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Teaching multidigit multiplication: combining multiple frameworks to analyse a class episode

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Abstract This paper provides an analysis of a teaching episode of the multidigit algorithm for multiplication, with a focus on the influence of the teacher's mathematical knowledge on their teaching. The theoretical framework uses Mathematical Knowledge for Teaching, mathematical pertinence of the teacher and structuration of the milieu in a descending and ascending a priori analysis and an a posteriori analysis. This analysis shows a development of different didactical situations and some links between mathematical knowledge and pertinence. In the conclusion, the contribution of the frameworks from both French and Anglo-American origins is briefly addressed.

Keywords Mathematical knowledge · Teacher · Elementary teaching · Algorithm for multiplication · Pertinence · Structuration of the milieu

1 Introduction

The question of the influence of teachers' mathematical knowledge on their students' achievement in mathematics is important when it comes to prioritising the resources and the time devoted to different subjects in teacher education. This question has occupied a lot of research since the 1990s, and many papers and reports addressing the question have been produced. Many studies have tried to quantify this influence, but "evidence about the relationship of elementary and middle school teachers' mathematical knowledge to students' mathematics achievement remains uneven and has been surprisingly difficult to produce" (National Mathematics Advisory Panel, 2008, p. 37). This paper and the related research (Clivaz, 2014) adopt another point of view on this question. The focus here is not to measure but to describe and analyse the influence of the mathematical knowledge of primary school teachers on their management of school mathematical tasks. I believe this research offers specific value

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by combining analytical frameworks from both Anglo-American mathematics education research and the French *didactique des mathématiques*. Ball's categories of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) are used to describe teacher's knowledge in practice. In this paper, I describe the effect of Mathematical Knowledge for Teaching (MKT) on teaching through the mathematical pertinence of the teacher (Bloch, 2009). I analyse the teacher's knowledge and the effect of this knowledge in ordinary classroom situations, utilising the model of structuration of the milieu (Margolinas, Coulange, & Bessot, 2005) in order to take into account the complexity of the teacher's activity. The present case study focuses on an episode about the teaching of the multidigit algorithm for multiplication in a grade 4[1] class in the Lausanne region (French-speaking part) of Switzerland.

After a brief overview of these three frameworks, I highlight some points about the algorithm for multiplication.[2] The frameworks are then combined through an episode about this topic. Finally, I discuss interactions of these frameworks in analysing the teacher's knowledge and teaching.

2 Relevant theoretical perspectives and research

The nature of teacher mathematical knowledge has been a controversial subject since Shulman's (1986, 1987) seminal work. A body of solid research work regarding teacher mathematical knowledge has now been produced by international scholars (Tchoshanov, 2011, p. 141), and many authors have summarised these studies (e.g., Davis & Renert, 2013; Hart, Oesterle, & Swars, 2013; Tchoshanov, 2011; Zazkis & Zazkis, 2011). As the review by the National Mathematics Advisory Panel (2008) notes, many studies have concentrated on the attempt to provide evidence for the relationship between elementary and middle school teachers' mathematical knowledge and students' mathematics achievement. Evidence of such outcomes has been surprisingly difficult to produce and the search for such evidence has given rise to many categorisations of this knowledge. The most influential reconceptualisation of teachers' pedagogical content knowledge is the categorisation of Mathematical Knowledge for Teaching (Depaepe, Verschaffel, & Kelchtermans, 2013, p. 13).

2.1 Categories of mathematical knowledge for teaching

Refining Shulman's (1986, 1987) categories of teacher knowledge for mathematics, Ball et al. (2008) provide a practice-based division of MKT (Fig. 1).

One of the special features of this categorisation is the existence of a *Specialised Content Knowledge* (SCK), defined as

> the mathematical knowledge and skill unique to teaching. In looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general, […] teachers have to do a kind of mathematical work that others do not. […] This work involves an uncanny kind of unpacking of mathematics that

[1] Nine to ten-year-old students

[2] A detailed presentation of the three frameworks and of the epistemological analysis can be found in Clivaz (2011, 2014).

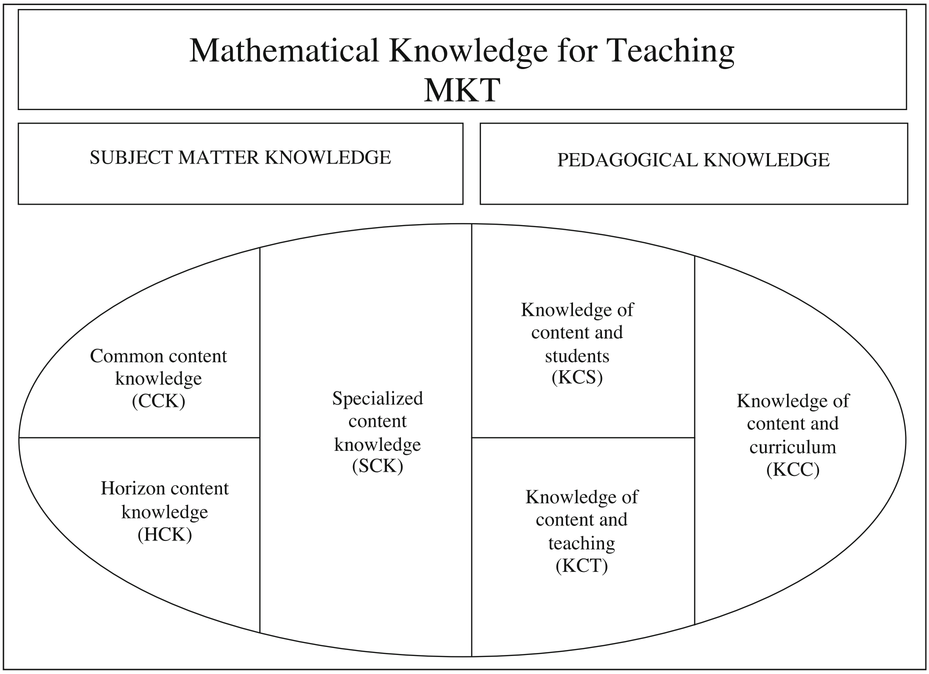


Fig. 1 Domains of mathematical knowledge for teaching (Ball et al., 2008, p. 403)

> is not needed—or even desirable—in settings other than teaching. (Ball et al., 2008, p. 400)

The categorisation of MKT was successful in showing the influence of Specialised Content Knowledge on students' results (Hill, Rowan, & Ball, 2005). It was also used to try to capture the mathematical quality of instruction (Hill et al., 2008) by the means of a quantitative classroom observation study. But, as Ball acknowledges, the framework remains quite static (Ball et al., 2008, p. 403) and is not focused on describing *how* teacher mathematical knowledge influences teaching and learning. As stated by Davis and Renert (2013), "this will require more fine-grained analyses than large-scale assessments" (p. 20).

