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A conceptual model of mathematical reasoning for school mathematics

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Abstract The development of students' mathematical reasoning (MR) is a goal of several curricula and an essential element of the culture of the mathematics education research community. But what mathematical reasoning consists of is not always clear; it is generally assumed that everyone has a sense of what it is. Wanting to clarify the elements of MR, this research project aimed to qualify it from a theoretical perspective, with an elaboration that would not only indicate its ways of being thought about and espoused but also serve as a tool for reflection and thereby contribute to the further evolution of the cultures of the teaching and research communities in mathematics education. To achieve such an elaboration, a literature search based on *anasynthesis* (Legendre, 2005) was undertaken. From the analysis of the mathematics education research literature on MR and taking a commognitive perspective (Sfard, 2008), the synthesis that was carried out led to conceptualizing a model of mathematical reasoning. This model, which is herein described, is constituted of two main aspects: a structural aspect and a process aspect, both of which are needed to capture the central characteristics of MR.

Keywords Mathematical reasoning · Theoretical model · Commognition · School mathematics · Anasynthesis · Structural aspect of mathematical reasoning · Process aspect of mathematical reasoning

1 Introduction: the conceptual blur regarding mathematical reasoning

What is mathematical reasoning? How might we characterize it? These questions underpinned the development of the model of mathematical reasoning (MR) (Jeannotte, 2015) that is the focus of this article. While curricular documents around the world emphasize the fostering of students' MR as an important goal (e.g., NCTM, 2000; OCDE, 2006), the way in which MR is

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described within these documents tends to be vague, unsystematic, and even contradictory from one document to the other. As well, within the mathematics education research community, the discourse on MR is not monolithic; it does not consist of a single voice. Various visions of mathematics, and of teaching and learning, confront each other. An additional factor adds to the confusion, as pointed out by Yackel and Hanna (2003): "Writing about reasoning in mathematics is complicated by the fact that the term *reasoning*, like understanding, is widely used with the implicit assumption that there is universal agreement on its meaning" (p. 228). But, as an in-depth search of the research literature makes amply clear, this assumption does not hold.

At the extreme end of the spectrum is that "most mathematicians and mathematics educators use this term [mathematical reasoning] without any clarification or elaboration" (Yackel & Hanna, 2003, p. 228). Nevertheless, among those who do define MR, various aspects are stressed by various authors. Arsac (1996) and Cabassut (2005) emphasize its double nature (product vs. process); Arsac (1996) and Lithner (2008), its function of producing new knowledge; and Duval (1995), its function of changing the epistemic[1] value of a certain proposition. Definitions of the classic forms of MR, such as deductive, inductive, and abductive, are all to be found but with various emphases. Duval (1995) insists that only deductive reasoning can be considered mathematical, while Reid (2003), Rivera (2008), and Meyer (2010) point to the importance of abductive reasoning in mathematical discovery. In contrast to the structural aspect related to the form of MR, the literature also discloses characterizations of MR that stress its underlying processes, with inferencing being at the heart of these processes: conjecturing (Mason, 1982; Stylianides, 2008), generalizing (Stylianides, 2008), exemplifying (Mason, 1982), proving (Duval, 1995; Stylianides, 2008), arguing (Pedemonte, 2002), and convincing (Cabassut, 2005). While the process aspect is well represented in the literature, it remains relatively unexplored from an epistemological perspective.

These few highlights from the research literature on MR suggest that this area is one that could benefit greatly from an attempt at coherent conceptualization. What Steen said in 1999 is still true today:

> [MR sometimes] denotes the distinctively mathematical methodology of axiomatic reasoning, logical deduction, and formal inference. Other times it signals a much broader quantitative and geometric craft that blends analysis and intuition with reasoning and inference, both rigorous and suggestive. This ambiguity confounds any analysis and leaves room for many questions. (Steen, 1999, p. 270)

The current state of the field renders difficult any comparison of not only the various approaches to, and characterizations of, MR but also the results of related studies. The importance of developing MR in the teaching and learning of mathematics at the different levels of schooling, as well as the need for substantive conceptual resources on MR in the training and professional development of teachers, also serve to motivate a deeper and more theoretically-sound study of MR.

