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# Examining early algebraic thinking: insights from empirical data

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**Abstract** The aim of this study is to better understand the notion of early algebraic thinking by describing differences in grade 4–7 students' thinking about basic algebraic concepts. To achieve this goal, one test that involved generalized arithmetic, functional thinking, and modeling tasks, was administered to 684 students from these grades. Quantitative analysis of the data yielded four distinct groups of students demonstrating a wide range of performance in these tasks. Qualitative analysis of students' solutions provided further insight into their understanding of basic algebraic concepts, and the nature of the processes and forms of reasoning they utilized. The results showed that students in each group were able to solve different number and types of tasks, using different strategies. Results also indicated that students from all grades were present in each group. These findings suggest the presence of a consistent trend in the difficulty level across early algebraic tasks which may support the existence of a specific developmental trend from more intuitive types of early algebraic thinking to more sophisticated ones.

**Keywords** Early algebraic thinking · Generalized arithmetic · Functional thinking · Modeling · Concepts · Processes · Reasoning forms

## 1 Introduction

Mathematics curricula have usually been treated as an assortment of isolated topics where arithmetic precedes and algebra follows (Carraher & Schliemann, 2007). However, researchers and policy makers tend to agree that students should engage with algebra in a coherent way throughout their schooling, reflecting a consensus that algebra represents the gateway for reforming K-12 mathematics (e.g., NCTM, 2000). This idea was considered important for at

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least two reasons; first, the mere focus of elementary mathematics on arithmetic and computational fluency confined the conceptual development of mathematical ideas in the early grades (Blanton & Kaput, 2005); secondly, the abrupt introduction of students to algebra through traditional courses in middle school resulted in serious difficulties in understanding algebraic concepts (Cai & Knuth, 2005).

The need for enhancing elementary mathematics through the integration of algebra has led to a wealth of studies that contributed to the identification of certain characteristics of algebraic thinking in all grades (e.g., Blanton et al., 2011; Kieran, 2007; Radford, 2014). Blanton and Kaput (2005) defined algebraic thinking as a "habit of mind" that enables students to identify and express mathematical structure and relationships, such as the structure in arithmetical and symbolic expressions (Radford, 2003), the relationships in numerical and geometrical patterns (Mulligan & Mitchelmore, 2009; Warren & Cooper, 2008), and the numerical and geometrical structure in tables, graphs, and number lines (Carraher, Schliemann, Brizuela, & Earnest, 2006). In an earlier study, Driscoll (1999) claimed that there are three basic algebraic "habits of mind:" doing and undoing mathematical processes, identifying and representing functional rules, and thinking about computations independently of particular numbers. More recently, Radford (2014) described algebraic thinking as a form of mathematical reflection and action, highlighting the need to recognize elementary forms of algebraic thinking that are not exclusively based on alphanumeric symbolism.

These approaches and others illustrate the complexity in defining early algebraic thinking, clarifying its differences from arithmetical thinking, and describing the way it develops. This article takes a step towards this direction by examining grade 4–7 students' early algebraic thinking while dealing with a variety of algebraic tasks. To fulfill this goal, students' responses are analyzed along four dimensions; algebra core content strands, basic algebraic concepts, algebraic processes, and reasoning forms. Specifically, this paper explores whether there are groups of students who exhibit significant differences in their early algebraic thinking and whether there is a possible developmental trend from more intuitive types of early algebraic thinking to more sophisticated ones.

To this end, Section 2 provides an overview of previous research regarding algebraic thinking. In Section 3, the aims of the study are presented along with a description of the participants, tasks, and analyses employed. Section 4 presents the results, while findings are discussed in Section 5 and conclusions in Section 6.

## 2 Theoretical framework

### 2.1 Different approaches to the notion of algebraic thinking

The association of algebra with numerous mathematical features makes it challenging to extract a simple definition for algebraic thinking (Driscoll, 1999). In this section, we present important perspectives through which research addressed algebraic thinking.

**Algebra content strands and concepts** Kieran (2007) was among the first to conceptualize algebra as a multidimensional activity in the context of secondary education. Kaput (2008) elaborated the multiple facets of algebra from K-12 grades, proposing three algebra core content strands: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of generalizations as modeling languages. Generalized arithmetic refers to the identification of relationships between numbers, the manipulation of operations and their properties, and the transformation and solution of equations. Functional thinking refers to the generalization of relationships between co-varying quantities; this strand was related to the ability for expressing numerical and figural patterns as functions and algebraic expressions (e.g., Mulligan & Mitchelmore, 2009; Warren & Cooper, 2008). Modeling languages refers to the generalization of regularities that are presented implicitly through various problem contexts.

Each of the aforementioned strands is associated with several algebraic concepts. The generalized arithmetic strand involves the concepts of numbers, operations, the equals sign, equality, equation, expression, and variable. Within functional thinking the concepts of variable, expression, and equation are also central but with a different interpretation from the one held in the generalized arithmetic strand (Kieran, Pang, Schifter, & Ng, 2016). The functional thinking strand also involves the concepts of co-variance, correspondence, and change. All of these concepts are also incorporated in the strand of modeling languages.

**Algebraic processes and forms of reasoning** Several studies suggested that examining algebraic concepts requires a range of processes. For example, Blanton et al. (2011) illustrated that noticing, generalizing, representing, and justifying with mathematical structure and relationships are crucial within early algebra. Jeannotte and Kieran (2017) suggested that these kinds of processes are related to the search for similarities and differences and to validating. Similarly, it has been shown that the extraction of generalizations, which is considered as a core algebraic process, depends on noticing "the same and the different" (Radford, 2000).

However, algebraic processes cannot be separated from basic forms of reasoning. Both processes and reasoning are present while students deal with mathematical tasks and are related dialectically (Jeannotte & Kieran, 2017). Rivera and Becker (2007) showed that abductive reasoning is necessary at the stage where individuals develop a prediction about a plausible generalization; abductive reasoning boosts the generation of new data which in turn facilitates the adoption of a hypothesis which is considered testable. Inductive reasoning also has a significant role in identifying commonalities and extracting generalizations, while deductive reasoning is important for moving from limited to more accurate generalizations (Ellis, 2007). Hence, the conclusions which students reach while dealing with algebraic tasks depend on the forms of reasoning they utilize (Jeannotte & Kieran, 2017).

**Conditions that characterize algebraic thinking** Radford (2008) claimed that the way students act on and represent new relationships and objects in order to articulate generalizations might differ, varying from concrete numerical actions, to situated descriptions of the objects of the actions, to actions with symbols or signs. Radford (2014) further specified that there are three basic conditions that characterize algebraic thinking: (a) indeterminacy, the problem involves unknown quantities; (b) denotation, the unknown quantities are represented in various ways which are not only restricted to alphanumeric symbols; and (c) analyticity, the unknown quantities are added, subtracted, multiplied or divided, as if they were known.