The framework for such a qualitative, fine-grained and mobile analysis can be found in the *Theory of didactical situations* (Brousseau, 1997). At its beginning in 1970, this theory first modelled a learning situation where the teacher was rather absent (Bloch, 2005). But, since the 1990s, the importance of the teacher's role has been increasingly evident in the study of ordinary classroom situations (Bloch, 1999; Dorier, 2012; Roditi, 2011). This development applies to the mathematical pertinence of a teacher's actions and for the structure of the milieu I used in my study.

2.2 Mathematical pertinence of a teacher's actions

In order to identify the effects of a teacher's mathematical knowledge, Bloch (2009) suggests considering the *mathematical pertinence of a teacher's actions*. A teaching action is pertinent if it allows the student to grasp the functionality of mathematical objects, if that action is stating mathematical properties and mathematical arguments

Table 1 Levels of a teacher's activity (Margolinas et al., 2005, p. 207)

+3 Values and conceptions about learning and teaching
Educational project: educational values, conceptions of learning, conceptions of teaching
+2 The global didactic project
The global didactic project, of which the planned sequence of lessons is a part: notions to study and knowledge to acquire
+1 The local didactic project
The specific didactic project in the planned sequence of lessons: objectives, organisation of work
0 Didactic action
Interactions with pupils, decisions during action
−1 Observation of pupils' activity
Perception of pupils' activity, regulation of pupils' work

for the validity of procedures or for the nature of mathematical objects.[3] The main criterion to evaluate this pertinence in teaching is the "ability of interacting with the students on mathematical aspects of the situation and to encourage their activity by the means of interventions and feedback on their mathematical production"[4] (p. 32).

2.3 Structure of the milieu

The concept of *milieu*[5] is central to the theory of didactical situations. The milieu is defined by "all of the pertinent features of the student's surroundings, including the space, the teacher, the materials and the presence or absence of other students" (Warfield, 2014, p. 66). To describe the teacher's activity, Margolinas (2002) has developed a model structuring the milieu, based on Brousseau (1997). This model was designed to take into account the complexity of the teacher's activity and in particular to capture the elements the teacher is dealing with (Margolinas et al., 2005, p. 207). A simpler version of this model is presented by Margolinas and her colleagues as a model of the teacher's activity (cf. Table 1).

At every level, the teacher has to deal not only with the current level but at least also with the levels directly before and after the current level. These multidimensional tensions make a linear interpretation of teachers' work inaccurate (Margolinas et al., 2005, p. 208).

In fact, a more complete model can be considered, including the student (E, for *élève*), the teacher (P, for *professeur*) and the milieu (M). Brousseau characterised the milieu as an "onion", in a recursive structure. Each milieu M_i is constituted at each level i by the lower E_{i-1}, P_{i-1} and M_{i-1} components. Then, the situation S_i is made at each level i by E_i, P_i and M_i. This can be written as $S_i = (M_i; E_i; P_i)$ and $M_i = S_{i-1}$ (Margolinas, 2002, p. 146). At every level i, P_i is the teacher in the level of activity described in Table 1.

This model can be represented in an onion diagram (Fig. 2) or in a table (Table 2) where the teacher's levels range from +3 to −1 and the student's levels range from −3 to +1.

[3] « Une intervention mathématique est pertinente si elle rend compte dans une certaine mesure de la fonctionnalité de l'objet mathématique visé ; ou, s'agissant d'enseignement, si elle permet au moins de progresser dans l'appréhension de cette fonctionnalité, avec des énoncés de propriétés mathématiques contextualisées ou non, des arguments appropriés sur la validité de procédures ou sur la nature des objets mathématiques. » (Bloch, 2009, p. 32)

[4] « […] capacité à interagir avec les élèves sur des éléments mathématiques de la situation et à encourager l'activité des élèves par des interventions et des retours sur leur production mathématique. » (Bloch, 2009, p. 33)

[5] Milieu is the usual translation for Brousseau's French term "milieu", but, in French, it refers not only to the sociological milieu but it is also used in biology or in Piaget's work. A more accurate translation would be "environment".

Fig. 2 Structure of the milieu, level −1 to +1

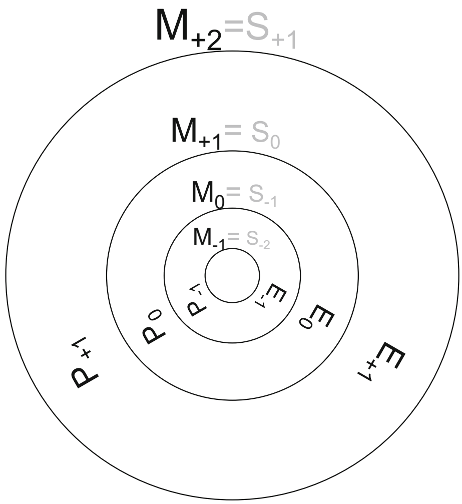


Table 2 represents the fundamental idea of interlocking but in a manner that is less clear than Fig. 2. The advantage of Table 2, however, is that it shows that teacher's levels P_i in Table 2 are the levels of teacher activity i in Table 1. Table 2 also better presents the symmetry of the model around level 0. It is perhaps rather difficult to understand the precise definitions of every level of the milieu without an example. Therefore, more precise definitions and examples of the situations $S_i = (M_i;\ E_i;\ P_i)$ and of the various teacher's and student's levels, which will clarify the terms used in each level, are given in the analysis below.