Thus, the objective of this research is to elaborate a conceptual model of mathematical reasoning for the teaching and learning of primary and secondary levels of school mathematics (see also Jeannotte, 2015). The article "a" is here very important since it implies that it is A model among other possibilities, in line with Sfard's (2012) principle of multivocality. As will

[1] The epistemic value of a proposition refers to the notion that an utterance can be true, probable, likely, or false.

be stated in the next section, our epistemological stance, which is central to the whole process, goes hand-in-hand with the adopted theoretical frame, which in turn plays the role of an interpretative lens for drawing out key features of MR from the research literature. Following the presentation of the theoretical framework, we describe the *anasynthesis* methodology (Legendre, 2005), which provided a rigorous tool for searching the mathematics education research literature. Then we present an overview of some of these key features of MR that were extracted from the literature, features that co-constituted the emerging conceptualization of a model of MR for school mathematics. It must be emphasized, however, that the result is not a unique model that unifies all of the divergences found in the literature. It is rather a model that systematizes the various converging features of MR within a theoretically coherent frame. As argued by Balacheff (2008, p. 501):

> The scientific challenge of research in mathematics education is […] to shape a body of knowledge which should be robust (which means theoretically valid) and relevant (which means instrumental for practitioners and other stake holders). Convergence should be the rule.

2 Theoretical considerations

2.1 Sociocultural assumptions of this research

The sociocultural turn in mathematics education research (e.g., Cobb, 2007) has sensitized the community not only to the phenomenon of the co-constructive emergence of the classroom mathematical culture by its participants – both teacher and learners – but also to the role played by the ways of reasoning and communicating that draw upon historical practices of the discipline. These ways of reasoning mathematically are correspondingly reflected in the didactic literature on MR in schools.

Student discursive activity consisting of what students say, the way in which they say it, what they do, the representations and drawings they make, the ways in which they use these representations, and their intonations and gestures, all these are key levers in the activation of a teacher's input with respect to enculturating students to the ways of participating and the ways of reasoning that are expected of them. To provide this input, teachers must not only be aware of the nature of the forms and processes of mathematical reasoning that they wish students to learn to participate in but also recognize when students are engaging in the desired aspects of reasoning. This requires a well-elaborated vision of MR where discourse is fundamental and that not only reflects the didactical discourse of the discipline but also serves as a conceptual tool for teachers (and researchers) to analyze students' discursive activity.

2.2 The commognitive frame

From this sociocultural view, MR can be seen as discursive activity. To conceptualize MR as discursive activity, we turn to the commognitive framework developed by Sfard (2008, 2012). Sfard (2008) defines *commognition* as "the term that encompasses thinking (individual cognition) and (interpersonal) communicating. As a combination of the words communication and cognition, it stresses the fact that these two processes are different (intrapersonal and interpersonal) manifestations of the same phenomenon" (p. 296). Within the commognitive framework, discourse is central. According to Sfard (2008), discourse is a "special type of communication made distinct by its repertoire of admissible actions […] and discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives" (p. 297).

For a commognitive researcher, mathematics is a discourse, that is, a particular type of communication (Sfard, 2008, 2012). Development of mathematical discourse (i.e., development of mathematics) involves a change in discourse, which occurs within the mathematical community in which we evolve. In a movement of individualization and (re)communication, changes in discourse are proposed, rejected, and negotiated. For example, Sfard (2008) notes that algebra is a meta-discourse that subsumes arithmetical discourse and becomes a discourse in itself. From an epistemological vantage point, this position has implications for the way in which MR will be conceptualized: It is the given mathematics community in which we evolve that fixes the rules, the acceptable visual mediators, and word use.

Another point is that within the commognitive approach, there are two main kinds of discursive development: object-level and meta-level. Object-level discourse development refers to an extension of an existing discourse on already constructed mathematical objects. On the other hand, meta-level discourse development refers to the construction of a new discourse, changing the rules of the game, which goes hand in hand with the building of new mathematical objects. As will be seen later, the distinction between the type of discourse and the type of discourse development will prove crucial in discriminating those mathematical thinking processes that will be considered processes of MR.