## 2.2 Levels of sophistication of students' thinking about several algebraic concepts

Several studies investigated students' understanding of basic algebraic concepts and the way this understanding emerges. Regarding generalized arithmetic, Matthews et al. (2012)

indicated that students' understanding of the equals sign advances through progressive levels. At the initial level students interpret the equals sign operationally in equations of the type $a + b = c$. When students reach the final level, the equals sign is interpreted relationally. For example, students are able to compare the expressions on the two sides of an equation (e.g., $45 + 86 = 46 + 85$ ), without performing any operations.

Regarding functional thinking, Blanton et al. (2017) demonstrated that even Grade 1 students are able to understand variables and variable notations. They showed that students move from understanding a letter as a label for naming an object to understanding the letter as an unknown with a fixed value. Later, they conceptualize a letter as representing any fixed number that can be randomly chosen. At a higher level, students conceptualize variables as unknowns that vary. At the highest level, students use variables to represent functional relationships.

Blanton et al. (2015) also examined sophistication levels of Grade 1 students' functional thinking. Their findings suggested that students move from a "pre-structural" level where they are not able to notice structure and relationships to a level where they start noticing mathematical features. Later, they become able to notice recursive relationships. Students gradually understand the co-variational relationship between two explicitly noted quantities. Finally, they are able to understand function as an object.

Summing up, it seems that various levels can be identified regarding the nature of students' understanding of algebraic concepts. These types of studies provide evidence that the way early algebraic thinking emerges is a complicated process that moves from more intuitive forms of thinking to more formal ones.

### 2.3 Basic dimensions of algebraic thinking

Based on research results described in the previous section, it seems that algebraic thinking can be more coherently defined on the basis of four different dimensions:

1. Studying structure and relationships in three *algebra core content strands*: generalized arithmetic, functional thinking, and modeling (Kaput, 2008);
2. Understanding *fundamental algebraic concepts*, such as the equals sign, equality, equations, properties of numbers, properties of operations, variables, unknown quantities, symbols, co-variation, and correspondence;
3. Applying *processes* oriented towards the search of similarities and differences and validation of structure and relationships, such as noticing, conjecturing, representing, generalizing, justifying and validating;
4. Utilizing *forms of reasoning*, such as abductive, inductive and deductive reasoning, which lead to the extraction of conclusions (Fig. 1).

## 3 The present study

### 3.1 Aims of the study

The purpose of this study is to better understand the notion of *early algebraic thinking*. Specifically, this study aims to investigate whether differences in the way students perceive

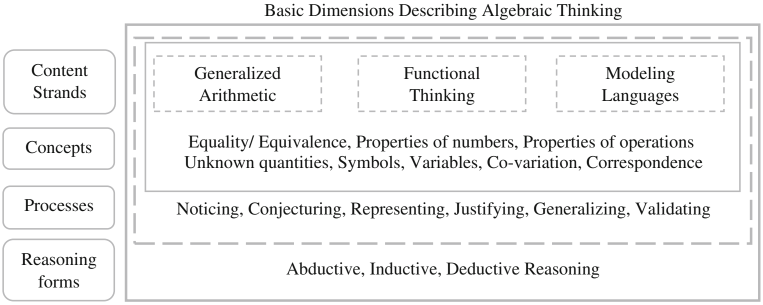



**Fig. 1** Basic dimensions describing algebraic thinking

different algebra content strands and basic algebraic concepts (Kaput, 2008), apply algebraic processes (Blanton et al., 2011), and utilize reasoning forms (Jeannotte & Kieran, 2017) explain variations in their performance in early algebraic tasks. Additionally, this study explores whether differences in students' performance provide insight to the way early algebraic thinking emerges and develops (Radford, 2008).

The specific aims are the following:

(a) To describe the characteristic features of distinct groups of students who exhibit different performance while dealing with various early algebraic tasks,
(b) To examine whether these kinds of differences provide an account of a developmental trend in early algebraic thinking from more intuitive forms towards more sophisticated ones.

Guided by the four basic dimensions of algebraic thinking that were reported in the previous section, students' responses in algebraic tasks will be analyzed based on the following questions: (1) Are students able to solve algebraic tasks that belong to different content strands? (2) What is the nature of students' understanding of basic algebraic concepts, such as the equals sign, equality, unknown quantities, symbols, variables, and functions? (3) What kind of strategies support students' algebraic processes while dealing with the tasks? (4) Which reasoning forms do students utilize?

### 3.2 Participants

The participants of the study were a convenience sample of 684 students from 10 schools (3 urban and 7 rural). In order to investigate early algebraic thinking across elementary school and early middle school students, the sample consisted of 170 fourth-graders (10 years old), 164 fifth-graders (11 years old), 184 sixth graders (12 years old), and 166 seventh graders (13 years old). There were approximately equal numbers of males and females in the sample.

### 3.3 Test on algebraic thinking

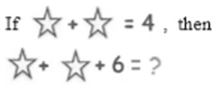
The test on algebraic thinking consisted of 18 tasks that were adapted from previous research studies (e.g., Blanton & Kaput, 2005; IEA, 2013) and mathematics textbooks (Altieri et al., 2008) and which captured the three algebra core content strands. Specifically, the test included eight generalized arithmetic tasks, five functional thinking tasks, and five modeling tasks. The number of generalized arithmetic tasks was larger than the number of tasks in the other two strands due to the fact that a larger number of different concepts are associated with this strand. Table 1 presents the tasks and their corresponding content strand and concepts. Table 2 presents indicative examples of the tasks.

