Educ Stud Math (2017) 96:327–348
DOI 10.1007/s10649-017-9771-6

Pre-service teachers' flexibility with referent units in solving a fraction division problem

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Published online: 13 July 2017
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Abstract This study investigated 111 pre-service teachers' (PSTs') flexibility with referent units in solving a fraction division problem using a length model. Participants' written solutions to a measurement fraction division problem were analyzed in terms of strategies and types of errors, using an inductive content analysis approach. Findings suggest that most PSTs could calculate fraction division and make equivalent fractions procedurally but did not have the quantitative meanings of measurement division with fraction quantities or of making equivalent fractions. Implications are discussed for the improvement of PSTs' specialized knowledge for teaching fraction division.

Keywords Fraction division · Referent units · Visual representations · Pedagogical content knowledge · Quantitative approach

According to the NCTM (2000) and the CCSSI (2010), the ability to use multiple representations appropriately demonstrates mathematical understanding and is a critical characteristic of a good problem solver. Nevertheless, teachers tend to use only symbolic notation in their classrooms. Even if teachers use other representations, they frequently use them to illustrate solutions rather than promote students' mathematical understanding of rational numbers (Izsák, 2008; Lee, Brown, & Orrill, 2011), for example, by using them to model concepts or solve word problems. This limited use of representations may be related to teachers' lack either of their own understanding of alternative representations or of the knowledge to help students use them to learn mathematical concepts (Lee et al., 2011), both of which are issues to be addressed in teacher education. Accordingly, the present study is an investigation of elementary pre-service teachers' (PSTs') knowledge about representations for modeling fraction divisions and ability to draw them, specifically, to use referent units in solving a fraction division problem through a length model. The goal is to suggest what specialized knowledge

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for teaching with visual representations should be emphasized in the domain of fraction divisions.

1 Teachers' knowledge of division of fractions

Shulman (1986) proposed the concept *Pedagogical Content knowledge* (PCK) to refer to specialized knowledge for teaching. PCK represents the merging of content and pedagogy into an understanding of how particular topics should be organized, represented and presented for diverse learners. For example, in mathematics such knowledge includes unpacking compressed mathematical ideas, representing mathematical ideas, explaining why rules and procedures work, and comprehending students' unusual solutions (Hill, Ball, & Schilling, 2008).

Knowledge of fractions is particularly important in the mathematics curriculum because it is integral to algebraic reasoning and essential for further study in mathematics (cf. Hackenberg & Lee, 2015; Lee & Hackenberg, 2014; NMAP, 2008). To teach fractions effectively, teachers need to know not only what procedures to use in solving fraction problems, which is content knowledge, but also why the procedures work, which is relevant to PCK. However, research has demonstrated that even teachers who are procedurally fluent in fraction computation often do not understand why their computation works. For example, Olanoff, Lo, and Tobias's (2014) critical review of 43 studies of teachers' fractional knowledge found that their computation knowledge was relatively strong but they generally had difficulty understanding the meanings behind the procedures. Rosli, Han, Capraro, and Capraro (2013) also found that pre-service teachers' content knowledge of fractions related to computation was much better than their PCK, such as using multiple representations to support students' mathematical learning.

In particular, with regard to teaching fraction division, many studies have reported that teachers are insufficient in both mathematical content knowledge and PCK. (Borko et al., 1992; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). Borko et al.'s (1992) case study demonstrated a teacher's difficulty with explaining the invert and multiply rule for fraction division. To explain the rule after computing the answer to $\frac{3}{4} \div \frac{1}{2}$, the teacher used area representation that illustrated $\frac{3}{4} \times \frac{1}{2}$ instead of fraction division. Similarly, Ma's (1999) comparative study of Chinese and U.S. in-service teachers reported U.S. teachers' lower mathematical knowledge for understanding and teaching fraction divisions. For example, after correctly calculating $1\frac{3}{4} \div \frac{1}{2}$, many U.S. teachers explained the calculation with a story for $1\frac{3}{4} \times \frac{1}{2}$. Sowder et al. (1998) also reported a case of a middle grade teacher who had trouble appropriately connecting operations of fractional quantities to the arithmetic operations of multiplication and division.

Pre-service teachers (PSTs) have been found to have similar lack of knowledge about fraction division as well as low PCK (Ball, 1990; Jansen & Hohensee, 2016; Lo & Luo, 2012; Tirosh, 2000). Ball (1990) found that only five of nineteen PSTs could provide appropriate word problems for $1\frac{3}{4} \div 2$. Tirosh (2000) found that PSTs knew how to divide fractions but could neither explain the procedure nor predict major sources of students' incorrect responses in fraction division. More recently Lo and Luo (2012) found that PSTs were challenged with the task of representing fraction division through either word problems or pictorial diagrams even though they were able to calculate fraction division problems correctly. Jansen and Hohensee (2016) reported that PSTs demonstrated *disconnected* conceptions of division with fractions by incorrectly translating between representations. The PSTs also revealed *inflexible* concepts of division with fractions through their apparent unawareness that the process of iterating could be associated with the operation of division.

The majority of these studies (e.g., Ball, 1990; Ma, 1999; Tirosh, 2000) tended to measure PSTs' or in-service teachers' understanding of fraction division by asking them to create word problems for given division expressions or to explain why a procedure for fraction division worked. Even though some researchers (e.g., Jansen & Hohensee, 2016; Lo & Luo, 2012) asked participants to draw a model to show how they solved the fraction division problems, they mainly addressed different types and frequencies of models drawn by the participants rather than analyzing their models by focusing explicitly on *referent units*. Also, Borko et al.'s (1992) and Lo and Luo's (2012) research particularly highlighted PSTs' and in-service teachers' challenges in using visual representations to solve or explain fraction division problems.

In order to represent fraction divisions, teachers need to implicitly or explicitly understand the units to which numbers refer in their representations. Jacobson and Izsák (2015) identified these *referent units* as one of the important components of fractional reasoning. A *unit* is a standard for measurement, which could be one whole (e.g., 1 in.) or a part that is either contained in a standard for measurement (e.g., 1/3 in.) or contains a standard for measurement (e.g., 2 in.). *Referent units* are units that are needed when numbers are embedded in problem situations.

To represent fraction divisions using pictorial models, teachers' flexibility with referent units is critical (Lamon, 1994; Lee et al., 2011). Lee et al. defined *flexibility with referent units*[1] as "a teacher's ability to keep track of the unit to which a fraction refers and to shift his/her relative understanding of the quantities as the referent unit changes" (p. 204). For example, to solve the problem $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ using representations, teachers should know that the $\frac{1}{3}$ and $\frac{1}{6}$ refer to a whole while the $\frac{1}{2}$ refers to a part of the $\frac{1}{3}$. If teachers do not have flexibility with referent units, they may focus on the referent unit (i.e., a whole) as a fixed value rather than considering that a different whole (i.e., one-third) can be embedded in the problem.