The choice of Sfard's (2008) commognitive frame as the main underpinning of this research project on the elaboration of a model of MR for school mathematics has certain implications and limits. The narrative form of the model is open to different interpretations, especially if approached from an incommensurable discourse. Furthermore, the aim here is not to construct a model that provides specific practical advice with respect to classroom task sequences that are designed to encourage the development of MR. Nor is the aim to construct a model that illustrates the research-based ways in which students have been found to develop and communicate certain modes of MR. Rather the aim is to construct, in harmony with the commognitive frame, a coherent theoretical model that synthesizes and builds upon the convergences to be found in the main types and characteristics of MR described in the mathematics education research literature and that can thereby serve as a conceptual tool for both teachers and researchers, that is, a narrative model that connects semantically different concepts in the same coherent network – thus, a means to improve communication with the help of a shared vocabulary (Lee, 1997).

The Oxford Living Dictionary (https://en.oxforddictionaries.com/definition/narrative) defines *narrative* both as a spoken or written account of connected events and as a representation of a particular situation or process in such a way as to reflect or conform to an overarching set of aims or values. The connected events that are represented by the model – connected and interpreted by means of the discursive framework that underpins the model – are those aspects of mathematical reasoning that have become a part of the current narratives in the mathematics education community. And because narratives invite interpretation, the model, which is itself a narrative, will evolve as others from other vantage points read it, and take it on and adapt it to their own practice.

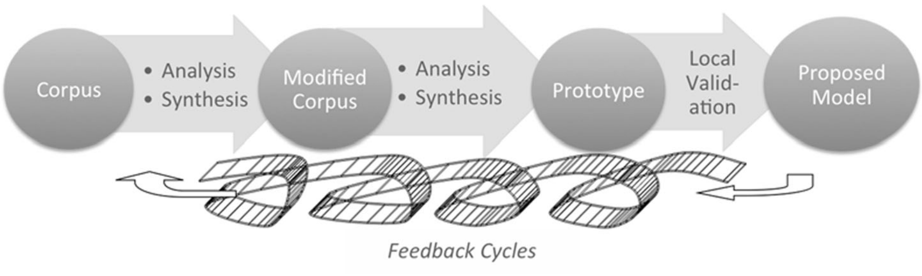

Fig. 1 The methodological process of anasynthesis

3 Methodology

The commognitive stand taken here creates methodological implications. For a commognitive researcher, research development is equivalent to development of research discourse. A researcher has to build on other researchers' works and attempt to develop a common discourse. This supports the justification that the data for this study are the mathematics education literature resources dealing with MR. Those data allow for constructing with and grounding upon the discourse already built by members of the mathematics education community.

The methodological process that framed this research is *anasynthesis* (Legendre, 2005) – *anasynthesis* being a neologism coined from the words analysis and synthesis. We present this process linearly but readers should consider it a cyclical process (see Fig. 1).

First, we created a corpus from a review of databases and selected texts that had MR as a keyword or associated keywords such as mathematical thinking, deductive reasoning, inductive reasoning, and so on. Additional cycles provided new keywords that helped to elaborate the corpus. Four criteria were used to select the corpus: access, completeness, recency, and authenticity (Van der Maren, 1996). We also added famous texts and those cited by authors who have studied MR but were not referenced in the database. To assure the quality of the sources, the Toerner and Arzarello (2012) classification was used to refine the selection of scientific journals. By the end of the process, 145 English and French texts[2] (books, chapters, articles, and research reports in proceedings) constituted the corpus.