**Generalized arithmetic tasks** These tasks addressed the concepts of equality/inequality, unknown quantities, symbols, properties of numbers, and properties of operations. Task ga1 (adapted from Stylianides & Stylianides, 2008) required students to examine whether the statement “the sum of two even numbers is odd” is true. This task could be approached by adding specific pairs of even numbers (e.g., $2 + 4$) or by interpreting even and odd numbers as variables. In task ga2 (adapted from Bastable & Schifter, 2008), students were expected to find missing quantities in equalities (e.g., $8 + 5 = \_\_\_ + 9$) by interpreting the equals signs relationally. In task ga3 (adapted from Blanton & Kaput, 2005) students had to check the result of a two-digit multiplication, considering place-value and the distributive property of multiplication. Task ga4 (adapted from Blanton & Kaput, 2005) presented a pawn’s movements in the hundredths table; students had to translate the movements into mathematical expressions, based on the structure of the table. Task ga5 (adapted from Chester, 2012) involved an inequality ($12 < 3 \times b$); students were expected to identify possible values for the letter $b$, indicating understanding of the use of symbols which stand for various values and not a single one. Task ga6 (adapted from Carraher & Schliemann, 2014) required the identification of the value of a missing quantity in a single variable equation ($N + 4 = 12$) where students had to apply “undoing”. Task ga7 (adapted from Blanton & Kaput, 2005) asked students to determine whether the sum of two multi-digit numbers would be odd or even based on odd and even

**Table 1** Description and coding of the tasks in the algebraic thinking test

| Task | Content strand | Concepts |
|---|---|---|
| ga1: Justifying if the sum of two even numbers is odd<br>ga4: Analyzing the structure of the hundredths table<br>ga7: Determining if the sum of two multi-digit numbers is odd | Generalized arithmetic (ga) | Properties of numbers |
| ga3: Justifying the result of a two-digit multiplication | | Properties of operations |
| ga2: Identifying missing quantities in equality expressions<br>ga5: Solving an inequality<br>ga6: Solving a single variable equation<br>ga8: Analyzing equalities that involve symbols and numbers | | Equality, equals sign,<br>Inequality,<br>Equation,<br>Unknown quantity,<br>Symbols |
| ft1: Choosing the appropriate graph for representing a correspondence relationship<br>ft2: Identifying distant terms in a numerical pattern<br>ft3: Choosing the appropriate verbal expression for representing a correspondence relationship<br>ft4: Calculating a distant term in a figural pattern<br>ft5: Calculating the next term and a distant term in a figural pattern | Functional thinking (ft) | Variable, Co-variation, Correspondence, Equation |
| mod1: Expressing the relationship between Celsius and Fahrenheit degrees<br>mod2: Using the known area of a square to find the area of new squares<br>mod3: Determining the best offer (price reductions)<br>mod4: Determining the best offer (computer lessons)<br>mod5: Determining the best offer (song downloads) | Modeling (mod) | Variable, Equation |

**Table 2** Examples of tasks included in the algebraic thinking test

| Example | Content strand | Concepts | Anticipated Processes | Reasoning Forms |
|---|---|---|---|---|
| *Task ga8*<br><br> | Generalized Arithmetic | Equality, Equals sign, Unknown quantity, Symbols | Noticing commonalities in the two expressions. Conjecturing about the relationship between the expressions. Justifying based on properties of equalities. | |
| *Task ft5*<br>At a table that has the shape of a trapezium, 5 children can be seated. If two tables are connected, then 8 children can be seated.<br><br><br>(a) How many children can be seated at 3 tables?<br>(b) How many children can be seated at 10 tables? | Functional thinking | Variable, Co-variation, Correspondence | Noticing commonalities among particular terms. Noticing variables. Conjecturing about the relationship between the variables. Representing, justifying, generalizing the relationship between variables. | Abductive, Inductive, Deductive |
| *Task mod4*<br>Joanna will take computers lessons twice a week. Which is the best offer? Justify your answer.<br>OFFER A: €8 for each lesson<br>OFFER B: €50 for the first 5 lessons of the month and then €4 for every additional lesson | Modeling | Variable, Equation, Correspondence | Noticing variables. Conjecturing about the relationship between the variables. Representing, justifying, generalizing the relationship between variables. | |

number properties. Task ga8 (adapted from Blanton & Kaput, 2005, see Table 2) presented an equality expression which included two identical symbols and numbers and asked for the value of a new expression that involved the same symbols; students were expected to notice similarities among the expressions and apply properties of equalities.

**Functional thinking tasks** These tasks addressed the concepts of variables and functional relationships. Task ft1 (adapted from Chester, 2012) involved graphing; students had to use a relationship described verbally (money earned per hour) to identify which graph represented two-variable data sets that fitted this relationship. Task ft2 (adapted from IEA, 2013) required students to identify terms that would appear if a numerical pattern was extended. In task ft3 (adapted from IEA, 2013), students were presented with examples of a correspondence relationship (e.g., 3⟶8,5⟶12) and they had to choose the verbal expression that described the general rule of this relationship. Task ft4 (adapted from IEA, 2013) presented the figural pattern of even numbers and asked for a distant term. Task ft5 (adapted from Rivera & Becker, 2007, see Table 2) required students to find the next term and a distant term in a figural pattern involving tables and seats.

**Modeling tasks** These tasks were comprised of problems where a situation had to be modeled and algebraic processes needed to be used as a tool for unfolding the general form of the situation. Task mod1 (adapted from Blanton & Kaput, 2005) asked students to study a set of data to model the relationship between Fahrenheit and Celsius degrees. In task mod2 (adapted from Blanton & Kaput, 2005) students were expected to use the given area of a small square to build a model for calculating the area of new enlarged squares. In task mod3 (adapted from Mason, Graham & Johnston-Wilder, 2005) students had to put statements about price reductions in order of increasing reduction. The modeling tasks mod4 and mod5 (adapted from Altieri et al., 2008) involved the comparison of offers. For example, in task mod4 (see Table 2) students had to compare two offers for computer lessons by translating each offer into a mathematical expression in order to calculate the total cost per month for any number of lessons.

### 3.4 Analysis

The MPLUS structural equation-modeling program and latent class analysis (LCA) was used to analyze the data (Muthén & Muthén, 1998). LCA is one of the Mixture Modeling techniques, which aims to find groups of people who give similar responses to specific variables. In the present study, LCA is used to examine whether we can find groups of students who gave similar responses to the tasks in the algebraic thinking test among the 684 students that participated in the study. Furthermore, descriptive results of students' performance were measured using the SPSS statistical package.

In order to further elaborate on the special characteristics of each group of students, qualitative information from students' tests was analyzed. This method addresses the aim of the study for describing the characteristic features of distinct groups of students who exhibit different performance. In particular, ten students from each identified group were selected and their solutions were examined. The selection of these students was based on the method of purposeful sampling (Patton, 2002) which suggests the selection of cases that provide rich information about the phenomenon under investigation. In the present study, we selected students from each group whose answers provided a wealth of information about their understanding of algebraic concepts, and the processes and forms of reasoning they applied. Additionally, following a comparative approach (Grove, 1988), the selection of the students was based on whether their answers were indicative of the way the group responded to the tasks.

## 4 Results

### 4.1 Identification of different groups of students based on their performance

This section presents the results of the quantitative analyses, concerning the extent to which the participants differed according to their performance in the test.