2 Referent units and representations

As noted above, few studies have examined teachers' understanding of referent units and representations (Izsák, 2008; Lee et al., 2011). In an investigation of how two sixth-grade teachers used representations to explain fraction multiplication problems, Izsák (2008) found some relationships between teachers' *unit structures* and pedagogical purposes for using drawings. For example, one teacher struggled to compare $\frac{2}{3}$ and $\frac{3}{4}$ by drawing two squares, one partitioned into thirds and one into fourths, because she attended only to two levels of units (i.e., a unit of three units / a unit of four units) instead of three levels of units (e.g., a unit of three units, each of which contains four units / a unit of four units, each of which contains three units). Izsák also found that teachers' insufficient reasoning with referent units affects their ability to effectively incorporate length or area representations into their teaching of fraction multiplication. For instance, to represent $\frac{1}{2}$ of $\frac{2}{3}$ using an area model, a teacher may draw a

[1] This ability is related to *norming*, which is described as "the process of reconceptualzing a system in relation to some fixed unit or standard" (Lamon, 1994, p. 94) – more specifically – identifying the standard unit with which to measure and representing quantities of an object with the unit.

rectangle, partition it into thirds, and shade two parts. Then to figure out $\frac{1}{2}$ of $\frac{2}{3}$, the teacher may cross-partition the two shaded parts into halves, producing four smaller pieces. Then the teacher may shade two of the little pieces (i.e., ). Here, although the teacher appears to have two little pieces out of four in his/her drawing, in order to provide a correct answer, the teacher needs to know that the referent unit of the two little shaded pieces is the original whole rather than $\frac{2}{3}$, and thus the answer is $\frac{2}{6}$ rather than $\frac{2}{4}$.

Similarly, in an investigation of 12 middle grade teachers' strategies for completing tasks that required analysis of drawn representations of fractions and decimals, Lee et al. (2011) found that teachers who correctly identified or flexibly used the referent units could justify their choices better than teachers who did not attend to the referent units. For example, when given the following two number line problems (see Fig. 1), the four teachers who attended to the referent unit determined that the first number line was incorrect because the unit to which one-fifth referred in the drawing was the whole rather than one-fourth.

Lee et al. also found that teachers who lacked flexibility with referent units struggled with making sense of representations. For instance, when teachers were asked whether two shaded parts out of three (i.e., ), could be interpreted as modeling $\frac{3}{2}$ as the quotient for $1\div\frac{2}{3}$, the teachers showed inflexibility in their interpretation of this drawing by not identifying that the model can be interpreted as showing $\frac{2}{3}$ of one whole shaded, $\frac{1}{2}$ of $\frac{2}{3}$ unshaded, or the quotient $\frac{3}{2}$ shaded in all. In particular, the teachers showed inflexibility in seeing the model as the quotient because they failed to recognize that the quotient referred to a different referent unit compared to the divisor and dividend.

Both Lee et al.'s (2011) study focusing on fraction comparison, fraction multiplication, and decimal multiplication and Izsák's (2008) study of the use of representation in teaching fraction comparison and fraction multiplication showed the importance of referent units and representations. However, both studies focused only on in-service teachers, and neither addressed fraction division in detail. Accordingly, the present study focuses on the importance of figuring out the appropriate referent units for solving fraction division problems and targets PSTs.

3 Theoretical framework

This study is grounded in the notion of specialized knowledge for teaching mathematics, one key element of which is to know how to use representations to unpack compressed mathematical ideas. The theoretical framework of this study comprises the construct of pedagogical content knowledge and the quantitative approach to developing students' rational number concepts. Because pedagogical content knowledge has already been addressed, in this section,

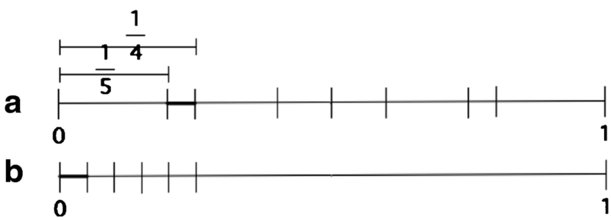

Fig. 1 Number line model of $\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$ and $\frac{1}{4}\times\frac{1}{5}=\frac{1}{20}$ (Extracted from Lee et al., 2011, p. 209)

I discuss the quantitative approach and another key construct, representations and referent units in fraction division.

3.1 The quantitative approach

The quantitative approach emphasizes operating with quantities and their relationships (Thompson, 1993, 1995). *Quantities* are defined as a group's conception of attributes of objects or phenomena that can be measured, which requires a unit and a process for assigning a numerical value to each attribute. Because this perspective emphasizes the relationship between units of measurement and magnitudes of quantity, it is important for learners to experience meaningful *referents* for understanding quantitative relationships and operations (e.g., Carraher, Schliemann, & Schwartz, 2008; Smith & Thompson, 2008). In the quantitative approach, representations, which show the referent units, are often used to demonstrate quantitative relationships visually (Thompson, 1993). For this reason, the NCTM's (2014) publication *Principle to Actions* emphasizes the use of visual representations, asserting that "representations embody critical features of mathematical constructs and actions, such as drawing diagrams and using words, to show and explain the meaning of fractions, ratios, or the operation of multiplication" (p. 24).

Smith and Thompson (2008) emphasized that quantitative reasoning is important because it provides "conceptual content for powerful forms of representation and manipulation in algebra" (p. 100) and specified two roles quantitative reasoning plays in complex problem solving: (1) providing the content for algebraic expressions and (2) supporting flexible reasoning without relying on symbolic notation. They also distinguished between reasoning about numbers, which they labeled *numerical/computational*, and reasoning about quantities, which they labeled *quantitative/conceptual.* Numerical/computational reasoning is based on a sequence of operations on the right pairs of numbers to produce the correct final answer, or an algorithm, whereas quantitative/conceptual reasoning begins with conceptualizing situations and moving onto reasoning about quantities, their properties, and relationships among quantities before linking the quantitative relationships to numerical operations.

While this dual classification of ways of reasoning helps clarify the meaning of conceptual understanding in mathematics, it does not include cases in which students can also reason through computations from their prior knowledge rather than simply using algorithms, which Star (2005) suggests labeling as "deep procedural knowledge.[2] For example, Son and Crespo (2009) suggested multiple ways of solving a fraction division problem, some based on formal strategies involving algorithms (e.g., invert and multiply strategy) and others on informal strategies involving reasoning through computations (e.g., common denominator strategy, repeated subtraction strategy, decimal strategy, and a unit rate strategy).

In Table 1, the first strategy is based on a formal algorithm, which students can follow without understanding why the rule always works. The other five strategies are derived from mathematical knowledge that has been learned prior to fraction division operations, such as finding common denominators, subtracting fractions repeatedly, converting fractions into

[2] According to Star (2005), deep procedural knowledge is defined as "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts" (p. 408).

Table 1 Formal and informal strategies in solving fraction division problems (Adapted from Son & Crespo, 2009, p. 238)

	Meaning of division	Strategies	Mathematical Notation (b, c, d ≠ 0)
Formal Strategy	Division as the inverse of multiplication	Invert and multiply strategy	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$
Informal Strategies	Division as measurement	Common denominator strategy	$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = ad \div bc = \frac{ad}{bc}$ $\frac{5}{3} \div \frac{1}{2} = \frac{10}{6} \div \frac{3}{6} = 10 \div 3 = \frac{10}{3} = 3\frac{1}{3}$
	Division as measurement	Repeated subtraction strategy	$3 \div \frac{3}{4} = 3 - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} = 0$ (Counting how many instances of ¾ there are in 3)
	Division as measurement	Decimal strategy	$1\frac{3}{4} \div \frac{1}{2} = 1.75 \div 0.5 = 3.5.$
	Division as a determination of a unit rate	Unit rate strategy	$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$
	Division as the inverse of a Cartesian product	Strategy of dividing numerators and denominators	$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$

decimals, and using a concept of unit rate. Thus, these solutions show students' reasoning through computations although they are not explicitly tied to quantities as usually denoted by pictorial representations.

3.2 Representations and referent units in fraction division

In the following, I address quantitative/conceptual reasoning in a discussion of how fraction division problems can be solved based on the quantitative approach by using representations and referent units. To specify referent units in fractions, researchers who explore fractional knowledge based on Piagetian scheme theory have recommended using a length representation such as a rectangular bar or a number line (cf. Hackenberg, 2013; Lee & Hackenberg, 2014; Steffe, 2002) because the use of length (as opposed to area) demands more explicit attention to the coordination and distribution of units. Therefore rectangular bars are used in the following explanation of referent units in fraction division.