Secondly, at every cycle, we analyzed the resulting corpus for relevant information related to formal, axiological, and praxis characteristics of MR (Jeannotte, 2015). The formal characteristic refers to accuracy, description, expression, and definition of concepts, terms, notions; the axiological characteristic refers to aims, goals, principles; and the praxis characteristic refers to norms, prescriptions, theoretical or experimental practices, habits, and custom. Content and conceptual analyses supported the model building process. Successive and repeated readings allowed for locating the MR units that were to be categorized. Every unit received three codes: i) descriptors linked to the nature of the information (formal, axiological, or praxis); ii) descriptors linked to its content (e.g., deductive reasoning, abductive reasoning, inference, conjecturing, proving, …) – descriptors that helped to constitute the corpus; and iii) descriptors linked to the emergent characteristics of MR (e.g., structure, process) – this third level of codification going beyond the content and helping in the building of the model. For example, the unit "To reason is to infer a proposition, called conclusion, from certain

[2] In French, the word *raisonnement* is usually translated as *reasoning* (see, e.g., Duval, 1991).

premises" (Cabassut, 2005, p. 24, our translation) provided not only *formal* information, but was also linked to the keyword *inference*, and highlighted both the structural aspect (i.e., premises, proposition, and conclusion) and the process aspect (in defining reasoning by an action verb, i.e., inferring). The analysis was guided by commognitive principles (Sfard, 2012) that led to searching for discursive elements within the texts of the corpus, but that also took into account the context and epistemological positions behind the text.

Thirdly, this information was then synthesized so as to highlight convergences, divergences, and to point out areas where there were theoretical gaps that would need to be filled in by any model proposing to represent the central aspects of MR for school mathematics. Via the commognitive framework that underpinned this research, a theoretical prototype was then developed, which became a self-standing, theoretically-coherent model of MR after multiple anasynthetic cycles (see Fig. 1). As the goal was to provide a portrait of the concept of MR, we stopped the process when no new information emerged and when the model respected internal coherence, relevance to the issue raised, and heuristic value (local validation). We emphasize that the coherence of the model is constituted by its commognitive, discursive framing that, in conjunction with the methodological procedures of anasynthesis, provided the theoretical tools to discern both the gaps and overlaps within the mathematics education research literature related to MR and thereupon to move forward in the creative act of conceptualizing the model.

Each element that emerged was either explicitly or implicitly linked to MR by multiple texts. With the help of the commognitive frame, we reformulated them with a sound vocabulary reflecting a discursive approach to MR. From this perspective, the resulting theoretical model of MR is designed to enable a better and fuller understanding of MR in the context of school mathematics, as well as a tool for improved communication by providing a shared vocabulary. In addition, the model aims to nourish reflection on MR among researchers and teachers – those who can, by their action, influence directly or indirectly student learning. Discourse is what teachers rely on to judge whether MR is occurring in the class. Furthermore, as Lee (1997) has pointed out, "without a theoretical model, research remains piecemeal or eclectic and without empirical research the diverse models or theories stagnate" (p. 42). With this model, we respond to Reid's (2002) appeal to the mathematics education research community: "The aim of developing mathematical reasoning in classrooms calls on the research community to clarify what is mathematical reasoning and what it looks like in school contexts" (p. 7).

4 Mathematical reasoning: what are its central aspects[3]

Four major elements emerged from the analysis of the mathematics education literature, elements that helped to clarify the conceptual blur found therein: the activity/product dichotomy, the inferential nature of MR, the goal and functions of MR, and what we came to refer to as the structural and process aspects.

The activity/product dichotomy relates to reasoning activity that is considered inaccessible and for which the product is but an imperfect hint (e.g., Balacheff, 1988). The inferential nature of MR is emphasized by many authors who point to the novel ideas that result from such inferencing; however, the precise nature of this novelty has yet to be clarified. The

[3] Please note that, in the interests of space, only a small number of the references that were actually consulted and that served to help in the development of the model are included herein.

element of the goal and functions of MR also leads to questions, such as whether the goal of MR is restricted to proving (e.g., deVilliers, 1999) or whether the function of all MR processes is to change the epistemic value of a narrative (e.g., Duval, 1995). Finally, MR is traditionally defined in terms of structure, that is, the form in which the reasoning is expressed, be it deductive, inductive, or abductive. On the other hand, the process perspective, which is espoused by others, tends not to be defined or explored epistemologically.