The results of the latent class analysis suggested that the participants of the study formed four distinct groups which had different performance while dealing with the early algebraic tasks.

The results of the descriptive statistics analysis indicated that students' overall performance in the test was .44 and the standard deviation was .25. The means of each group are reported in Fig. 2. The mean performance of each group was significantly higher than the corresponding mean of the previous group.

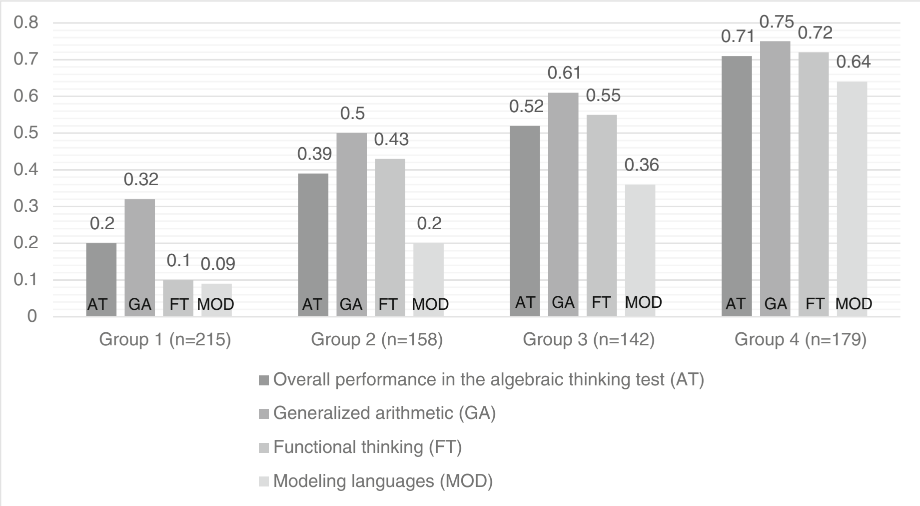



**Fig. 2** Descriptive results of the performance of the four groups in the early algebraic thinking test

The performance of Group 1 students was below .44 in all types of tasks. Group 2 students had better performance in the generalized arithmetic tasks ($\bar{x} = .50$) compared to Group 1 students; however, they appeared to have difficulties in functional thinking and modeling tasks ($\bar{x} = .43$ and $\bar{x} = .20$, respectively). Group 3 students had better performance in the generalized arithmetic and functional thinking tasks compared to Group 2 students ($\bar{x} = .61$ and $\bar{x} = .55$, respectively); yet, they seemed to struggle in modeling tasks ($\bar{x} = .36$). Group 4 students outperformed students in Groups 1, 2, and 3 in all types of tasks ($\bar{x} = .75$, $\bar{x} = .71$, and $\bar{x} = .64$, respectively). Figure 3 reports the specific tasks that were successfully performed by students in each group.

Figure 4 presents the percentage of students in each group according to grade level. The majority of Grade 4 students were classified within Group 1 (37.20%). Almost equal percentages of Grade 5 students were classified within Groups 1, 2, and 3 (22.80, 22.80, and 23.50%, respectively); 26.80% of Grade 5 students were classified within Group 4. Almost equal percentages of Grade 6 students were classified within Groups 3 and 4 (31.80 and 31.30%, respectively). The highest percentage of Grade 7 students was classified within Group 4 (30.70%), while 23.50% of Grade 7 students were classified within Group 3, 25.30% within Group 2, and 18.60% within Group 1.

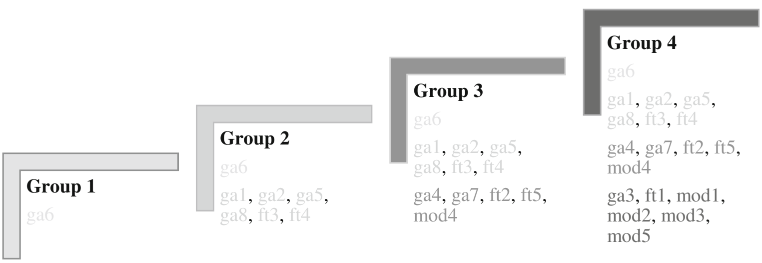



**Fig. 3** Tasks that were solved by students in the four groups; ga_generalized arithmetic; ft_functional thinking; mod_modeling languages

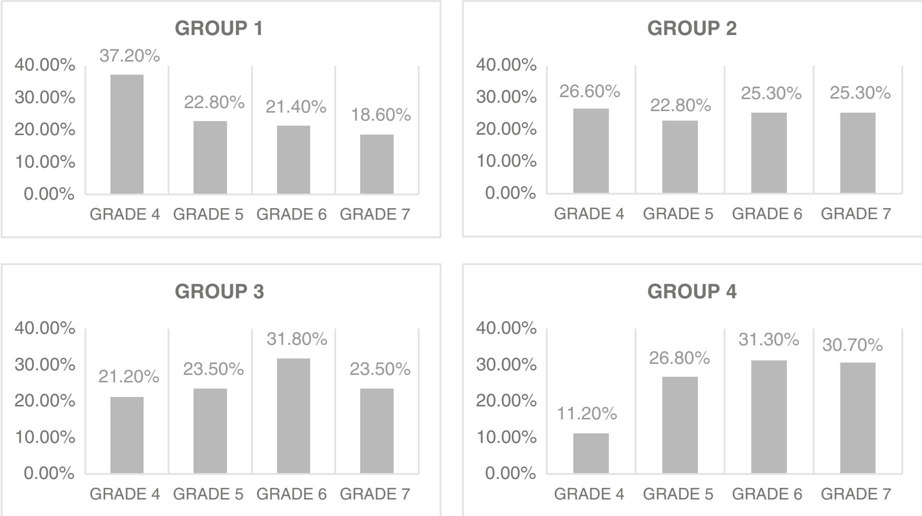



**Fig. 4** Percentages of students in the four groups

Although the focus of the present study was not to examine the effect of age, it should be mentioned that students from all grades were present in all groups. This result indicates that students with different characteristics regarding their responses to the tasks are found in each grade, independently of their age. It seems that there are younger students with more sophisticated early algebraic thinking which resembles that of older students. At the same time, there are older students whose early algebraic thinking is similar to that of younger students. Still, the majority of younger students in Grade 4 are classified within Groups 1 and 2. At the same time, the percentage of students classified within Groups 3 and 4 appears to become larger as students move to Grade 5 and Grade 6. This result indicates that as students grow older, they tend to become more successful in solving these tasks.