There are two types of fraction division: *partitive* fraction division and *measurement* fraction division (Lo & Luo, 2012). In partitive division, the dividend, quotient, and remainder will have the same referent unit, but the divisor will have a different one. For example, when asked "how much flour can I buy with one dollar if $\frac{1}{2}$ of a pound of flour costs $\frac{3}{4}$ of a dollar," students need to consider the concept of unit rate (Jansen & Hohensee, 2016). To figure out the answer using a rectangular bar, they need to first partition it into halves. Because $\frac{1}{2}$ of a pound costs $\frac{3}{4}$ of a dollar, if the half is partitioned into thirds again, one little piece (see the third bar in Fig. 2) represents the amount of flour that can be purchased with $\frac{1}{4}$ of a dollar. If the other half is partitioned into thirds, the original whole ($\frac{2}{2}$) can be represented as $\frac{6}{6}$. Thus, with one dollar (i.e., $\frac{1}{4}$ dollar × 4), four little pieces (i.e., one little piece × 4) can be purchased, which can be represented $\frac{4}{6}$ (see the fourth bar in Fig. 2). Here, the referent unit for $\frac{1}{2}$ and $\frac{4}{6}$ is the whole, but

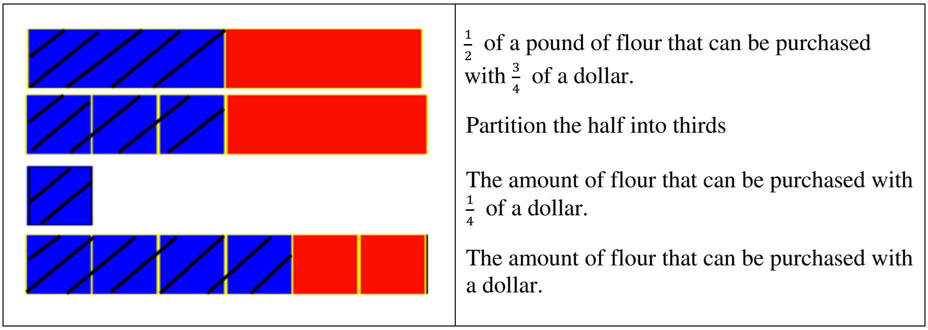

Fig. 2 Referent units in partitive division, $\frac{1}{2} \div \frac{3}{4}$

the referent unit for $\frac{3}{4}$ is $\frac{4}{6}$. Therefore, to solve this problem correctly using a length representation, flexibility with referent units is required.

In measurement division, the dividend, divisor, and remainder will all have the same referent unit but the quotient will have a different one. For example, in the measurement fraction division problem, $\frac{3}{4} \div \frac{1}{2}$, students need to consider how many times "$\frac{1}{2}$" occurs in $\frac{3}{4}$. To figure this out, one can draw two rectangular bars and partition each into fourths. Then one can shade two pieces on one bar to represent ½, and three pieces on the other to represent $\frac{3}{4}$ (see Fig. 3). By comparing the two bars, one can notice that there is *one* $\frac{1}{2}$ and a *half* of $\frac{1}{2}$ in $\frac{3}{4}$ (i.e., $\frac{3}{4} \div \frac{1}{2}$ = one and a half). Or, by using improper fractions, one can observe that the size of a piece in $\frac{3}{4}$ equals half the size of a piece in $\frac{1}{2}$ and thus there are three halves in $\frac{3}{4}$ (i.e., $\frac{3}{4} \div \frac{1}{2}$ = three halves). Here the referent unit for $\frac{3}{4}$ and $\frac{1}{2}$ is the one whole, but the referent unit for one and a half (or three halves) is $\frac{1}{2}$. It can be reasonably assumed that changing the nature of the unit from one original whole to $\frac{1}{2}$, referred to here as *flexibility with referent units,* could cause students to experience a great deal of cognitive complexity in their efforts to link meaning, symbols, and operations (Izsák, 2008; Lee et al., 2011).

Despite the importance of referent units and representations for teaching operations with fractions, in-service teachers have been found to have difficulty with referent units in both multiplication and division of fractions (Izsák, 2008; Lee et al., 2011). To better prepare future teachers for the challenge of teaching fraction division, it is critical to investigate PSTs' flexibility with referent units in solving fraction division problems, but few studies have addressed this issue. Accordingly, the present study focused on examining PSTs' flexibility

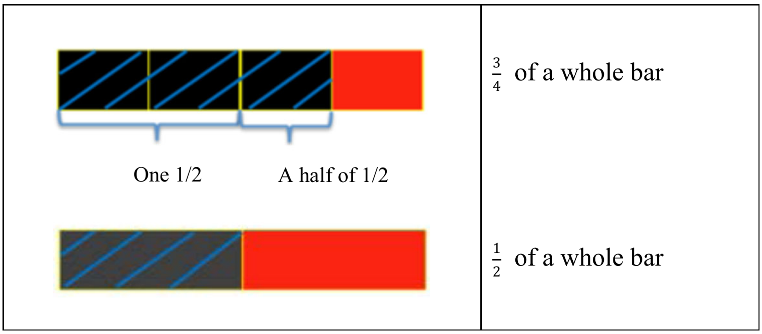

Fig. 3 Referent units in measurement division, $\frac{3}{4} \div \frac{1}{2}$

with referent units by asking them to solve a fraction division problem using a length representation. The research questions that guided the study were: (1) How do PSTs solve a measurement fraction division problem correctly? (2) What types of errors do PSTs demonstrate in solving a measurement fraction division problem? (3) What pictorial representations do PSTs provide in solving a measurement fraction division problem in relation to the correctness of their solutions, and when their solutions are correct, for what purposes do PSTs appear to use representations?

4 Method

4.1 Contexts and participants

The setting for this study was a mathematics content course for elementary PSTs at a large university in the southwestern U.S. This course was the third in a sequence of required mathematical content courses in the elementary teacher education program. Course 1 dealt with number and operations; Course 2 with geometry and spatial reasoning; and Course 3, the target course, with patterns, functions, and modeling. Relevant to the current study, Course 1 included a chapter on fractions and fraction operations, in which PSTs were encouraged to use set, length, and area models. For example, to teach fraction multiplication and division, the course began with the use of number lines, fraction bars, and area models to solve word problems before moving to the use of algorithms. Also in Course 1, proportional drawing was emphasized to clearly represent relationships among magnitudes of quantities in using models for fractions and operations. After completing the three-content-course sequence, PSTs were scheduled to take a mathematics pedagogy course.

The participants in this study were 111 undergraduate students majoring in elementary education and enrolled in four sections of Course 3. The group comprised 91 females and 20 males, with an ethnic distribution of 60 Whites, 35 Latinas, 11 Blacks, and 5 Asians. The mean age was 20.59 years, and they had studied in college about 2.81 years.

4.2 Data sources

The source of the main data set for this study was a written assessment of PSTs' fractional knowledge, consisting of 10 items, some of which were derived from Norton and Wilkins' (2009) research and the rest developed by the author. Nine questions focused on measuring PSTs' fractional knowledge involving partitioning and iterating (e.g., when the given bar is a whole, draw a picture to show $\frac{5}{6}$ of the bar and explain) while one question, the one of interest in this study, focused on measuring PSTs' flexibility with referent units. At the beginning of the assessment, students were instructed to "solve all problems using a drawn length model (e.g., strips or fractional bars) and show your process along with your explanation." Also, it was noted that partitions in the drawings were to be marked clearly, and pictures were to be drawn proportionately to accurately show relationships between quantities. As noted, only answers to the question measuring PSTs' flexibility with referent units in solving a *measurement* type of fraction division problem (see Fig. 4) were examined in this study, but because this was the last question, solving the nine previous problems served to remind PSTs of what they had learned from Course 1.