The articulation of these four elements, combined within the commognitive perspective, leads to defining MR as a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances. So the second major element is captured in this definition, that is, the inferential nature of MR – the element that will be seen to play a key role in both the structural and process aspects of MR. Also, we can say from the analysis of the literature that MR develops the discourse by extension *(endogenous discursive expansion*, in the terminology of Sfard, 2008), that is, there is no change in meta-discursive rules; there is no new mathematical object in a commognitive sense. The novelty is situated within the objet-level utterances themselves. This definition also allows us to avoid the first major element, that is, the activity/product dichotomy, in that discourse within our framework is seen as both activity and product. Every communicational act presents both the activity and product aspects, which are captured respectively by the process and structural aspects of MR.

As will be seen, the dual aspects related to the structural and process element allow for refining our above definition and at the same time integrate the different functions (the third element) of MR. Thus, the structural and process element, in conjunction with key discursive features of the underlying commognitive framework, will be seen to capture all four elements of MR that emerged from the literature. These two aspects are elaborated in the next sections, with less space devoted to the structural because it has already been relatively well explored from an epistemological perspective in the mathematics education literature. It is important to emphasize, before continuing, that the structural and process aspects of MR represent two different ways of looking at a given discourse. Both aspects are present and are related dialectically: structures are part of the process aspect of MR and processes contribute to the construction of those structures.

4.1 Structural aspect of mathematical reasoning

The structural aspect of MR refers in general to a more static aspect that is related to the form of a given piece of MR. More specifically, the structural aspect refers to the way in which the discursive elements combine in an ordered system that describes both the elements and their relation with each other. The more cited forms are deduction, induction, and abduction. Likewise, Toulmin (2007) and Peirce (n.d.) are the more widely used references for discussing the structural aspect within the literature. The Toulmin model schematizes the basic elements (data, claim, warrant) along with the qualifier (linked to the epistemic value), the backing (to further support the warrant), and the rebuttal (to pre-empt possible counter arguments to the claim). All those elements are narrative in nature and serve to structure the mathematical discourse. The Peirce model involves three basic, one-step, modes of inference: the deductive, the inductive, and the abductive. Every step is composed minimally of data, claim, and warrant (to use the same terminology across the two models). The deductive, the inductive, and the abductive each infer a different conclusion.

4.1.1 Deductive step

Deductive reasoning is, for some authors, synonymous with MR. Duval (1995), for example, describes deductive reasoning as the only form of reasoning that can change the epistemic value of mathematical knowledge from likely to true. As a structural aspect, the deductive step infers a claim from data and warrant. The nature of the qualifier attached to the claim (which is the conclusion for the deductive step) depends on the epistemic value of the data and the warrant. The deductive form of reasoning plays an important role within the processes of proving and formal proving, both of which require deductive restructuring (see Section 4.2.2).

4.1.2 Inductive step

Inductive reasoning is the second most common step in the literature linked to MR. It is defined inconsistently, partly because it refers to every reasoning that is not deductive (Reid, 2010). In our model, the inductive step infers a warrant from the data and the claim about the data. The epistemic value (i.e., the qualifier) that is allowed with respect to the conclusion of the inductive step is that of *likely* (or *probable*). Inductive reasoning is linked to the soon-to-be-described process of generalizing (Pedemonte, 2002; Rivera, 2008) in that this process can, at one moment or another, be structured inductively.

4.1.3 Abductive step

Researchers interested in the study of exploration activity, such as Reid (2003) and Pedemonte (2002), introduce the abductive step. It is a less discussed structure that is sometimes mingled with the inductive step (Rivera, 2008). According to Eco (1983, in Pedemonte & Reid, 2011), the abductive step can take two forms. The first infers data from the claim and the warrant. The second infers data and warrant from the claim. For Peirce (n.d.), the abductive step infers elements that can explain the claim. The abductive reasoning structure can be an element of every MR process by generating data and warrant in the search for similarities and differences as in, for example, generalizing (Rivera, 2008), conjecturing (Pedemonte, 2002), and also validating (Pedemonte & Reid, 2011).

