that associated with NRA. Figure 3 shows that the students in elementary cycle (grades 4–6; corresponding ages 10–12), on the average, noticed a numerical or figural commonality (abduction that matches U level) or generalized a local commonality observed on some numeric or figural facts (arithmetic generalization that matches M level). Students' level of generalization in intermediate school cycle (grades 7–9; corresponding ages 13–15) was classified, on the average, at the arithmetic numerical or figural generalization level (matches M level); however, students using the FRA also tended to use factual generalizations (match R level). Students who used NRA in secondary school cycle (grades 10 and 11; corresponding ages 16 and 17) were classified, on the average, at the arithmetic numerical generalization level (matches M level), and they showed a tendency to use factual generalizations (match R level) based on numerical actions and in the form of numerical schemes. Nevertheless, the level of generalization for students who used the FRA was classified, on the average, at the arithmetic or higher layers of generality (factual, contextual, or symbolic generalizations that correspond to R or E levels).

Why did the student LOR associated with NRA progress from U to M level but hardly reached the R level; whereas student LOR associated with FRA developed from U to M and then to the R and E levels? A plausible explanation is that the progression of student LOR in PG is due more to experience than maturation. In particular, the involvement of Lebanese students in pattern-related topics may partially explain their progression. Findings in the literature support the hypothesis that students' experience in generalizing patterns leads to the development of their abilities to generalize patterns (e.g., Jurdak & El Mouhayar, 2014; Lannin et al. 2006; Rivera, 2010; Stacey, 1989). In support of this explanation, we note that the students in this study were indirectly exposed to pattern-related topics in different grades. Students start to deal with variables, equations, algebraic expressions in grade 7 (first grade in the intermediate cluster) and with functions and their representations in grade 9 (last grade in the intermediate cluster).

Another plausible explanation is the relationship between student approach and strategy use. Findings in a previous study (El Mouhayar & Jurdak, 2016) suggest that students who adopt NRA frequently use the recursive strategy, whereas those who adopt FRA frequently use the functional strategy. Accordingly, further analysis showed that, across grades, the development of student LOR was moderated by the recursive strategy in NRA and by the functional strategy in FRA. The LOR for the recursive strategy was classified, on the average, at the U level in grades 4–7, and it progressed to the M level in grades 8–11, whereas the LOR for functional strategy was classified at the M level in grades 4–9, and it developed to R level in grades 10 and 11.

A third plausible explanation may be made by drawing on the logical and empirical correspondence between SOLO taxonomy and Radford's classification of levels of generalizations. According to this correspondence, the R level matched factual generalizations, which allow to go beyond the first several steps of the pattern and to determine the number of squares in any particular step of the pattern (e.g., steps 9 or 100). However, factual generalizations, which consist of "numerical actions in the form of numerical scheme" (Radford, 2003, p. 65), do not allow going beyond the realm of specific steps of the pattern in order to deal with the generic objects (Radford, 2010b). Contextual generalizations, which correspond to E level, were not limited to generalizing numerical actions but also to generalizing objects of actions. Consequently, students who used FRA were able to "go beyond the realm of specific figures and deal with generic objects that cannot be perceived by our senses" (Radford, 2003, p. 65). According to Radford (2003), students in contextual and higher layers of generality, name general objects by using situated and embodied depiction of those objects (like the top row, bottom row, the figure, the next figure). In fact, students who used FRA, on the average, reached the realm of algebraic generalization in grades 10 and 11, since they were able to provide a direct rule that allowed them to calculate the number of squares in any step of the pattern.

The findings regarding the second research question support the assumption that the type of generalization (immediate, near, and far generalizations) influences the performance of students. The findings show that the LOR associated with NRA for near generalizations was higher than that for far generalizations, which was higher than that for immediate generalizations in each of the grades 4–11. A plausible explanation may be made by drawing on Radford's (2008) comment that students are more likely to fail in employing the NRA to address far generalizations:

> Numeric patterns are reputedly difficult not because of the difficulties that the students encounter in grasping the commonality, but because the students tend to fail at using it to form a direct and meaningful rule. (p. 93)

However, the LOR in FRA for far generalizations was higher than that for near generalizations, and the latter was higher than that for immediate generalizations in each of the grades 4–11. From a SOLO perspective, the amount of used elements and relationships in the FRA approach increased as the task demands progressed. This may be due to the nature of generalization tasks, where far generalization tasks require higher levels of reasoning compared to near generalization tasks that involve higher levels of reasoning compared to immediate generalizations.

The findings in Fig. 4 show that the variation of student LOR associated with each of NRA and FRA followed recognizable trends of progression in immediate, near, and far generalizations. From Radford's (2003, 2008, 2010b) perspective, this finding suggests that the type of generalization influenced the variation of levels of generalization, which followed identifiable progressive trends across clusters of grades and the one associated with FRA was higher than that associated with NRA in each generalization type.

This study informs researchers, teachers and curriculum designers about the progression of student level of reasoning and generalization in numerical and figural approaches across grades and the influence of PG types on the progression. The findings show that the development occurred across clusters of grades, which may be helpful to inform instruction on how to help students to reach the optimal range of level of reasoning in specific developmental stages at school. For example, in the intermediate school cycle, students using FRA are more likely to reach higher optimal range of LOR compared to those using NRA. The findings show that students using FRA reached factual generalization level, whereas the level of generalization for those using NRA was classified at the arithmetic generalization level. The findings regarding the influence of PG on students' LOR may be helpful to inform instruction that students using NRA may reach their optimal range of LOR in near generalizations, whereas students' using FRA may reach their optimal range of LOR in far generalizations in different school cycles.

In conclusion, this study outlines a logical and empirical correspondence between two theoretical frameworks, a cognitive realm and a social-cultural semiotic realm. Findings from the cognitive perspective gave strong support to the validity of the progression of SOLO levels across stages and the influence of pattern generalization types on those trends. Nevertheless, findings from the social-cultural semiotic perspective provided a framework to characterize different levels of generalizations in correspondence with SOLO levels. A future research direction may include using both frameworks to explore the means of objectification that teachers refer to in their classrooms to help students to develop their level of generalization and reasoning in different types of generalization.

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