The stick shown below is $\frac{3}{5}$ of a whole stick. How many $\frac{1}{20}$ sticks can you make from the $\frac{3}{5}$ stick? Solve the problem and provide a pictorial representation by using the given length model to show your reasoning to reach your solution.

Fig. 4 The target question used in this study

I purposefully selected measurement division rather than partitive division to assess. Measurement division problems call for determining how many groups of an intended quantity are contained in a given quantity. That is, when the divisor represents the amount in one group and the quotient represents the number of groups, the divisor can be repeatedly subtracted from the dividend. In contrast, partitive division problems ask how many units are in one group, so solving such problems is closely related to generating a unit rate. Many U.S. mathematics textbooks tend to introduce fraction division in a measurement context (Jansen & Hohensee, 2016; Lo & Luo, 2012) rather than a partitive context, which is more cognitively challenging. Thus the PSTs in this study were more likely to be familiar with measurement than with partitive fraction division. Also, because it makes sense to use rectangular bar representations in measurement fraction division, I assumed that this type of problem would better reveal PSTs' flexibility with referent units in drawing representations modeling fraction division.

4.3 Data collection and data analysis

A written assessment about fractional knowledge that included the question in Fig. 4 was administered to all PSTs in four sections of a mathematics content course at the beginning of fall 2014 and spring 2015. Enough time (about 40 min) was given to PSTs so that everyone could complete the assessment within the given time.

Both qualitative and quantitative analyses were conducted based on an inductive content analysis approach (Grbich, 2007), which involves five processes: (a) organizing raw data into an Excel spreadsheet and reading each PST's response, (b) identifying correctness of the responses, (c) drafting and finalizing coding schemes, (d) coding all data using the finalized coding schemes (refer to Tables 2, 3, and 4 presented in the findings section), and (e) interpreting the data quantitatively and qualitatively (Creswell, 2014). For the reliability of the coding, I asked two other mathematics education researchers to code twenty randomly selected responses, resulting in 100% agreement on the coding of 98% of the examples. To address the three research questions, PSTs' responses to the target task were categorized based on their correctness and the approach used to solve the problem. I then coded the data in terms of correct or incorrect solutions and representations. I developed three coding schemes as discussed below.

4.4 Operationalization of the coding schemes

The coding scheme for PSTs' correct solutions Based on Son and Crespo's (2009) and Smith and Thompson's (2008) studies, I classified PSTs' correct solutions into three categories: "reasoning with quantities," "reasoning through computations not tied explicitly to quantities," and "computing based on algorithms without quantitative reasoning."

Reasoning with quantities One way to approach fractions based on quantitative reasoning is to consider fractions as constituents of the measureable quantity, length, in which a fractional length is compared with a length that is identified as a referent unit. From this perspective, fractions can be depicted using length representations such as rectangular bars, which can be operated on mentally or manually. Important operations for fractional knowledge are *partitioning* (i.e., dividing a fraction bar into equal parts), *disembedding* (i.e., extracting a part while mentally preserving the whole), and *iterating* (i.e., repeating a fractional part to make a larger fraction such as a whole) (Hackenberg, 2013; Steffe & Olive, 2010). Students' level of fractional knowledge is determined by the extent of their command of these operations. Thus, PSTs' solutions are categorized as reasoning with quantities when they provide correct solutions based on drawings with attention to appropriate referent units and fractional operations. Because this category focuses on use of pictorial representations to deal with quantities, PSTs who (1) depended entirely on using their drawings, or (2) used their drawings and further connected them with numerical operations of fraction division belonged to this category. Following is an example in the context of fraction division.

> **Mary Stick Problem**: Mary has a stick, which is $\frac{2}{3}$ of a whole stick. How many $\frac{1}{9}$ sticks can Mary make from the $\frac{2}{3}$ stick?

To solve this problem, PSTs categorized as reasoning with quantities might first have partitioned the given stick into two equal parts and added a part to make a $\frac{3}{3}$ stick, using their fractional knowledge that the referent units of the two fractions ($\frac{2}{3}$ and $\frac{1}{9}$) are both one whole, and then partitioned each third into thirds to get nine equal parts. Finally, some PSTs might have implicitly used an informal strategy of fraction division such as "repeated subtraction" or explicitly linked their quantitative reasoning to the numerical operation of fraction division, i.e., $\frac{2}{3} \div \frac{1}{9}$.

Reasoning through computations not tied explicitly to quantities PSTs assigned to this category solved the problem using informal strategies of fraction division, as shown in Table 1, without using an invert and multiply algorithm. That is, because they produced an incorrect drawing, used their drawing to show only the final result, or produced no drawing, their solutions were not tied explicitly to quantities but were based on previously learned computational reasoning. These PSTs might not have used drawings as a tool for quantitative reasoning, but they might have used a decimal strategy or a common denominator strategy to solve the Mary-Stick-Problem. That is, they might have changed the given fractions into decimals and divided 0.666666… by 0.111111 or used their knowledge about equivalent fractions to find the common denominator of the two fractions (i.e., $\frac{6}{9} \div \frac{1}{9}$) and divided the numerators of the fractions (i.e., $6 \div 1 = 6$).

Computing based on algorithms without quantitative reasoning PSTs assigned to this category solved the problem using a formal algorithm for fraction division without using either a drawing as a tool for quantitative reasoning or informal strategies based on computational reasoning. They might also have provided an incorrect drawing, a drawing showing only the result, or no drawing. For instance, these PSTs might have solved the Mary-Stick-Problem using a conventional "invert and multiply" algorithm (e.g., $\frac{2}{3} \div \frac{1}{9} = \frac{2}{3} \times \frac{9}{1} = 6$) without a drawing.

The coding scheme for PSTs' incorrect solutions Isiksal and Cakiroglu (2011) grouped PSTs' awareness of children's common misconceptions about fraction operations into five categories: Algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge of fraction operations, misunderstanding of the symbolism of a fraction, and misunderstanding of problems. *Algorithmically based mistakes* arise from misapplying basic operational rules. For example, PSTs might solve the Mary-Stick-Problem by inverting the first term instead of the second term and multiplying them (i.e., $\frac{2}{3} \div \frac{1}{9} = \frac{3}{2} \times \frac{1}{9} = \frac{1}{6}$). *Intuitively based mistakes* result from overgeneralizing the properties of operations of whole numbers to fractions. For example, PSTs might think that it is impossible to get a larger number, six, when you divide $\frac{2}{3}$ by $\frac{1}{9}$ because whole number division usually produces a smaller number. *Mistakes based on formal knowledge of fraction operations* are related to students' insufficient knowledge of the properties of operations with fractions. For example, some PSTs might use multiplication (i.e., $\frac{2}{3} \times \frac{1}{9} = \frac{2}{27}$) although the Mary-Stick-Problem requires division of fractions. *Misunderstanding the symbolism of a fraction* indicates limited knowledge of the notation of fractions. For example, some PSTs might consider the top part as denominator and the bottom as numerator in the fraction notations given in the Mary-Stick-Problem, $\frac{2}{3}$ and $\frac{1}{9}$. *Misunderstanding the problem* can derive from either lack of care or mathematical knowledge and language, which is needed to understand the mathematical ideas in a word problem and to transfer the given information into symbolic representation. Isiksal and Cakiroglu used the five categories to analyze PSTs' predictions of the common misconceptions students would demonstrate when given fraction operation problems. In this study, based on a pattern I found in the preliminary data analysis, I used the framework to categorize PSTs' *own* errors in solving a measurement division problem with fractions.