4.1.4 Concluding remarks on the structural aspect of mathematical reasoning

From a commognitive standpoint, the structural aspect highlights the construction rules of mathematical discourse as well as its diverse components. It foregrounds the nature of the conclusion and its epistemic value. According to Peirce (n.d.), the validity of the reasoning is not judged only from its structure, but from the epistemic value attached to the conclusion, thus giving a special status to the deductive step. (This position stands in contrast to standard definitions of *logical validity* where an argument is valid if and only if it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless to be false; it is not required that a valid argument have premises that are actually true.) The meta-rules that control mathematical discourse (within a mathematical community) demand that the discourse of reasoning be eventually structured into the deductive step. It is the unique structure that allows for theorization of mathematical discourse.

Even if necessary to a model of MR for the teaching and learning of school mathematics, the structural aspect is not sufficient for fully understanding the nature of MR in school. While it puts forward, in a static way, the narrative elements, relations, and epistemic values that constitute MR, it neglects the temporality and axiological nature of MR that are central to reasoning activity. The process aspect, less developed in the literature, will fill that gap.

4.2 Process aspect of mathematical reasoning

Several action verbs are linked to MR in the literature, verbs that put forward the temporal nature of MR. However, few texts conceptualize MR as a process. From the commognitive perspective adopted here, the following definition of the process aspect of MR emerges:

> MR processes are commognitive processes that are meta-discursive, that is, that derive narratives about objects or relations by exploring the relations between objects.

Of the several overlapping MR processes found within the literature, nine distinct processes emerged. Eight of these were classified into one of two categories: the processes related to the search for similarities and differences, or the processes related to validating. These categories, which are similar to those described by Stylianides (2008), materialized after several feedback cycles. The ninth process, that of exemplifying, was classified as a support for both of the other two categories. The description of each of these nine processes follows. It includes: i) some background literature that gave rise to the inclusion of the given process; ii) the creation of a definition of the given process that is consistent with the commognitive framework; and iii) a brief discussion of the process from a commognitive point of view.

4.2.1 Processes related to the search for similarities and differences

The following five processes relate to the search for similarities and differences: generalizing, conjecturing, identifying a pattern, comparing, and classifying.

Generalizing For Stylianides (2008) and Artzt (1999), MR is all about generalizing and arriving at valid conclusions. According to Stylianides (2008), generalizing is "the transportation of mathematical relations from given sets to new sets for which the original sets are subsets" (p. 9). For Pedemonte (2002), the processes of argumentation that involve generalizing allow for passing to a larger set and also provide the reasons for believing in the narrative: "Mathematical argumentation always has the objective of seeking truth" (p. 30, our translation). However, it is the passage from a given set to a larger one that is highlighted by our anasynthesis (e.g., Dreyfus, 1991). The inferential and expansion aspects of generalizing are considered its main features, thereby leading to the following definition of generalizing:

> **Generalizing:** A process that infers narratives about a set of mathematical objects or a relation between objects of the set from a subset of this set.

From a commognitive viewpoint, we can link generalizing to MR because the process is clearly associated with inference and discourse, without necessarily creating a new incommensurable discourse. In contrast, abstracting (Jeannotte, 2015) produces a meta-level development of discourse and is thereby not considered a process of MR from a commognitive perspective.

Conjecturing According to Stylianides (2008), conjecturing has to lead to a reasoned conjecture. The conjecture, as a narrative, is then always associated with the epistemic value, *probable* or *likely*. Other MR processes are thus needed to determine whether the conjecture is true or false. For Stylianides, conjecturing also leads to an utterance that generalizes the cases generated, that is, that expands its domain of application. Mason (1982) describes conjecturing as a cyclical process involving i) enunciating clearly a conjecture, ii) verifying that the conjecture covers all known cases and examples, iii) being wary of the conjecture by trying to refute it, and iv) finding out why it is true or modifying it (which brings us back to i). A related element pulled from the literature is the link between conjecture and theorem. Pedemonte (2002), who proposes a parallel between the two, states that argumentation is linked with conjecture and formal proving is linked with theorem.

Thus, several elements are retained in order to build the definition of conjecturing as a MR process. First, a central element is the search for regularity, the search for some relation. In fact, it is the search for similarities and differences that emerges from the analysis. This search allows for building a relation – around objects or other relations, that is, around some mathematical phenomenon. Second, conjecturing leads to a narrative with *probable* or *likely* as its epistemic value. There is an uncertainty about the built narrative. Third, conjecturing can lead to a general discourse when it expands an observed relation to a larger set. What distinguishes conjecturing from generalizing is that it has an epistemic value attached to it. Conjecturing can thus be defined as follows:

> **Conjecturing**: A MR process that, by the search for similarities and differences, infers a narrative about some regularity with a likely or probable epistemic value and that has the potential for mathematical theorization.