As illustrated in Fig. 4, there was a slightly higher percentage of Grade 6 students in Groups 3 and 4 than of Grade 7 students. This result might be considered unexpected, since the Grade 7 mathematics curriculum included algebra teaching and learning. This is not the case for the Grade 6 mathematics curriculum, which did not address algebra as a distinct domain. However, the fact that the types of tasks included in the test were not commonly used either in the Grade 7 curriculum or textbooks, which were mostly focused on transformational algebra, might be a possible reason for not observing a higher performance by seventh graders compared to sixth graders.

## 4.2 Characteristics of the responses provided by students in each group in specific tasks

The quantitative analysis identified four distinct groups of students with different performance in the algebraic thinking test. In this section, the results of the qualitative analyses of students' responses to specific tasks are described, in order to further illustrate which are the characteristic features of students' thinking in each group. The focus of this analysis is on the ways in which students perceived particular algebraic concepts, the nature of the processes they applied, and the conclusions they were able to reach based on the reasoning forms they utilized.

**Group 1** Students in Group 1 were successful only in one generalized arithmetic task (ga6) which required the identification of the value of an unknown quantity in an equation with one variable. The responses of Group 1 students in this task (see Fig. 5) seemed to be based on an inverse operation. This action reveals the application of an arithmetical process where students recognized and reconfigured a simple relationship between known and unknown quantities (Carraher & Schliemann, 2014).

Their response to another generalized arithmetic task (ga8), exemplified in Fig. 6a, indicates the difficulty Group 1 students had when operating with unknown quantities and reconfiguring the relationship between two expressions. They appeared not to notice similarities and differences between the two expressions so that inferences could be made. They merely focused on conjecturing which specific numbers the symbols might represent in the first expression.

Students in Group 1 were not successful in any functional thinking or modeling task. Figure 7a presents an example of their response to a pattern task (ft5). Their answer did not seem to be achieved by noticing or conjecturing the existence of a regularity. As shown, they gave wrong answers about the number of seats in the next term and a distant term, without explaining the way they obtained them or showing any attempt to develop a generalization.

On the whole, students' thinking in Group 1 appeared to be restricted to arithmetical contexts. They were not able to notice the underlying structure or relationships in any type of task. No relational thinking seemed to occur.

**Group 2** Students in Group 2 were able to solve a set of generalized arithmetic tasks (ga1, ga2, ga5, ga6, and ga8) and two functional thinking tasks (ft3 and ft4). Their response in task ga8 (see Fig. 6b) shows that they had probably noticed commonalities in the structure of the two expressions. However, they appeared not to be able to operate with unknown quantities. They used the "guess and check" strategy to build conjectures about possible values of the symbol. Furthermore, they seemed not to have actually grasped the meaning of symbols. They did not understand that the same symbol stands for the same number, and seemed to view a symbol as a fixed unknown number that can be chosen arbitrarily. Moreover, they used the equals sign operationally to calculate the value of the second expression. Hence, the expression was not perceived as representing an equality relationship but as an addition where the sum has to be calculated.

In pattern tasks, Group 2 students appeared to notice commonalities between the successive terms of the pattern. Their response to task ft5 (see Fig. 7b) indicates that they built conjectures about a recursive relationship. Moreover, they represented this relationship based on the construction of drawings and the counting strategy. While they noticed the way the number of seats

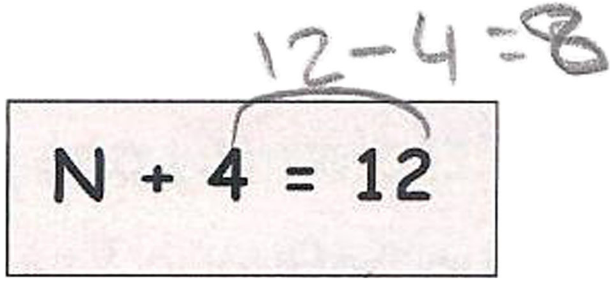



**Fig. 5** Indicative example of response to a generalized arithmetic task (ga6) by students in Group 1

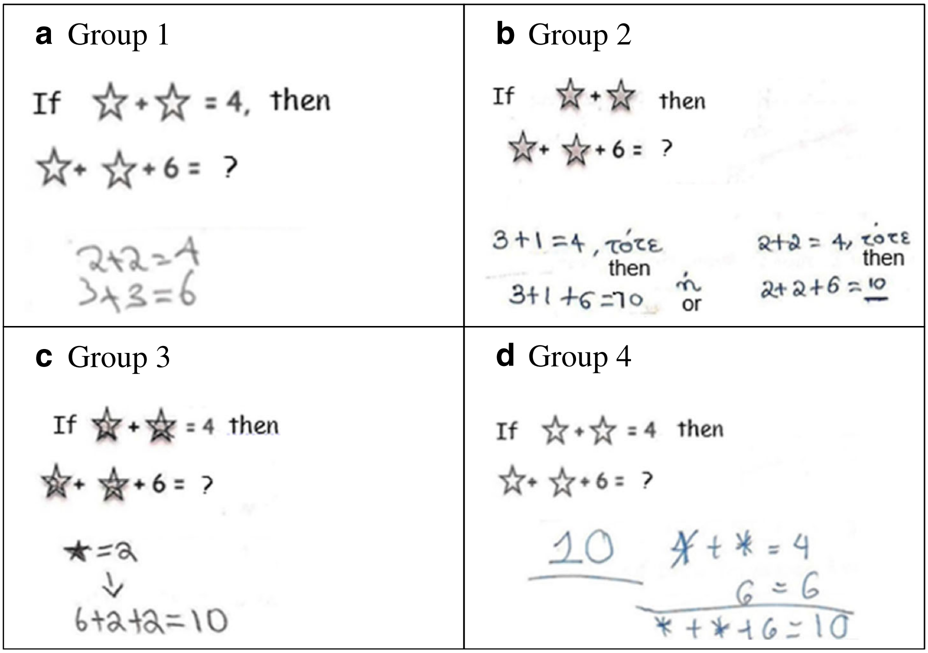



**Fig. 6** Indicative examples of response to a generalized arithmetic task (ga8) by students in Groups 1, 2, 3, and 4

increased, they did not notice the general relationship between the number of seats and the number of tables. They followed a step-by-step arithmetical process, relying on visual representations and repeated counting which led them to a wrong answer in the second question. Hence, the strategies they used guided them to develop a process of generalizing which employed concrete actions and did not allow the shift from the given set of data to a broader one.