The coding scheme for PSTs' purpose for drawing Lee et al. (2011) and Izsák (2008) identified two pedagogical purposes for using drawings in fraction and decimal multiplication: (1) as a means of reasoning about fractional quantities and (2) as a support for numeric computation. Izsák presented examples of teachers' drawings representing the first and second categories. In determining $\frac{1}{2}$ of $\frac{2}{3}$, Ms. Reese used the area model as a tool to reason about fractional quantities (see Fig. 5, left) by drawing a square, partitioning it into thirds horizontally, and shading $\frac{2}{3}$ of the square. Then she divided the square vertically and cross-shaded a half of $\frac{2}{3}$ to notate the result, $\frac{2}{6}$. In the present study, teachers who used a length representation quantitatively to show their process of getting the answer (refer to Fig. 6) were assigned to the first

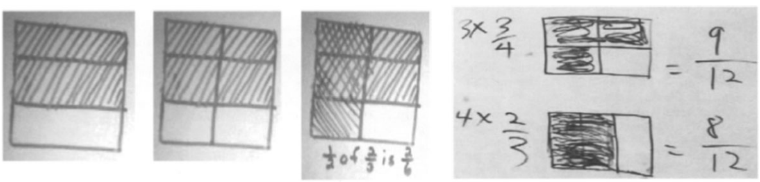

Fig. 5 Examples of drawings used as a means of reasoning about fractional quantities (Extracted from Izsák, 2008, p. 130) and of drawings used as a support for numeric commutations (Extracted from Izsák, 2008, p. 118)

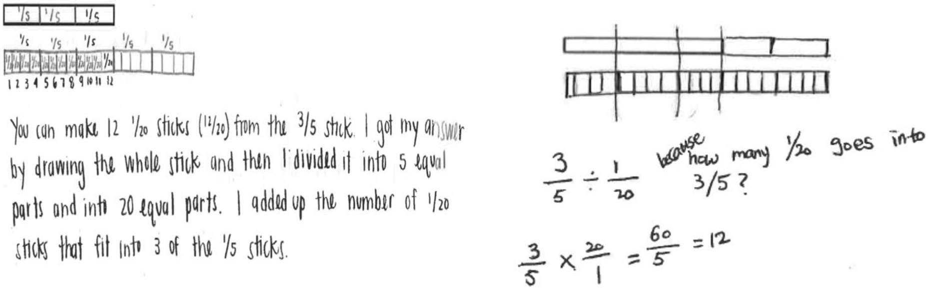

Fig. 6 Examples of PSTs' solutions based on quantitative reasoning without fraction division expression (*left*) and with an explicit fraction division expression (*right*)

category, *the use of drawing as a means of reasoning about fractional quantities*, which also included most teachers assigned to the category of "reasoning with quantities" in the coding of their correct solutions.

However another teacher in Izsák's study, Ms. Archer, used numeric computation with her drawing in comparing two fractions, $\frac{3}{4}$ and $\frac{2}{3}$ (see Fig. 5, right). That is, she first drew two squares and shaded three out of four equal parts in one and two out of three parts in the other. Then to make equivalent fractions, she switched to numeric computations by multiplying the denominators 3 and 4 from the fractions $\frac{3}{4}$ and $\frac{2}{3}$ for least common denominator. Thus her drawing did not reflect her reasoning but just showed the quantities of two fractions given in the problem. In this study, teachers who used their drawings only to represent numbers given in the problem or to illustrate the problem situation but solved it through computations were assigned to the second category, *the use of drawings as a support for numeric computation* (refer to Fig. 11).

In the current study, I derived one more category from the preliminary findings, "the use of drawings as a means of illustrating numeric answers." Teachers in this category did numeric computation first and then started drawing to illustrate their answer. Thus, to find $\frac{1}{2}$ of $\frac{2}{3}$, the teachers first multiplied $\frac{1}{2}$ by $\frac{2}{3}$ to obtain $\frac{2}{6}$ or $\frac{1}{3}$ and then drew and partitioned their squares to illustrate this answer. Teachers who provided a length representation to show the final answer without any supplementary marks or explanations (refer to Fig. 10) were assigned to this category.

5 Analysis and findings

In this section, to address three research questions, first, I describe types of correct solutions the PSTs provided based on the three categories, reasoning with quantities, reasoning through computations not tied explicitly to quantities, and computing based on algorithms without quantitative reasoning. Then I present errors they made using Isiksal and Cakiroglu's (2011) framework. Finally, I discuss the features of PSTs' representations in relation to the correctness of their solutions and address the purposes for which representations were used with correct solutions.

5.1 PSTs' correct solutions for a measurement fraction division problem

Out of 111, 52 PSTs (47%) provided correct solutions while 42 (38%) provided incorrect solutions, and 17 (15%) provided no solutions. Among the 52 PSTs who got correct solutions,

Table 2 Number of PSTs who got correct solutions but correct or incorrect representations

Types of correct solutions	Correctness of representations	Frequency
Reasoning with quantities (13)	1. Quantitative reasoning without explicit fraction division expression	2
	2. Quantitative reasoning with explicit fraction division expression	11
Reasoning through computations not tied explicitly to quantities: Informal strategies (8)	1. Common denominator strategy (or making equivalent fractions) with explicit fraction division expression	6
	2. Repeated subtraction with explicit fraction division expression	0
	3. Decimals with explicit fraction division expression	2
	4. Unit rate with explicit fraction division expression	0
	5. Dividing numerators and denominators with explicit fraction division expression	0
Computing based on algorithms without quantitative reasoning: Formal strategies (31)	1. Invert and multiply in an explicit fraction division expression	28
	2. Cross multiplication in a proportion without an explicit fraction division expression	3

13 solved the problem by reasoning with quantities through drawing and 8 solved the problem by reasoning through computations although they were not tied explicitly to quantities. The remaining 31 PSTs solved the problem using the invert and multiply algorithm (see Table 2).

The 13 PSTs who solved the problem by flexibly reasoning with referent units through a pictorial representation partitioned the given bar into three parts, added two more fifths to create a whole bar, partitioned it into 20 pieces, aligned it with a $\frac{3}{5}$ bar, and counted the number of twentieths corresponding to the $\frac{3}{5}$ bar. Interestingly, however, two of these PSTs did not connect their drawings to a fraction division expression (see Fig. 6, left) while 11 PSTs provided the fraction division expression, $\frac{3}{5} \div \frac{1}{20}$, along with their drawings (Fig. 6, right).