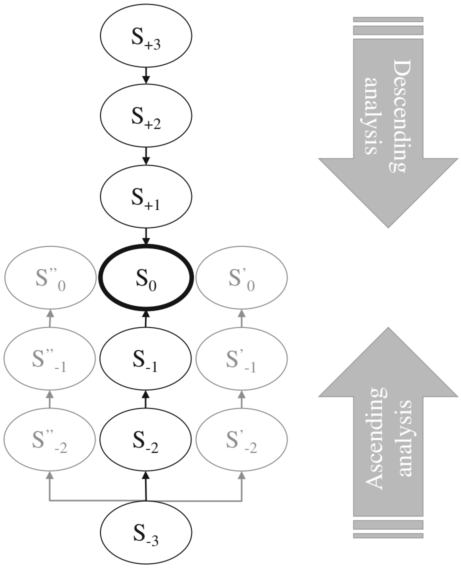
In an a priori analysis, the didactical classroom situation S_0 should be determined. Therefore, this situation can be determined either from the teacher's point of view, by a *descending analysis* from level +3 down to level 0, or from the student's perspective, by an *ascending analysis* from level −3 to 0. In some cases, interpretation of the objective situation S_{-3} by the generic active student (E_{-2}) can be diverse and divergent, as illustrated in the analysis below. In that case, the analysis may lead to several didactical situations S_0 which may or may not be the same as the situation determined by the descending analysis. The consequence is that students and teacher would "inhabit" various didactical situations. Margolinas (2004b) calls this a *didactic bifurcation* (Fig. 3).

Table 2 Milieu's structuring (Margolinas, 2002, p. 145)

M+3: M-Construction		P+3: P-Ideological	S+3: Ideological situation
M+2: M-Project		P+2: P-Constructor	S+2: Construction situation
M+1: M-Didactical	E+1: E-*Reflective*	P+1: P-Projector	S+1: Project situation
M0: M-*Learning*	**E0**: **Student**	**P0**: **Teacher**	**S0**: **Didactical situation**
M−1: M-*Reference*	E−1: E-*Learning*	P−1: P-Observer	S−1: *Learning situation*
M−2: M-*Objective*	E−2: E-*Active*		S−2: *Reference situation*
M−3: M-*Material*	E−3: E-*Objective*		S−3: *Objective situation*

Translation from Margolinas (2004a), my translation for the *italicised terms*

Fig. 3 Descending and ascending analysis in case of a didactic bifurcation



The descending analysis usually uses an interview with the teacher, conducted before the lesson. The ascending analysis is a priori "in the sense that it doesn't depend on experimental or observational facts[6]" (Margolinas, 1994, p. 30). Every student's E_i represents a level of activity of a generic, impersonalised student. Both descending and ascending a priori analyses are then tools to analyse the classroom observations in an a posteriori analysis by a comparison between the potential S_0 situations and the actual ones.

Several case studies using the structuration of the milieu have been published in French (e.g., Bloch, 1999; Bloch & Gibel, 2011; Comiti, Grenier, & Margolinas, 1995; Coulange, 2001; Margolinas, 1995, 1998, 1999; Perrin-Glorian, 1999; Perrin-Glorian & Hersant, 2003). They fruitfully use the fine-grained analysis of the structuration to analyse a posteriori a problematic classroom situation and find its roots through the a priori analysis.

2.4 The algorithm for multiplication

The choice of a mathematical topic for performing my analysis was guided by the typical example used in Ball and colleagues' analysis of MKT: the teaching of the multiplication of whole numbers algorithm (Ball, Hill, & Bass, 2005, pp. 17–21). I have made a detailed epistemological analysis of the algorithm for multiplication and the possible didactical choices for its teaching in Clivaz (2014), mostly from a Swiss and French point of view. From a more international perspective, one can note the "substantial evidence […] that algorithms should be learned with understanding" (Ambrose, Baek, & Carpenter, 2003, p. 67). This understanding is important for accuracy of the algorithm, as errors arise when procedures are separated from meaning (Lampert, 1986). Particularly, the advantage of working with non-standard algorithms, especially invented algorithms, is emphasised by many authors (e.g., Ambrose et al., 2003; Baek, 1998; Van De Walle, Karp, Bay-Williams, Wray, & Rigelman, 2014). One reason is that "the compact notation of standard algorithms tends to mask the underlying principles

[6] « dans le sens qu'elle ne dépend pas des faits d'expérience ou d'observation », my translation.

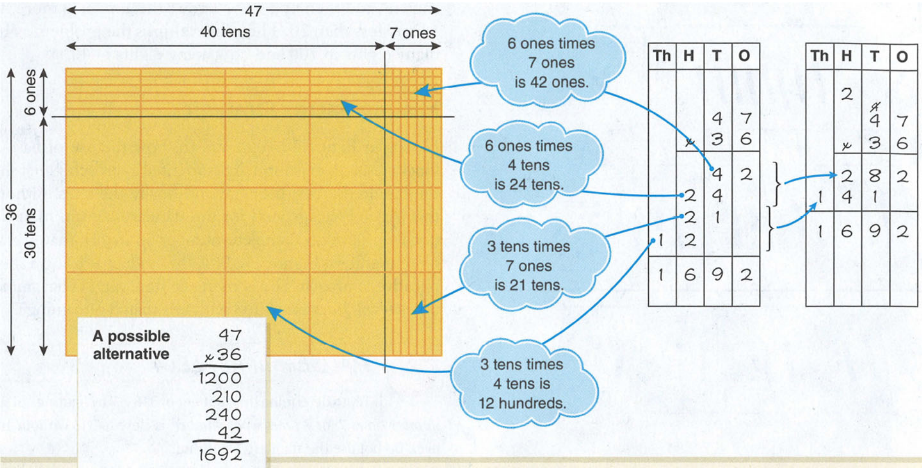


Fig. 4 Area model for 47 × 36 (Van De Walle et al., 2014, p. 259)

that make them work" (Ambrose et al., 2003, p. 66). This is particularly true for place value, since "standard algorithms tend to make students think in terms of digits rather than the composite number that the digits make up, so students lose the essence of the actual place value of a digit" (Van De Walle et al., 2014, p. 252). It is also the case for the other properties for which non-standard algorithms "require a strong understanding of the operations and the properties of the operations, especially the commutative property, the associative property, and the distributive property of multiplication over addition" (Van De Walle et al., 2014, p. 252). In many present materials and guidelines for teachers, the repeated addition model and the area model are presented (e.g., Ashlock, 2010; Van De Walle et al., 2014), but

> The area model encourages a visual demonstration of the commutative and distributive properties (unlike the number line or a model of equal groups). The area model can also be linked to successful representations of the standard algorithm for multiplication and future topics such as multiplication of fractions. […] When you move to two-digit multipliers, the area model has some advantages. (Van De Walle et al., 2014, pp. 256–257)

In the Lausanne region, the mandatory textbook used (Danalet, Dumas, Studer, & Villars-Kneubühler, 1999) surprisingly has no activity or exercise to introduce or provide practice on algorithms for multiplication and teachers have to find or create their own exercises (Clivaz, 2014, pp. 128–129). This is particularly relevant to the episode analysed in this paper.