Overall, these students appeared to use operational procedures and pre-acquired arithmetical strategies intuitively, both in generalized arithmetic and functional thinking tasks. However, these seemed to have facilitated primitive abductive steps in their reasoning that triggered conjecturing, which might signify a smooth transition from arithmetical to algebraic processes and reasoning. These steps facilitated the generation of new data from the given data (e.g., the possible values of the symbols, the drawings, the numerical examples) and indicated an awareness of "sameness" relationships. However, this kind of data appeared insufficient to support inductive steps of reasoning which would lead to generalizations. Hence, the thinking they exhibited unfolded through a constructive conjecture process, involving the creation of new evidence, which, nonetheless, remained in concrete, arithmetical contexts.

**Group 3** Students in Group 3 were able to solve more tasks from the strand of generalized arithmetic (ga1, ga2, ga4, ga5, ga6, ga7, and ga8) and functional thinking (ft2, ft3, ft4 and ft5) in relation to Group 2 students. In task ga8, they seemed to have noticed similarities between the two expressions. As shown in Fig. 6c, these students appeared to have developed a better understanding of the use of symbols compared to Group 2 students, since they wrote down one single value in order to specify what the symbol represented. Nevertheless, these students also appeared unable to operate with unknown quantities and, used the equals sign operationally. They justified their answer based on the result of an operation with specific numbers.

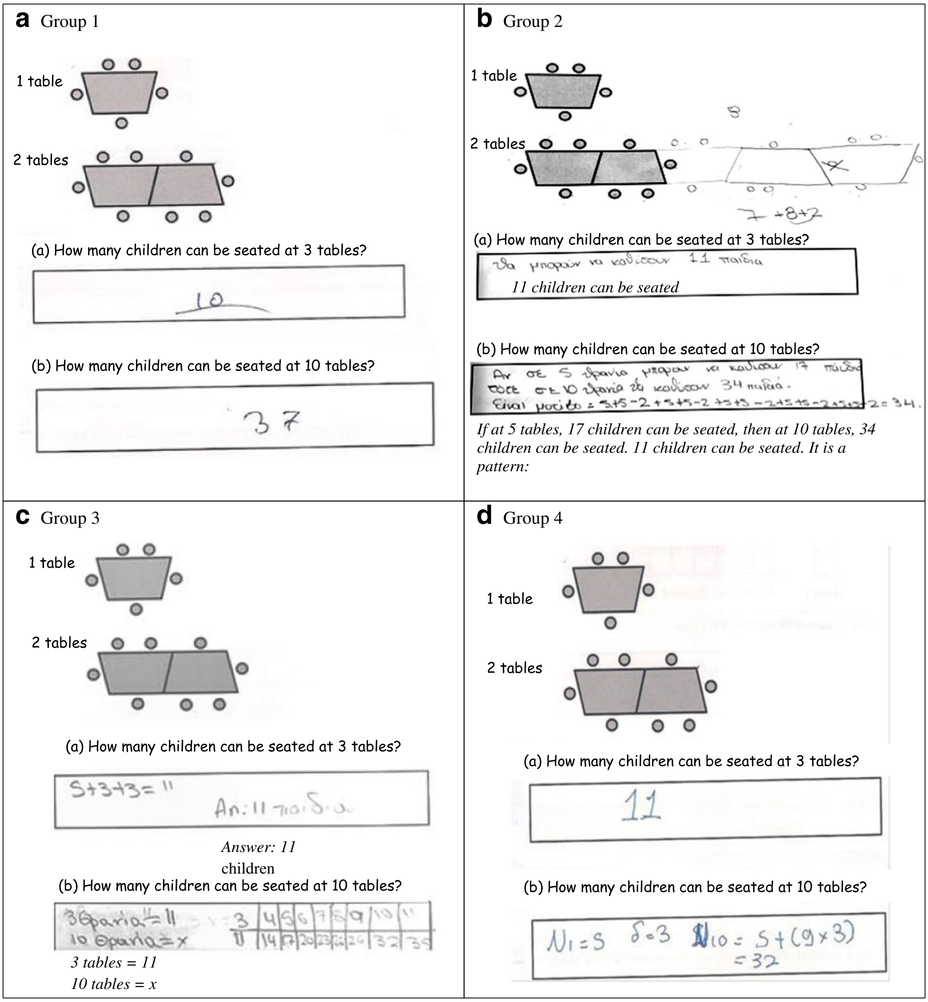



**Fig. 7** Indicative examples of response to a functional thinking task (ft5) by students in Groups 1, 2, 3, and 4

In the pattern task ft5 (Fig. 7c), these students explicitly noticed two quantities, the number of tables and seats, as well as similarities and differences in the way these quantities changed across the pattern. They seemed to build a conjecture about a co-variational relationship between them and created a t-chart to represent this relationship. Their justification process seemed to be based on a situated description of the relationship between several numerical examples of the two variables. Hence, these students appeared to uncover the co-variational relationship over a particular class of data.

Generally, Group 3 students exhibited abductive steps in reasoning which facilitated the generation of new data (e.g., the tabular data extracted from a pattern). This kind of data triggered primitive inductive steps in reasoning which led to the extraction of a plausible generalization that confirmed the relationship between the two variables. However, this generalization was articulated based on a situated description of the objects of students' actions (Radford, 2008) and was not extended in cases beyond the available ones.

**Group 4** Students in Group 4 were able to solve all of the tasks in the test successfully. As shown in Fig. 6d (task ga8), these students noticed the structure of each expression and the commonalities between them. Moreover, they appeared to operate with symbols as if they were known. They seemed to notice a new object (★+ ★ =4) and used it to reconfigure the relationship ★ + ★+ 6 =. Based on this relationship and on properties of equalities, they justified their answer. In this case, the equals sign appeared to be used relationally.

As shown in Fig. 7d, in the pattern task ft5, these students noticed two quantities. Moreover, they used both numbers and symbols to formulate a general rule that represented the correspondence relationship between them. Thus, their process of generalizing was more abstract and based on the use of symbols. The justification of their answer was based on a general rule which appears to apply not only to the particular set of data but over a larger set of data.

In contrast to students in the previous groups, students in Group 4 managed to solve modeling tasks. In the task mod4 (Fig. 8), students noticed the dependent variable (the monthly cost of the lessons) and the independent variables (the cost for each lesson and the number of lessons during a month). Furthermore, students made conjectures about the way they could compare the two situations and used two equation models. In this sense, they recognized a general structure for the problem that enabled the calculation of the monthly cost for computer lessons.

These students indicated similar behavior in all types of tasks. Their reasoning seemed to be based on abductive steps which created new data. These data (e.g., a general relationship, a rule, a model expressed with symbols and numbers) enabled inductive steps which led them to infer a generalization.

Table 3 summarizes the results of both the quantitative and qualitative analyses.