Of the eight PSTs who solved the problem by reasoning through computations not tied explicitly to quantities, six used the common denominator strategy by making equivalent fractions (Fig. 7, left). Two changed the given fractions into decimals and performed division (Fig. 7, right). However, none used the strategies of repeated subtraction (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{12}{20} - \frac{1}{20} - \frac{1}{20} - \frac{1}{20} \ldots\ldots = 0$), of unit rate (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{\frac{3}{5}}{\frac{1}{20}} = \frac{\frac{3}{5} \times \frac{20}{1}}{\frac{1}{20} \times \frac{20}{1}} = \frac{\frac{60}{5}}{1} = \frac{60}{5} = 12$), or of dividing numerators and denominators (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{3 \div 1}{5 \div 20} = \frac{\frac{3}{1} \times 20}{\frac{5}{20} \times 20} = \frac{60}{5} = 12$). This result

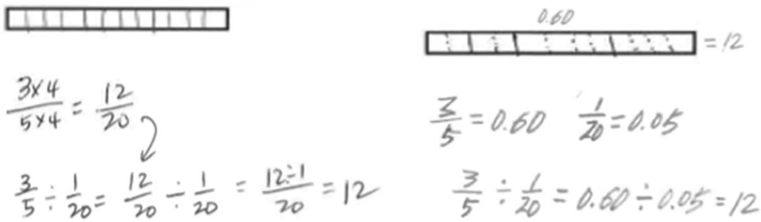

Fig. 7 Examples of PSTs' solutions based on the strategies of common denominator (*left*) and decimals (*right*)

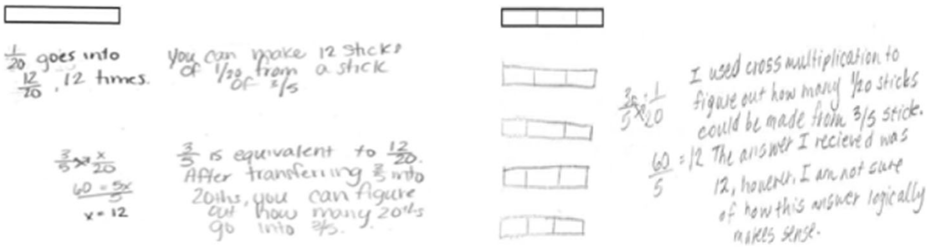

Fig. 8 Examples of PSTs' solutions based on the strategy of cross multiplication

suggests that PSTs in the U.S. may lack sufficient experience working with alternative strategies for solving fraction division problems.

Regarding the category of computing based on algorithms without quantitative reasoning, 28 PSTs used the invert and multiply strategy from fraction division expression (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{3}{5} \times \frac{20}{1} = \frac{60}{5} = 12$) and three used the formal strategy of cross multiplication related to proportion expression (Fig. 8), for example, by creating a proportion such as $\frac{3}{5} = \frac{x}{20}$ to figure out how many twentieths of a whole stick can be found in $\frac{3}{5}$ of the stick without explicitly realizing that they were solving a measurement division problem. The invert and multiply solution and the cross multiplication solution neither connected well to length quantities nor showed reasoning through computations.

5.2 Types of PSTs' errors in solving a measurement fraction division problem

I categorized the 42 incorrect solutions into three categories according to Isiksal and Cakiroglu's (2011) framework, the most frequent of which was misunderstanding the problem, followed by mistakes based on formal knowledge of fraction operations and algorithmically based mistakes (see Table 3).

Nineteen PSTs showed they misunderstood the problem by either verbalizing erroneous reasoning or providing depictions which did not make sense as solutions (Fig. 9). Thirteen PSTs made the formal knowledge mistake of using multiplication rather than division in their expression, for example, creating an expression such as $\frac{3}{5} \times \frac{1}{20}$ instead of $\frac{3}{5} \div \frac{1}{20}$. Ten PSTs made algorithmically based mistakes derived from operational errors. For example, some created a correct fraction division expression such as $\frac{3}{5} \div \frac{1}{20}$, but instead of using the invert and multiply algorithm properly, they multiplied without inverting (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{3}{5} \times \frac{1}{20} = \frac{3}{100}$), inverted the dividend instead of the divisor and multiplied (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{5}{3} \times \frac{1}{20} = \frac{1}{12}$), or inverted both the dividend and the divisor and multiplied (e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{5}{3} \times \frac{20}{1} = \frac{100}{3}$).

Table 3 Types of incorrect solutions (Adapted from Isiksal & Cakiroglu, 2011, p. 220)

Types of incorrect solutions (42)	Frequency
Algorithmically based mistakes	10
Mistakes based on formal knowledge of fraction operations	13
Mistakes resulting from misunderstanding the problem	19
Intuitively based mistakes	0
Mistakes resulting from misunderstanding the symbolism of a fraction	0

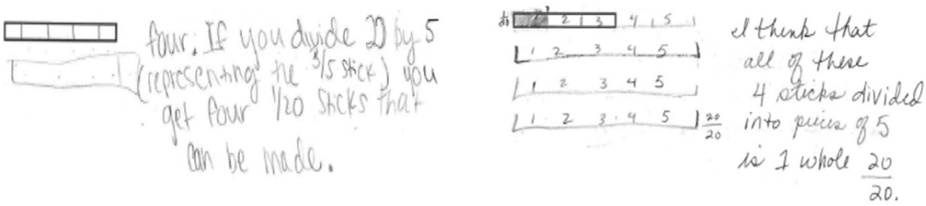

Fig. 9 Examples of PSTs' incorrect solutions based on misunderstanding the problem

5.3 PSTs' pictorial representations in relation to the correctness of their solutions

In this section, I discuss features of PSTs' representations in relation to the correctness of their solutions and the purposes for which they used representations when their solutions were correct. I classified PSTs' purposes for drawing into three categories: (1) as a means of reasoning about fractional quantities; (2) as a support for numeric computation; and (3) as a means of illustrating numeric answers.

Representations of PSTs who provided correct solutions and the apparent purposes for their use of representations The 13 PSTs who answered by reasoning through correctly drawn fractional quantities used their drawings as a means of reasoning about fractional quantities and identified an appropriate referent unit to arrive at correct solutions (refer to Fig. 6). That is, they first partitioned the given bar into thirds, each representing a fifth of a whole bar and added two more fifths to create the whole as a referent unit. Then they partitioned the whole into twentieths and aligned it with the 3/5 bar to count the number of twentieths that fitted into the 3/5 bar. Thus their drawings, which were represented with a length model, reflected their reasoning with fractional quantities in that measurement fraction division can be interpreted as finding how many times a divisor fits into a dividend. Also, counting the number of twentieths to find the answer demonstrated PSTs' flexible changing of the referent unit from the original whole to a twentieth.

However, all eight PSTs who solved the problem by reasoning through computations used their drawings only as a means of illustrating numeric answers (Table 4). That is, they first found the answer by using informal strategies and then partitioned the given $\frac{3}{5}$ bar into twelfths to show the result. Here, their drawings were neither tied explicitly to fractional quantities that were represented by a bar (refer to Fig. 7) nor indicative of what they considered to be a referent unit in solving a measurement fraction division problem.

Table 4 Frequency of PSTs' purposes when appearing to use representations according to types of correct solutions

Types of correct solutions		Purpose for using representations	
Quantitative reasoning	13	Using drawings as a means of reasoning about fractional quantities	13
Reasoning through computations not tied explicitly to quantities	8	Using drawings as a means of illustrating numeric answers	8
Computing based on algorithms without quantitative reasoning	31	Using drawings as a means of illustrating numeric answers	20
		Using drawings as a support for numeric computation	6
		No representations	5

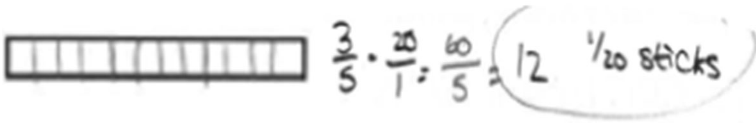

Fig. 10 An example of PSTs' representation to show the result

Also, none of 31 PSTs who solved the problem by using algorithms provided representations tied explicitly to fractional quantities. Twenty partitioned the given bar into 12 pieces to illustrate the answer they had already found using an invert and multiply strategy (Fig. 10), and five provided no representation.

Six PSTs provided inaccurate representations reflecting inappropriate fractional mental operations. Also they seemed to use their drawings as a support for numeric computation. Four erroneously partitioned the given $\frac{3}{5}$ bar into twentieths (Fig. 11, left), and the other two created a whole by partitioning, disembeding, and iterating, but their drawings did not clearly relate the bar of twentieths representing the constructed whole (the second bar in Fig. 11, right) to the bar of fifths representing the original whole (the first bar in Fig. 11, right). In this example, the PST started out trying to use her drawing quantitatively by partitioning the given bar into thirds and adding two more parts to create a whole. Then the PST drew another bar, labeled it "20th," and partitioned it into two parts, one aligned with the given $\frac{3}{5}$ bar, which she labeled "x." But then, rather than partitioning the second bar into twentieths to find how many twentieths would fit into $\frac{3}{5}$, she tried to seek the answer through algorithms by creating a proportion using the variable x in the drawing.