2.5 Research questions

The study aims to identify the influence of primary school teachers' mathematical knowledge on their didactical management of school mathematical tasks. For the purpose of the study, lessons that focused on the topic of teaching multidigit multiplication were observed. The research questions were: What choices does a teacher make when presenting the multidigit multiplication algorithm to his students? How does the teacher's MKT influence these choices? What are the consequences of these choices in terms of mathematical pertinence?

As an attempt to answer these questions, I present in this paper a fine-grained analysis of one of the classroom episodes observed. In order to acknowledge the complexity of the teacher's activity and capture the elements the teacher is dealing with, this analysis will use the structure of the milieu and compare a priori (descending, from the point of view of the teacher and ascending from the student's point of view) and a posteriori analyses.

3 The case of Dominique

3.1 Methodology

Focusing on the influence of mathematical knowledge of primary school teachers on their management of school mathematical tasks, I observed four teachers in the Lausanne region during their teaching of the multidigit algorithm for multiplication. The four teachers were chosen randomly and all of them volunteered to participate in the study. This was a convenience sample, but efforts were made to ensure that those teachers varied in terms of teaching experience (from 6 to 18 years), geographical location (rural and urban) and gender (three females and one male). All the lessons (between two and nine for each teacher) were observed and video-recorded by the researcher. Teachers were interviewed before and after the series of lessons according to a semi-structured framework of questions. The video recordings were transcribed and coded with Transana software (Fassnacht & Woods, 2002–2011) according to the categories of mathematical knowledge for teaching, mathematical pertinence and the levels of the teacher's activity described above. For all the data, features of MKT were coded first and, for each MKT segment, the mathematical pertinence and the level of activity were then coded (Clivaz, 2011, pp. 142–150).

A more detailed analysis was then conducted focusing on one particular episode from the four segments of explanation of the algorithm by the four teachers. I decided to choose this particular episode for more detailed analysis because, even after the coding and the analysis with MKT, pertinence and level of teacher activity, the situation still appeared incomprehensible. Hence, the decision to analyse this episode with the more complete model of the structure of the milieu was made.

The episode, which lasted for 27 min, was taken from a series of seven lessons conducted by Dominique, a grade 4 generalist teacher with 11 years of teaching experience, that were devoted to teaching this algorithm to his class. The episode analysed in this paper features the segments where Dominique demonstrated the multidigit algorithm of multiplication to seven students.

The a priori descending analysis is presented in the following sections of this paper. The a priori ascending analysis will then be presented and the a posteriori analysis will suggest reasons for some highlighted issues, in terms of types of MKT and mathematical pertinence.

3.2 A priori descending analysis: teacher's point of view

Based on the interview with the teacher, the descending analysis goes down from level +3 to level −1 and determines a priori the didactical situation S_0 from the teacher's point of view. The topics Dominique addresses are various, so, starting from level +3, I focus only on the question about the type of algorithm presented, the MKT linked to the type of algorithm presented and the consequence resulting from the determination of the S_0 didactical situation. For each aspect of the teaching and learning of the algorithm, this will allow us to better understand Dominique's choices about the didactical situation. It will also reveal some possible causes of these choices: in his MKT in a general level (+3), in his MKT about multiplication and algorithm (+2), in his MKT about the moment of explaining the multidigit algorithm (+1) and in his MKT about observing his students (−1).

I will first analyse aspects linked to the choice of the algorithms. On the ideological level (+3), Dominique thinks that pupils should understand what they do in mathematics. He also feels this way about the algorithms, but he views algorithms as tools for problem solving (+3). So, for the series of lessons about multidigit multiplication, his main goal is for pupils to perform the algorithm and use it efficiently (+2). The type of algorithm is not important if it is efficient for the pupils (+2). Dominique knows that there are several kinds of algorithms for multidigit multiplication, and he plans to demonstrate two of them: the "table algorithm" (Fig. 5, left) and the algorithm *en colonnes*[7] (EC) (Fig. 5, right). This way of demonstrating more than one algorithm is consistent with the official regional instructions (DFJ, 2006) and the teacher's guide for the textbooks (+3) (Danalet et al., 1999). At the end of the chapter of the textbook on this topic, Dominique will ask students to only retain and use the EC algorithm, with the justification that this is the algorithm everybody learns at school, and it is more efficient than the table algorithm (+2).

Dominique plans to demonstrate first the table algorithm on a two-digit by two-digit multiplication and then to show the EC algorithm on the same example. He knows that the two algorithms give the same results and that the partial results can be compared line by line (Fig. 5, SCK, +2), and he plans to show that (+1). However, he does not mention any other link between the table algorithm and EC—in general (+2), when planning the lesson (+1) or when envisioning teaching the lesson (+1). In addition, he does not observe the students using the two algorithms on the same multiplication (−1). To make the line-by-line comparison possible, he plans on asking the students to "put the tens below": "It's not very logical, but it allows the student to have the two (lines) in front of each other".[8] He never mentions any other reason or justification for this step and never mentions the possibility (and the effects) of the inversion of the two factors (SCK, +2).

Considering pupils' possible difficulties, Dominique thinks that the only problems pupils will face in the EC algorithm are multiplication facts and the correct placing of zero in the second line (Knowledge of Content and Students, KCS, +2). He foresees that he will observe many errors about this zero (KCS, −1). So he plans to repeat the *zero rule*: "when one works with tens, a zero must be added"[9] (Common Content Knowledge, CCK, +1).