## 5 Discussion

The main aims of the present study were (a) to describe the characteristic features of distinct groups of students who exhibit different performance while dealing with early algebraic tasks, and (b) to examine whether these kinds of differences provide an account of a developmental trend in early algebraic thinking from more intuitive forms towards more sophisticated ones. Concerning the first aim, the quantitative analysis of the data showed that four different groups of students could be identified. The students in each group managed to solve different number and types of tasks

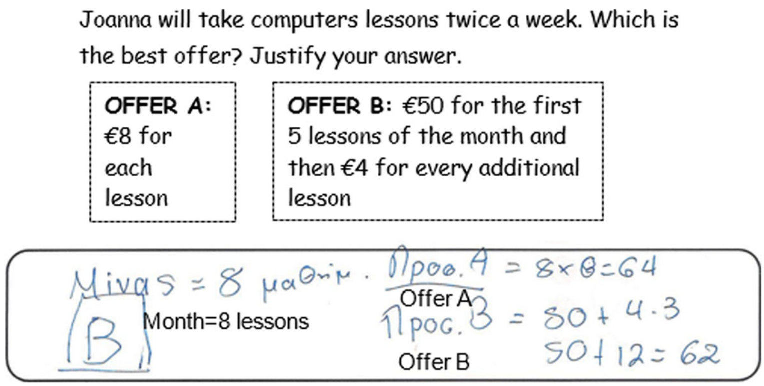



**Fig. 8** Indicative example of response to a modeling task (mod4) by students in Group 4

**Table 3** General description of the performance of students in different groups while dealing with early algebraic tasks

| | Content strands | Concepts | Processes | Reasoning |
|---|---|---|---|---|
| Group 1 | Generalized arithmetic | Identify the value of an unknown quantity in an equation with one variable. | Justify an answer based on calculations. | Do not exhibit any kind of relational reasoning. |
| Group 2 | Generalized arithmetic | Understand a symbol as a fixed value which may be chosen arbitrarily. Do not operate with unknown quantities. Use the equals sign operationally. | Notice similarities between equality expressions. Conjecture about the value of a symbol using the "guess and check" strategy. Justify an answer by replacing the symbol with its possible values and performing operations. | Exhibit abductive steps in reasoning which facilitate the generation of new data from the given data. This kind of data is not sufficient to support inductive steps which would lead to the extraction of a generalization. |
| | Functional Thinking | Find successive terms in patterns. | Notice similarities between successive terms. Conjecture about the recursive relationship between them. Represent and justify the next term using visual-concrete representations. Do not build a generalization about the relationship between variables. | |
| Group 3 | Generalized arithmetic | Understand symbols as fixed-numbers to be identified. Do not operate with unknown quantities. Use the equals sign operationally. | Notice similarities between equality expressions. Identify the value of the symbol using arithmetical known facts. Justify an answer by replacing the symbol with the identified value and performing operations. | Exhibit abductive steps in reasoning which facilitate the generation of new data. This kind of data facilitate primitive inductive steps in reasoning which lead to the extraction of a plausible generalization. This generalization is not extended to cases beyond the available ones. |
| | Functional thinking | Understand the co-variational relationship between variables in patterns. | Notice two quantities that change along the pattern and conjecture about their relationship. Create a table to represent and justify the relationship. Build a reasonable generalization about the co-variational relationship over a particular class of data. | |
| Group 4 | Generalized arithmetic | Interpret symbols as variables. Operate with unknown quantities. Use the equals sign relationally. Apply properties of equality. | Notice similarities between equality expressions. Use symbols and numbers as objects to represent relationships. Justify an answer based on properties of equalities. | Exhibit abductive steps in reasoning. Generate new data that facilitate inductive steps in reasoning, which lead to the extraction of a generalization. Use the |

**Table 3** (continued)

| Content strands | Concepts | Processes | Reasoning |
|---|---|---|---|
| Functional thinking | Understand the correspondence relationship between variables in patterns. | Notice two related variables. Represent their relationship by formulating a general rule using numbers and symbols. Justify the value of a distant term using the rule. Build a reasonable generalization about the correspondence relationship between any set of two variables. | general relationship to produce inferences. |
| Modeling | Identify the relationship between the involved variables in a problem situation. | Notice, model and justify relationships between variables in contextualized situations. | |

regarding content strands. Group 1 students appeared to have difficulties across all types of tasks. Group 2 students responded to the majority of generalized arithmetic and to some functional thinking tasks, while those in Group 3 responded to the majority of both generalized arithmetic and functional thinking tasks. Group 4 students responded to all three types of tasks. Concerning the second aim, the presence of a consistent trend in the difficulty level across generalized arithmetic, functional thinking, and modeling tasks suggests a specific developmental trend. Specifically, the results imply that students might be able to carry out generalized arithmetic tasks first, and then manage to carry out functional thinking tasks. Students seem able to deal with modeling tasks once they have been successful in both generalized arithmetic and functional thinking tasks.

While the quantitative analysis revealed that there were four distinct groups of students based on their achievement in the three algebra content strands, the qualitative analysis allowed a more insightful look at the characteristics of students' thinking in each group. Specifically, the qualitative results showed that students in different groups differed not only because they managed to solve different number and types of tasks, but also because they differed in the ways in which they perceived basic algebraic concepts, the extent to which the processes they applied were concrete or abstract, and the conclusions they were able to reach based on the reasoning forms they utilized.

Group 1 students' solutions did not show understanding of algebraic concepts, use of algebraic processes or relational thinking. Hence, their thinking seemed to be restricted to arithmetical contexts. If we were to use Blanton et al.'s (2015) terms, their thinking was at a "pre-structural" level, where students are not able to describe or use any relationship or structure implicitly while dealing with algebraic tasks.

Group 2 students were able to solve most of the generalized arithmetic tasks. However, these students seemed to rely on pre-acquired arithmetical knowledge. Their understanding of the equals sign seemed to be at the level of "rigid operational" (Matthews et al., 2012). In addition, they could not operate with unknown quantities represented by a symbol but needed to identify the value of the symbol, before performing operations. They also seemed not to understand that the same symbol represents the same quantity. Therefore, they understood a symbol as a fixed value which could be chosen arbitrarily (Blanton et al., 2017). In the functional thinking tasks, these students were only able to notice recursive relationships, and, similarly to the generalized arithmetic tasks, they applied arithmetical strategies. Hence, they appeared to conceptualize a pattern as a sequence of particular "instances" (Blanton et al., 2015).