Representations of PSTs who provided incorrect solutions Of the 42 PSTs who solved the problem incorrectly, 8 provided no representations and 34 provided inappropriate representations, which I categorized based on the types of fractional mental operations that were misused in the process of creating the representations (Table 5). The most frequent type of misused mental operations was partitioning, followed by a combination of partitioning, disembeding, and iterating.

To create appropriate representations, the PSTs needed to use a correct referent unit implicitly or explicitly along with proper fractional mental operations. For example, they might partition the given $\frac{3}{5}$ bar into three parts, disembed one fifth, and iterate the fifth five times to create a whole bar as a referent unit. Then they might partition the whole bar into 20 parts to figure out how many twentieths occur on the $\frac{3}{5}$ stick. Or they might focus directly on how many twentieths are in a fifth and combine three of these groups without explicitly constructing the whole. However, even in the latter case, the PSTs needed to implicitly know that the referent unit for both the twentieth and the fifth was the same whole.

However, in this study, 34 PSTs provided inappropriate representations by using incorrect referent units or fractional mental operations. Although 11 of them created a whole bar ($\frac{5}{5}$) by

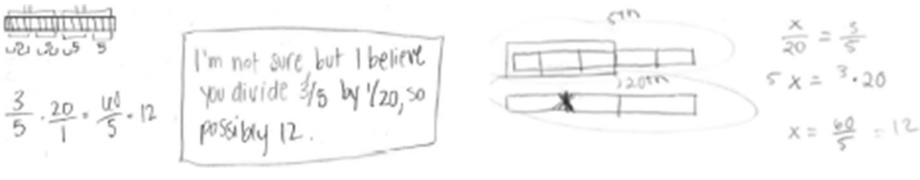

Fig. 11 Examples of PSTs who used drawings as a support for numeric computation

Table 5 Types of misused fractional mental operations and referent units

Mental operations	Explanation	Referent unit for twentieths	Frequency
Partitioning	Partitioning the given bar into twentieths	The given bar ($\frac{3}{5}$)	14
Partitioning, Re-partitioning	Partitioning the given bar into thirds and re-partitioning a fifth into twentieths	A fifth ($\frac{1}{5}$)	3
Iterating	Iterating the given bar multiple times (e.g., 3, 6, 10, 20 times etc.)	Unclear	6
Partitioning, Disembeding, and Iterating	Using three mental operations to create a whole but failing to relate the drawing explicitly to twentieths	No evidence of a referent unit	11

partitioning the $\frac{3}{5}$ bar into three parts, disembedding one fifth, and iterating it five times (Fig. 12, left)), they did not proceed to consider the relationship between a $\frac{3}{5}$ bar and a $\frac{20}{20}$ bar, both of which have a whole as a referent unit. Meanwhile, 14 PSTs tried to construct some relationship between a $\frac{3}{5}$ bar and a $\frac{20}{20}$ bar by partitioning the given bar into 20 parts (Fig. 12, right). However, they did not project a whole bar but treated the $\frac{3}{5}$ bar as a whole and directly partitioned it. Consequently, their $\frac{3}{5}$ bar and $\frac{20}{20}$ bar had different referent units, the $\frac{5}{5}$ bar and $\frac{3}{5}$ bar respectively, although in measurement division the dividend and divisor should have the same referent unit.

Also, six PSTs iterated the given $\frac{3}{5}$ bar multiple times to obtain 20 pieces in total. For example, the PST who provided the representation on the left side of Fig. 13 seemed to interpret the problem numerically as $3 \times \square = 20$, and iterated the given $\frac{3}{5}$ bar seven times to get to about 20 fifths by adding the numerators ($\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}=\frac{21}{5}$). Another three PSTs partitioned the given bar into three parts and re-partitioned each fifth into 20 pieces (Fig. 13, right) to reach the answer of 60 twentieth pieces in the $\frac{3}{5}$ bar. This representation shows that the PSTs considered a fifth as a referent unit for twentieths.

6 Discussions and implications

This study investigated the knowledge that PSTs need to solve a measurement fraction division problem. To analyze PSTs' solutions and representations, three frameworks were adapted and merged to categorize their correct solutions, incorrect solutions, and apparent purposes for drawing representations. Consistent with prior research (Borko et al., 1992; Rosli et al., 2013), the number of PSTs who solved the problem correctly through computation based on algorithms without quantitative reasoning was greater than that of PSTs who solved the problem through reasoning with quantities. Son and Crespo (2009) identified six strategies

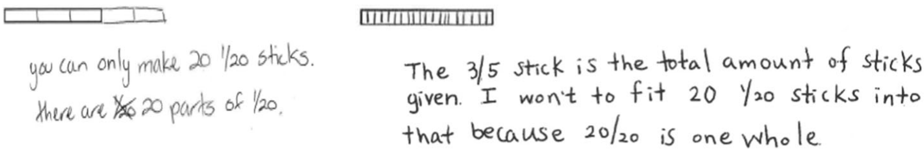

Fig. 12 Examples of PSTs' representations reflecting misused mental operations

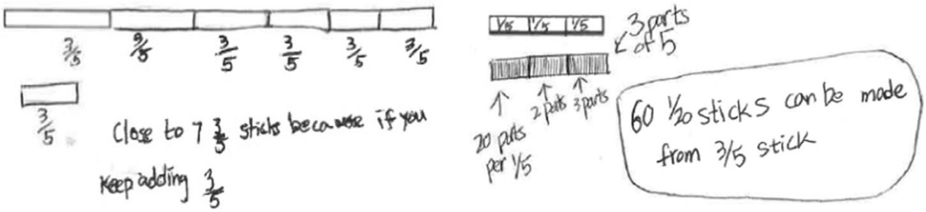

Fig. 13 Examples of PSTs' representations involving misused mental operations

based on numerical expressions to solve fraction division problems. However, in this study, two other types of solutions without using explicit fraction division expressions emerged, solutions based on quantitative reasoning with fractional quantities and solutions based on a proportion, an outcome which might be related to providing word problems with pictorial representations instead of asking students simply to solve numerical fraction division problems. This observation implies that providing a context with pictorial representations could encourage PSTs to think about a problem in more ways than one.

Also, 45% of the PSTs showed their misunderstanding of the problem, 30% demonstrated mistakes based on faulty formal knowledge of fraction operations by construing the problem as a multiplicative situation, and 23% made algorithmically based mistakes in the process of using the invert and multiply strategy. However, two types of mistakes identified in Isiksal and Cakiroglu's (2011) research, from which the original framework was derived, were not found: intuitively based mistakes and misunderstanding the symbolism of a fraction, which are related to lack of basic knowledge of fractions. This difference may reflect the different orientations of the two studies, as Isiksal and Cakiroglu investigated PSTs' understanding of children's misconceptions about fraction operations while the current study analyzed PSTs' own misconceptions about fraction operations. It is reasonable to assume that college-level PSTs have more basic fractional knowledge such as fraction symbols and rely less on intuitive knowledge than children, although some PSTs showed that they still did not have advanced fractional knowledge.