For Dominique, multiplication is a shortcut for addition (SCK, +2). For example, for Dominique, the definition of 3×4 is $4 + 4 + 4$. He never plans to mention any link with the measure of area (area of a 3 by 4 rectangle for 3×4 in our example) when explaining the table or EC algorithm (+1), even if he poses one area problem to introduce the necessity for building an algorithm (+2) and even if the table algorithm represents the area model. Challenged about his representation of multiplication during the post-lesson interview, he never gives any other

[7] Literally "in columns" for what is known in English as "long multiplication".
[8] « Mettre les dizaines en dessous. C'est. pas très logique, mais ça permet d'avoir les deux en face. »
[9] « Quand on travaille avec les dizaines, on ajoute un zéro. »

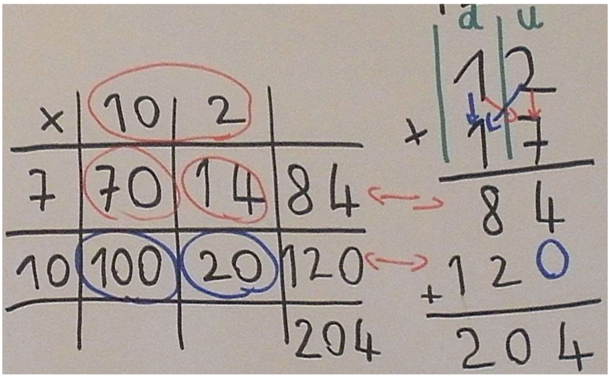


Fig. 5 Two algorithms for the multiplication 12 × 17, written by Dominique on a poster

representation (SCK, +2), and when asked about the link between multiplication and measure of the area, he answers that the area has to be computed with multiplication (CCK, +2).

These elements contribute to the determination of the didactical situation S_0 from the teacher's perspective. It can be summarised in four points:

1 Show the table algorithm for the example 12 × 17, requiring writing the units first for 17.
2 Show the EC algorithm for the example 12 × 17. Write the EC algorithm next to the table. After the first line, highlight the fact that the results of both algorithms' first lines are the same.
3 Write the zero at the right place in the second line, because "when one works with tens, a zero must be added". Carry out the second line and highlight the fact that the results of both algorithms' second lines are the same.
4 Finish by adding the two lines.

This way of presenting the algorithms has some common points with Van De Walle et al. (2014, p. 259), as in Fig. 4. Nevertheless, the area model is replaced by the table algorithm and the links between the algorithms are not explicit.

3.3 A priori ascending analysis: student's point of view

The ascending a priori analysis below is conducted from the student's point of view. The goal is to determine the didactical classroom situation S_0 from the student's point of view. It starts from level −3 (*material milieu* M_{-3}) in order to determine a priori all the ways the student could deal with the objective situation S_{-3} the teacher has put in place in the actual lesson and thus later to analyse a posteriori what happened in the actual lesson. In this case, the ascending a priori analysis leads to six didactic situations of level 0.

3.3.1 Level −3: objective situation

According to Margolinas (1995, p. 99), the *material milieu* M_{-3} for E_{-3} includes all necessary components to enter the question. The situation is not finalised (Comiti et al., 1995, p. 106).[10]

[10] There is no action and no learning intention at this level.

Therefore, M_{-3} consists of the raw description of the algorithms given by the teacher for an example (see above) and a generic student's prior and usable knowledge.

The generic student E_{-3} is able to compute a column addition and a long multiplication by a one-digit number. He knows that "to multiply by ten, a zero must be added". This *zero rule* has been recalled many times and inductively "proven" on several familiar products. It is completely automated for the students. E_{-3} knows that numbers (implicitly smaller than 100) can be decomposed into tens and units. Particularly, for a number between 11 and 19, "the one, it's not one, it's ten."

The material milieu also includes the description of the two algorithms (table and EC) given by the teacher for 12 × 17 (Fig. 5) and the description of the links between these two algorithms for that example. The first line of the table (84) provides the same result as the first line of EC algorithm, and the same observation can be made for the second line. The final sum is identical (204).

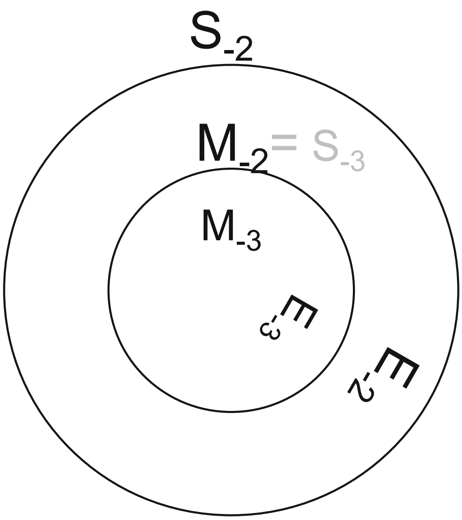
3.3.2 Level −2: reference situation

The generic student in position −2, E_{-2}, is an active student (Margolinas, 1995, p. 99), a student who will act on the milieu M_{-2} constituted by the situation S_{-3} (Fig. 6). His action is to compute the multiplication following the teacher who is interrogating the students at each step.

The M_{-2} milieu is composed of the S_{-3} situation which contains: the explanations and knowledge it comprises, the justifications given by the teacher, the student's knowledge of his "student job" and his habit in interacting with this teacher and to meet the ostensive expectations of this teaching. The possible answers can be determined a priori due to the elements of the milieu.

The table algorithm for 12 × 17 is presented as an extension of the table algorithm for one-digit multiplication, which was studied in the previous school year. Based on what the teacher said (in class as well as during the interview), it is purely a calculation layout. The link with the Cartesian product or to the area of the rectangle is never made. Each product is computed for itself. For E_{-2}, the *zero rule* is mechanically applied, and summing each line of the table, as was discussed during the previous lesson, seems obvious. E_{-2} applies this table multiplication algorithm automatically to one line. The decomposition of the second factor 17 into 10 and 7 and thus the creation of a

Fig. 6 Structure of the milieu, level −3 to −2



second line should not be questioned, since the teacher decided to put the units before the tens, justifying that by a future usefulness for the EC algorithm. After summing the two lines (as it was done in the one-digit table algorithm), the student is in an additive context and should add the two sums. However, it is possible that a prompt by the teacher would be necessary to make him compute 84 plus 120. The use of multiplication seems unlikely, since the two numbers are more difficult to multiply than the first numbers.