Accordingly, their processes of conjecturing, representing, and justifying were based on arithmetical strategies, such as counting, and concrete tools, such as drawings. Their process of generalizing employed "concrete-level actions" (Radford, 2003). To echo Mason's (1989) argument, these students needed first to experience the manipulation of objects before moving to more abstract forms of algebraic thinking. Regarding the utilization of reasoning forms, it seems that the strategies they used enabled abductive steps which in turn allowed the extraction of new data from the given data. Nevertheless, the nature of the extracted data did not support inductive steps which would lead to conclusions about general relationships. The thinking exhibited by students in Group 2, both in generalized arithmetic and functional thinking tasks, suggests that expressing the structure and relationships with concrete examples does not provide a sufficient stepping stone to more formal forms of algebraic thinking. As Radford (2014) suggested, arithmetical strategies fail to satisfy the condition of "analyticity," where "the indeterminate quantities are treated as if they were known numbers" (p. 4).

The responses of Group 3 students suggested a qualitative advancement that seemed to bridge arithmetical and algebraic ways of thinking. These students were successful both in generalized arithmetic and functional thinking tasks, indicating that they were able to deal with a variety of algebraic concepts. Regarding the equals sign, these students did not seem to move to a relational perception, but they still used it operationally. They needed to identify the value of an unknown quantity before performing any operations. Nevertheless, these students seemed to understand that a symbol represents a specific number, not simply any number chosen at random. In pattern tasks, they seemed to perceive the concept of variable and co-variation. The tool they created to represent these concepts was often a table with input-output values which explicitly depicted a general relationship between two quantities across a set of cases.

It appears that Group 3 students used strategies and tools that enabled them to start noticing, conjecturing, representing, and justifying structure and relationships. Their process of generalizing unfolded within the perspective of situated descriptions of the objects involved either in expressions or functions. Hence, their generalizations can be characterized as "contextual"; according to Radford (2003), these kinds of generalizations cannot yet be considered as algebraic because they are situated and entail access to specific objects. However, the regularities that students noticed supported abductive steps in reasoning which directed them to extract new data. These new data referred to a relationship, therefore they supported primitive inductive steps which led to a conclusion about a plausible generalization.

Students' responses in Group 4 reflect a more sophisticated level of early algebraic thinking. These students were able to solve all types of tasks successfully, indicating a deeper understanding of several algebraic concepts. The equals sign, in terms of Mathews et al. (2012), was interpreted relationally. In equalities, they operated with unknown quantities and compared the two sides of the relationship. In pattern tasks, they were able to identify two co-varying quantities and express the correspondence relationship between them. Moreover, in modeling tasks they were able to translate the information embedded in problem situations into an appropriate mathematical model.

These students explicitly noticed, conjectured, represented, and justified structure in expressions, variables in functions, and general relationships. They were able to combine both numbers and symbols to represent these relationships. Hence, the new data they produced, supported the development of a formula. This action enabled inductive steps in reasoning which culminated in "symbolic" generalizations (Radford, 2003), since symbols and signs were used to articulate the generalization. Group 4 students seemed to adopt the three characteristics of algebraic thinking described by Radford (2014); "indeterminacy", "denotation", and "analyticity". They were able to understand whether a problem involved unknown quantities, name, and symbolize variables using alphanumeric symbols, and operate with unknown quantities as if they were known.

Summing up, the characteristic features of students' responses in each group imply that their type of thinking in one content strand is inherently related to their type of thinking in other content strands. This observation indicates that the development of algebraic thinking from more intuitive forms of thinking to more formal forms explains students' behavior in all algebra content strands. Moreover, it is implied that content strands are connected and that teaching needs to address them coherently through the elementary grades. It is noteworthy that students from all grades belonged to Group 1, Group 2, Group 3, and Group 4. For example, students even from the age of 10 years old who belonged in Group 4, were able to exhibit algebraic thinking and solve generalized arithmetic, functional thinking, and modeling tasks. At the same time, the opposite also happened. We found some 13 year old students belonging in Group 1, who seemed to face great difficulties with all algebra content strands and concepts, demonstrating more primitive thinking processes and reasoning.

These results are important for researchers and teachers, since they support empirically the idea that there are no explicit compounds between arithmetic and algebra (e.g., Blanton & Kaput, 2005; Radford, 2012) and that younger students can be supported to think in advanced ways about basic algebraic concepts. In particular, the generalized arithmetic strand seems to offer students an appropriate opportunity to see algebra in arithmetic. Students might start dealing with generalized arithmetic tasks by applying arithmetical strategies. In these kinds of tasks, students have the opportunity to notice the relational meaning of the equals sign and to confront numbers in a generalized way. As Kaput (2008) highlighted, even the use of numbers can be qualified as algebraic in as much as its purpose is not on calculation per se but on the representation of a generic example.

Furthermore, the functional thinking strand, represents a perspective for examining the concepts of variable and function. At this point, arithmetical and algebraic modes of thinking seem to interact with each other. Arithmetical strategies and concrete tools become progressively more abstract and symbolic, helping students notice quantities that vary and general relationships. Specifically, it appears that from recursive relationships in patterns the students move to the identification of co-variational and correspondence relationships. A mixture of abductive, inductive, and deductive forms of reasoning seems to be associated with students' ability to come up with viable generalizations.

The modeling strand seems to be associated with a more sophisticated level of early algebraic processes and reasoning than the other two strands of algebra. This strand offers accessible entry points to students for further developing and establishing algebraic concepts and advanced algebraic thinking skills.

## 6 Conclusion

The overarching results of the present study show that the processes and reasoning forms associated with early algebraic thinking are utilized in different ways by different groups of students as they encounter different algebra content strands and concepts. These differences suggest a specific developmental trend, which proposes a movement from forms of thinking that are strictly arithmetical to forms of thinking that swing between arithmetic and algebra, and finally to forms of thinking that are algebraic in nature. Additionally, the results of the study showed that students who exhibit different forms of early algebraic thinking were found in each one of the grade levels investigated (grades 4–7) independently of their age.

In conclusion, the results of this study designate important parameters of early algebra teaching and learning. All of the three strands and their associated concepts should be addressed from elementary school, providing long-term and sustained experiences for students to develop their intuitive ways of thinking into more formal ones. Classroom culture and teaching practices should address students' intuitive strategies so that these become more robust over time.

Future longitudinal studies can be designed in order to examine and verify whether the different ways of thinking described in this study represent developmental stages. Furthermore, research could examine which types of teaching may facilitate students' progression from early algebra to more advance algebra. This kind of research might facilitate the description of a learning trajectory for early algebraic thinking which will offer clear and specific guidance for productive teaching activities in the mathematics classroom.

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