Findings from the analysis of representations in this study are consistent with prior research (Izsák, 2008; Lee et al., 2011), in which teachers were found to often depend mainly on using symbolic notation and use other representations such as diagrams only to illustrate answers. Similarly, Rosli et al. (2013) found that PSTs' representational knowledge for teaching fractions was much lower than their computational knowledge of fractions. PSTs in the current study also showed some tendency to use representations only to show final results. In addition, this study suggests further explanation for their difficulties by identifying how they misused fractional mental operations and referents in the process of creating pictorial representations, thus expanding the types of inappropriate representations.

Examining the purposes for which PSTs appeared to use pictorial representations revealed that 25% of those who provided correct solutions seemed to use their representations as a means of reasoning about fractional quantities, 54% tended to use them to illustrate numeric answers, and 12% used them to support numeric computation. Interestingly, in this study all the PSTs who solved the problem by reasoning through computations using informal strategies, and two thirds of those who solved it by computing based on formal algorithms, tended to use representations to show the result of their calculations rather than to support their mathematical reasoning with referent units. This phenomenon demonstrates that although using formal or informal strategies might be likely to result in a correct answer, it did not guarantee that PSTs understood the quantitative meaning of measurement division with fractions.

In particular, as only 13 out of 111 PSTs (12%) provided appropriate representations showing flexibility with referent units, the majority could be assumed to have difficulty in flexibly reasoning with referent units in solving a *measurement* fraction division word problem. This result is related to Lo and Luo's (2012) finding that PSTs who posed correct word problems involving *partitive* fraction division and appropriate pictorial representations tended to use referent units appropriately. Also, the present study expanded on Izsák's (2008) finding that reasoning with referent units influenced teachers' use of representations in teaching fraction multiplication by showing that PSTs' reasoning with referent units was related to creating representations while solving a measurement fraction division problem.

These findings raise the question of why PSTs struggle with reasoning with referent units. The answer might be that many PSTs have not constructed three levels of units and advanced fractional operations and schemes (Izsák, 2008), or that they have not had experience with this type of problem. Regarding the first possibility, in order to present an appropriate representation, PSTs first may need to be able to partition the given bar into three parts, disembed a fifth, and iterate it five times to create a $\frac{5}{5}$, i.e., a whole, and then be able to reason with the referent unit by re-partitioning each fifth into four parts. Then they may need to flexibly switch their view between $\frac{5}{5}$ and $\frac{20}{20}$ to figure out that 3 fifths correspond to 12 twentieths. In this process, constructing three levels of units (e.g., one unit of five units, each of which contains four units) is important to be able to reason about the referent unit. With regard to the second possibility, some implications for teacher education programs can be drawn as follows.

6.1 Specialized knowledge for teaching fraction division

Ma (1999) suggested a knowledge package for understanding the meaning of division with fractions, which includes the concept of units, the meaning of multiplication with fractions, the meaning of division with whole numbers, and the conception of inverse operation between multiplication and division. However, this study implies that two additional types of specialized knowledge need to be emphasized to prepare PSTs to teach fraction division, one of the areas with which students struggle the most. The first type of knowledge is about reasoning with referent units. While using linear representations can help students understand fraction concepts and operations, to create appropriate representations for fraction division problems, flexible reasoning with referent units is required. In this study, PSTs who solved the fraction division problem based on computation using informal or formal strategies did not always provide appropriate representations based on quantitative reasoning with referent units. However, PSTs who provided appropriate representations always offered correct solutions. Moreover, prior research has shown that teachers often teach fraction division procedurally by focusing on an "invert and multiply" algorithm because of their lack of the pedagogical content knowledge needed to use representations in order to support students' reasoning (Borko et al., 1992; Ma, 1999). Thus, mathematics educators need to design mathematics pedagogy courses in which PSTs gain enough experience to be able to readily use multiple representations, particularly linear representations, to support reasoning with referent units.

The second type of knowledge is about connecting mathematical thinking with symbolic notations. In this study, almost half of the PSTs who provided correct solutions used formal knowledge of invert and multiply [e.g., $\frac{3}{5} \div \frac{1}{20} = \frac{3}{5} \times \frac{20}{1} = 12$] to solve the given fraction division problem without providing appropriate representations. Some PSTs' representations did not match their solutions. In addition, many PSTs failed to provide linear representations that were associated with their symbolic notations, suggesting that they did not understand why the algorithm worked for the problem. This result implies the importance of connecting symbolic notations to quantitative reasoning based on appropriate representations. According to Jacobs and Empson (2016), connecting students' thinking to symbolic notation is an important teaching move in *responsive teaching,* in which teachers' instructional decisions are continually adjusted during instruction in response to children's content-specific thinking rather than being determined in advance. Thus, pre-service teacher education programs need to prepare PSTs for such teaching moves by having them practice connecting their mathematical thinking to symbolic notations.

In view of the above implication, mathematics educators should engage PSTs who use only algorithms to solve problems in discussion about why their solutions work. For example, when PSTs use the invert and multiply strategy to solve the problem without evidence of reasoning, mathematics educators can help them understand that the strategy can be derived from proportion sentences[3] [i.e., 1 whole: 20 twentieths = $\frac{3}{5}$: x number of twentieths]. That is, they can help PSTs understand how many twentieths are involved in a $\frac{3}{5}$ bar by encouraging them to project a whole bar from the given $\frac{3}{5}$ bar and dividing the whole into 20 twentieths. Also, when PSTs have solved a problem based on quantitative reasoning without symbolic notations, mathematics educators can guide them to represent their reasoning with referent units using symbolic notations. For instance, when PSTs have partitioned the $\frac{3}{5}$ bar into three parts, added two more parts to make a whole, re-partitioned each fifth into four parts to make a $\frac{20}{20}$ bar, and found how many twentieths fit into the $\frac{3}{5}$ bar, mathematics educators can help them draw the corresponding symbolic notations by directing their thinking to the procedure of finding an equivalent fraction of $\frac{3}{5}$ and then dividing the numerators of two fractions [i.e., $\frac{3}{5} \div \frac{1}{20} = \left(\frac{3}{5} \times \frac{4}{4}\right) \div \frac{1}{20} = \frac{12}{20} \div \frac{1}{20} = 12 \div 1 = 12$]. When PSTs connect symbolic notations to quantitative reasoning in this way, they can understand the notations as a way to record their mathematical reasoning efficiently rather than as something to be memorized arbitrarily.

This study has some limitations in that it used only one type of measurement fraction division problem, which had quotients that were whole numbers. Thus, although it provides insights into PSTs' flexibility with referent units in solving a measurement division problem, to expand on the results, follow-up studies should include multiple types of problems involving partitive division, different types of fraction divisors and fraction dividends, and quotients that are not whole numbers. Also, this study relied only on a written assessment and so lacked the human interaction, which might enable more in-depth examination of thinking processes. Because the analysis and inferences in this study were based on participants' written solutions, additional data could expand or change interpretations. Thus, to increase the credibility of analysis and inferences, future studies need to employ different research tools such as follow-up interviews. Finally, in this study, some PSTs drew two bars that were misaligned or disproportionate. It is possible that the results would change if the task were worded differently. For example, instead of being asked to solve the

[3] This is one among many possible ways to help PSTs think about why the invert and multiply strategy works.

problem using a pictorial representation to show their reasoning, PSTs might have been asked to use pictorial representations as if they were explaining their reasoning to students. If the task had motivated them to think about students, the PSTs might have taken more care to align the bars and demonstrated more specialized knowledge for teaching. In sum, although this study involved only one fraction division problem and the findings could be limited by the methodological approach, it warrants a closer examination of the role which flexibility with the referent unit may play in PSTs' learning and teaching fraction division meaningfully.

Acknowledgements An earlier version of this paper was presented at the NIMS (National Institute for Mathematical Sciences) and KSME (The Korean Society of Mathematical Education) International Workshop in 2017.

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