The EC multiplication 12 × 17 is presented directly after the 12 × 17 table algorithm and positioned next to the latter. Nevertheless, neither the drawing of the columns for the tens and ones nor the fact of writing two lines by separating the two digits of 17 and the way of computing these two lines is compared to the table algorithm. The two lines and the separation of 17 into 1 and 7 are simply stated by the teacher and the way of computing each line is brought back to a one-digit EC multiplication. The parallel structure of the two algorithms is exclusively noted by comparing the result of each line (84, 120) to the sums in the table.

Faced with this presentation, active student E_{-2} can choose between four potential attitudes (columns of Table 3).

- He could use the same rules as the ones the teacher gave: correspondence between each line.
- He could use the result of each cell (14 and 70, then 20 and 100) of the table to establish the corresponding line (84, then 120): correspondence between each product.
- He could directly write the result of each line (84, 120) or even the final result (204) and not carry on the EC multiplication: direct use of the table.
- He could also make no correspondence at all and carry on the two algorithms independently.

Regarding the zero in the second line, according to the explanations of the teacher, we have to "do 12 times 1, but, since it's not 1, but 10, we have to add a zero". For that part, three S_{-2} attitudes might be applicable: (lines of Table 3).

- Apply mechanically the "recipe", write the zero and carry on the rest of the multiplication: application of the "recipe".
- Relate this zero to the zeros that are the last digits in every product of the second line in the table: correspondence of zeros.
- Take into account the explanation of the teacher and therefore add a zero in the EC multiplication each time a digit is a tens digit: add a zero when working with tens.

By crossing the possible reactions about the link between the two algorithms and about the zeros, it is therefore possible to imagine the 12 situations shown in Table 3.

In Table 3, every situation matches one of the possible attitudes about the link between the algorithms (coded as superscript) and one of the possible attitudes about the zero (coded as subscript). Some of these situations are described below.

Situation $(\mathrm{S}_{-2})_r^L$ represents the situation of an E_{-2} student who notices a correspondence between each line of the EC multiplication and each line of the table. When beginning the second line, this E_{-2} student writes a zero without questioning the signification of this zero.

In the $(\mathrm{S}_{-2})_z^P$ situation, the student links every cell of the table to a partial product in the EC multiplication. He notices then that every line of the EC algorithm is the result of an addition. He also sees that every product of the second line in the table ends with a zero. He might even

Table 3 Possible S-2 situations

			Attitudes about the link between the two algorithms			
			corres-pondence between each line L	corres-pondence between each product P	direct use of the table. D	indepen-dent algorithms I
Attitude about the second line zero	application of the recipe.	r	$(S_{-2})^L_r$	$(S_{-2})^P_r$	$(S_{-2})^D_r$	$(S_{-2})^I_r$
	correspondence of zeros	z	$(S_{-2})^L_z$	$(S_{-2})^P_z$	$(S_{-2})^D_z$	$(S_{-2})^I_z$
	add a zero when working with tens	a	$(S_{-2})^L_a$	$(S_{-2})^P_a$	$(S_{-2})^D_a$	$(S_{-2})^I_a$

see that this zero is "added" because it comes from a multiplication by a number ending with zero, and therefore this zero is "added" the same way in the second line of the EC multiplication.

The $(S_{-2})^L_a$ situation is paradoxical. On the one hand, it respects the purpose of the teacher, which was determined by the descending analysis, since it presupposes a student making a line by line correspondence and applying the *zero rule*. On the other hand, it could lead this student to "add a zero" each time he is working with tens, that is for each partial product of the second line, as well as for the second product in line one. This situation is thus mathematically incoherent, but it is the one that best corresponds to the didactical contract.[11]

Other situations have an internal incoherence. For example, for $(S_{-2})^P_a$, if the student draws a correspondence between each cell of the table algorithm and the corresponding partial product of the EC algorithm, this correspondence will prevent him from applying the *zero rule* directly and without asking questions. Since these situations are incoherent and do not match the intention of the teacher, they are unlikely or even impossible. These cells are coloured grey in Table 3.

However, one situation does not appear in the combinations of Table 3. E_{-2} can copy the result of the lines of the table algorithm without worrying about the zero. This situation is labelled $(S_{-2})^D_*$. Hence, we can finally imagine six possible S_{-2} situations.

It is at level −2 that several situations appear in this analysis. This shows how, when acting on the same objective milieu M_{-2}, the generic active student E_{-2} has to choose between several potential actions. These choices of action will influence the student in his position as a learner (−1) and as an actual student in the didactical situation (level 0).

[11] "The didactical contract is the rule of the game and the strategy of the didactical situation. It is the justification that the teacher has for presenting the situation, [...] a relationship [...] which determines—explicitly to some extent, but mainly implicitly—what each partner, the teacher and the student, will [...] be responsible to the other person for. This system of reciprocal obligation [and expectation, we argue] resembles a contract." (Brousseau, 1997, p. 31)

3.3.3 Level −1, learning situation and level 0, didactical situation

Pursuing the ascending analysis enables the determination of the didactical situation from the student's point of view and therefore reveals six possible situations. One of these $(S_0)_a^L$ seems to match the S_0 situation. This match is however not complete since "adding a zero when working with tens" is only used by the teacher for the first partial product of the second line of the EC multiplication.

In this case, ascending analysis from the student's point of view leads to determining several didactical situations. Margolinas (2004b) speaks of *didactical bifurcations*. In this episode, the material situation S_{-3} set by the teacher cannot lead to the intended didactical situation $S_{0.}$. In this case, Margolinas speaks of *marginal branches*. In other words, none of the didactical situations determined by the ascending analysis from the student's point of view matches the situation determined by the descending analysis from the teacher's point of view.

3.4 A posteriori analysis

The a posteriori analysis confirms the tensions that these bifurcations may generate. In our observation, most the students are in a $(S_0)_r^L$ situation, since they do a line to line correspondence between the two algorithms and since they do not question the signification of the second line zero. Only one student, Armand, is in $(S_0)_a^L$ situation. He wants to add a zero each time he computes a multiplication by a tens digit.

This happens particularly during the first explanation of the EC algorithm. After having carried out a 12 × 17 table algorithm, Dominique writes multiplication and draws the columns labelled "u" and "d[12]" (Fig. 5). He announces that he will first work with units because this is what students already know. So he hides the 1 in 17 and carries out the first multiplication while drawing descending arrows. He emphasises that the result he obtains is the same as the one in the table (84) and indicates this using a horizontal arrow. He says: "we still have to do 12 times…". The students say "1", but Dominique makes them say "10". He reminds them that "when one works with tens, a zero must be added". He writes the zero in a different colour while saying that it was times 10 and not times 1, and adds: "2 times 1, 2; 1 times 1, 1". He again draws an arrow between both 120 s (Fig. 5). He carries out the addition. The students notice that it is the same result.

At this moment of the explanation, the S_0 situation considered by the teacher seems identical to that of the students. This is when Armand steps in.

1 A[13]: But why are we doing here… euh… there [Armand points at the 2 in 12 and the 1 in 17] … we did 2 times 10
2 D: Mm Mmm…
3 A: But why, there [he points at the 1 in 12 and the 1 in 17] we do 1 times 1? And not, for example, times 10?
4 E: Because, then, you just have to add a zero.
5 A: Oh, because we have…

[12] For "unités" and "dizaines" in French, "units" and "tens" in English.
[13] A stands for Armand, D for Dominique, the teacher, and E for another student in the class.

6 D (interrupting): But there, if I put it this way. I'll try to use colours, so you can see what the result is. We do 2 times 7.

7 A: Mm Mm…

8 D: OK? It's this answer, over there. (Dominique circles the 14 in the table, see Fig. 5)

9 A: Mm Mm…

10 D: This way (Dominique draws the descending arrow, from 2 to 7). OK?

11 A: Mm Mm…

12 D: Then, we did 1 times 7, but it's not 1 times 7, it's… This one, how many is that, this one (Dominique indicates the 1 in 12)?

13 E: 10 times 7.

14 A: That's 10.

15 D: That's 10! So, 10 times 7, what's the result over there (Dominique shows the table)?

16 A: 70.

17 D: So, which one is that?

18 A: 70, there (Armand shows the 70 in the table).

19 D: OK (Dominique circles the 70 in red colour). […] Then, we worked with the ten, so we did two times (Dominique draws again the descending arrow from the 2 to the 1 in 17)…

20 E: 10.

21 D: Which is?

22 A: 20.

23 D: So, where is it?

24 E: Well… there (student points at the 20 in the table and Dominique circles it).

25 D: This one. And then, to finish, we had to do… (Dominique draws again the descending arrow from 1 to 1)

26 E: 10 times 10.

27 D: And the result, it's this one (the student points at the 100 in the table and Dominique circles it). So when we multiply (Dominique points at the EC multiplication) how many calculations is that?

28 E: Four.

29 D: Four, plus the addition that we have at the end (Dominique shows the addition). Are we good?

The students nod, but Armand seems to sulk.

The intervention by Armand (1, 2) suddenly reveals that at least one student cannot follow the approach suggested by the teacher. More precisely, Armand wants to strictly apply the rule "add a zero when dealing with tens", which was suggested by the teacher. He is in a $(\mathrm{S}_0)_a^L$ situation. This situation is not the S_0 situation, since Armand wants to apply the rule "add a zero when dealing with tens" to every partial product where tens are used. Since Dominique notices that Armand is decomposing every product in the EC multiplication, he tries to use the table algorithm as his basis/reference. He can now provide a correspondence between each product in the table algorithm and each product in the EC multiplication and also indicate the correspondence between the zeros, that is to place himself in a $(\mathrm{S}_0)_z^P$ situation. But he chooses to simply indicate the correspondence between the products (interventions 8, 15–17, 21–24 and 27) and to notice that they are for "calculations" (27–29). He is in the $(\mathrm{S}_0)_a^P$ situation, which the a priori analysis determined as incoherent.

This situation cannot be maintained. It needs to evolve into either a $(S_0)_r^L$, by giving up the justification of "adding a zero when working with tens", or a $(S_0)_z^P$ situation, by indicating the correspondence between the partial products and the zeros to understand this justification. The first option is obstinately rejected by Armand who seems to refuse to apply a recipe without understanding it. So he tries again four times. The second option is systematically avoided by the teacher who tries to bring the student back to the S_0 situation. But this situation is incompatible with the material milieu he has set.

3.4.1 MKT and pertinence

The proliferation of didactical bifurcations and the inability of the teacher to notice that the Armand's S_0 didactical situation radically differed from his S_0 situation have their origin in the teacher's choices made at the +3 to 0 levels. Additionally, these choices may be understood as a consequence of the teacher's MKT, as described in this section.

The first choice was to use the table algorithm and particularly the correspondence between the lines' sum in the table and in the EC's lines but with no explicit correspondence between each partial product. As revealed in the interview, Dominique's MKT about the table was accurate (he knew the correspondence with EC, the role of the distributive property), but these aspects of his MKT were not pertinent, since they did not allow him to move to the $(S_0)_z^P$ situation and to interact with the students on the mathematical parts of the situation (the first criterion for pertinence, according to Bloch, 2009). The reason for this discrepancy between knowledge and pertinence was the knowledge of multiplication itself. For Dominique, multiplication was only a shortcut for repeated addition. He never considered it as a Cartesian product or as the area of a rectangle. Therefore, to Dominique, EC and table algorithm were two ways to perform multiplication; they were not linked to multiplication itself and they were only linked to each other because they gave the same result. As many teachers I have met in my research, and as the teachers individually met by Davis and Simmt (2006), Dominique's response to "what is multiplication" is "repeated addition" and nothing else. Contrary to observations by Leikin and Levav-Waynberg (2007), Dominique does not seem to learn mathematics from his students' responses.

The second choice was the "recipe" for *zero rule*. This rule is problematic in many ways: use of additive words (add a zero), lack of a link with place value and, above all, fallacy if literally applied. Regarding these two choices, Dominique had a working *Common Content Knowledge*, but he could not unpack it and could not use the corresponding *Specialised Content Knowledge* to explain why a zero appears when one multiply by tens.

Other choices are made by the teacher and had an influence on the didactical situation. Among the elements that influence the material milieu M_{-3} and thereafter the successive situations S_{-2} to S_0, I can mention the choices of the numbers for the first multiplications. They are all between 12 and 19. Dominique gave no reason for this choice, but it could be interpreted as his intention to begin with easy multiplications. This choice implies that the second line of the EC multiplication is trivial. The teacher does not mention this in class or during the interviews. This choice also implies that all the multiplications seems to be the same and it leads Armand to ask several times the same question about different multiplications: "Is it 1×1 or 10×10?[14]". This choices of tens digit is an example of "obscuring the role of the

[14] « c'est 1×1 ou 10×10 ? »

variable" (Rowland, 2008, p. 150) and a case of relation between teacher's knowledge and use of examples for teaching mathematics as studied by Zodik and Zaslavsky (2008). This knowledge, which allows teachers to see the mathematical implication of choosing a particular example, is an aspect of KCS in Ball's classification.

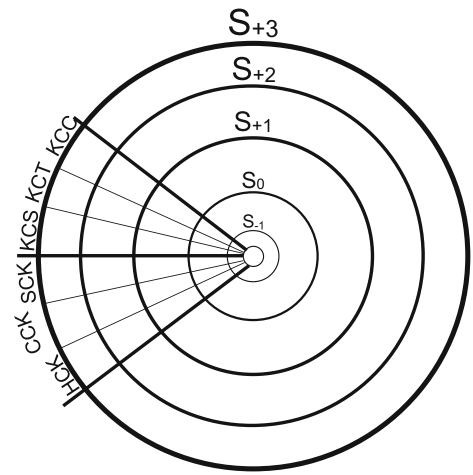
These choices (table instead of area, no correspondence between the two algorithms, zero rule and choices of example) all differ from Fig. 4 and the recommendations of Van De Walle et al. (2014, p. 259). All of them have their origins in Dominique's MKT and the teacher could find no help in any mathematical content book or method book or textbook explicitly linking student activity and unpacked mathematical knowledge. This lack of resources for teachers in Lausanne's region is certainly an aspect that needs serious consideration and should be perceived as a gap that needs to be filled in the near future.

4 Conclusion

This analysis of an episode used the structure of the milieu to highlight and to analyse links between MKT, pertinence and teaching choices of the teacher. The interactions between the different situations in the structuring of the milieu allowed a better understanding of what happened during this episode where the teacher and the students were inhabiting various situations. It also showed the origin of this didactical bifurcation in the divergent interpretation of the objective situation S_{-3}. Moreover, it showed the origin of the choices of the teacher about the situation S_{-3} in his MKT. More precisely, without the positioning of these aspects of MKT in situations S_{+3} to S_{-1} and the focus on their interaction across the levels, the didactical bifurcation could not have been explained. The analysis showed that more than Common Content Knowledge is necessary to apply pedagogical MKT but also "that each of these common tasks of teaching involves *mathematical* reasoning as much as it does pedagogical thinking" (Ball et al., 2005, p. 21). It is one illustration of the way one US mathematics education framework and elements of the *Theory of didactical situations* (Brousseau, 1997) can interact to analyse an issue in the teaching of mathematics.

This interaction was vital to answer the research questions and neither of the two frameworks was sufficient as a means to analyse the teaching situation. The original question was about the mathematical knowledge of the teacher, to show how the subtle analysis of the

Fig. 7 Descending analysis in accordance with MKT categories



structuration of the milieu was necessary to reveal the multiplicity of the various didactical situations and to demonstrate "how different categories of knowledge come into play in the course of teaching" (Ball et al., 2008, p. 403). On the other hand, the distinction of Specialised Content Knowledge within MKT was needed to analyse the causes of the phenomena, while the structuration of the milieu and the pertinence of the teacher were crucial to capture the dynamic movement of didactical situations beyond the static character of Ball's categories. The analysis has therefore illuminated not only mathematical knowledge for teaching but mathematical knowledge *in* teaching (Rowland & Ruthven, 2011; Watson, 2008).

The combination, in the sense of Bikner-Ahasbahs and Prediger (Bikner-Ahsbahs & Prediger, 2010; Prediger, Bikner-Ahsbahs, & Arzarello, 2008), of the frameworks from two cultural backgrounds were necessary for me to "get a multi-faceted insight into the empirical phenomenon in view" (Bikner-Ahsbahs & Prediger, 2010, p. 496). Unlike Huillet (2009), who uses Chevallard (1999) anthropological theory of didactics to formulate other categories of MKT, my two categorisations divide the analysis of the teacher point of view in two different directions as pictured in Fig. 7. These two directions were complementary as shown in the above analysis.

One key assumption in Shulman's and Ball's theories is that the teacher's knowledge makes a difference to students' learning. Along the same lines, Davis and Renert (2013) consider that the *Theory of didactical situations* includes "the body of mathematical knowledge within the space of teachers' transformative influence" (Davis & Renert, 2013, p. 254). This point is an argument for some degree of coherence between the two frameworks used in my analysis. Nevertheless, going further in networking these frameworks by *coordinating* them or locally *integrating* then would require a much more detailed study of the cores of both theories and "a careful analysis of the mutual relationship between the different elements" (Bikner-Ahsbahs & Prediger, 2010, p. 496).

The models of levels of a teacher's activity are a quite typical feature in French *didactique des mathématiques*, for example, invoking structuration of the milieu or the levels of didactic codetermination[15] (Artigue & Winsløw, 2010; Chevallard, 1999). Whereas categorisation of mathematical knowledge of the teacher is widely studied in the English-speaking mathematics education community, it is far less discussed[16] in the French-speaking *didactique des mathématiques*. In this sense, this combination can be seen not just as a combination of frameworks but also as a combination of two research cultures in the field of mathematics education.

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[15] Niveaux de codétermination didactique

[16] With the notable exception of Quebec and the presence of a Working Group on the topic in EMF congress (Clivaz, Proulx, Sangaré, & Kuzniak, 2012).

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