From a commognitive viewpoint, conjecturing leads to an extension of the discourse by the building of likely narratives, based on the search for similarities and differences.

Identifying a pattern Is identifying a pattern different from the process of conjecturing? For example, is there a probable epistemic value attached to the identifying of a pattern? According to Stylianides (2008), identifying a pattern (i.e., a recursive relation) can lead to conjecturing, but the two cannot be equated. For Cañadas, Deulofeu, Figueiras, Reid, and Yevdokimov (2007), the third stage of conjecturing (i.e., *empirical induction from a finite number of discrete cases*) is the search for and the predicting of a regularity in the pattern, which is clearly linked to identifying a pattern. Identifying a pattern, according to Stylianides (2008), goes further than observing a pattern. There is active searching, and then taking some distance from the phenomenon, which are necessary for MR. Furthermore, as for generalizing, there is no particular epistemic value associated with the inferred narrative. We thus define identifying a pattern as follows:

> **Identifying a pattern**: A MR process that, by the search for similarities and differences, infers a narrative about a recursive relation between mathematical objects or relations.

This process, from a commognitive viewpoint, differs from conjecturing and generalizing in that it is possible to identify a pattern that is applicable to a certain set without expanding it to a larger set.

Comparing The term comparing is linked by various researchers to certain elements of MR, such as inductive reasoning (Simon, 1996) and deductive reasoning (Duval, 1995). Pedemonte

(2002) links comparing to exemplifying and conjecturing. For her, MR must at some point involve comparing examples so as to be able to conjecture. The key element here is the inferential nature of comparing, which we define as follows:

Comparing: A MR process that infers, by the search for similarities and differences, a narrative about mathematical objects or relations.

Comparing can take place along with a plethora of other MR processes: generalizing, identifying a pattern, validating. For example, identifying a pattern necessitates comparing cases or examples so as to highlight the pattern. However, identifying a pattern goes beyond comparing because comparing only infers a narrative about similarities and differences.

Classifying Classifying is associated by some with MR. For Mason (2001), "classification is not just about making distinctions and describing properties, but about justifying conjectures that all possible objects with those properties have been described or otherwise captured" (p. 7). Mason (2001) highlights a meta-discursive rule upon which classifying is contingent: Mathematical properties and definitions are used to classify objects. The process of classifying is thus defined as follows:

Classifying: A MR process that infers, by the search for similarities and differences between mathematical objects, a narrative about a class of objects based on mathematical properties and definitions.

Classifying is an important process that allows for object-level development by putting together or pulling apart different discursive objects, thereby structuring a discourse. Classifying can be associated with comparing, conjecturing, and generalizing.

4.2.2 Processes related to validating

For the second set of processes, the change in epistemic value is foregrounded. Drawing on Duval (1995), Lithner (2008), and Cañadas et al. (2007), the term validating is linked to the epistemic value that an utterance in a given narrative can take on (e.g., likely, true, probable, false) and depends on the mathematical discourse community where it emerged. Unlike Duval (1995), from a discursive stand there is no difference between epistemic value and truth-value. The epistemic value of an utterance depends not only on the logical validity of its structure but also on the shared discourse (meta-rules and accepted narratives) of a given community. This epistemic value, already highlighted by the structural aspect, is very important in mathematics for systematizing discourses and thus for theorizing. The three processes below, which are related to validating, are defined inclusively as follows:

Validating: A MR process that aims at changing the epistemic value (i.e., the likelihood or the truth) of a mathematical narrative.

Contrary to conjecturing that infers a narrative that is likely, the validating processes aim at changing a narrative's epistemic value one way or another. This change can be from likely to true, from likely to false, or even from likely to more likely. The meta-discursive rules of mathematics constrain the possible changes of epistemic value. It is partly this definition of validating that will help us to highlight the different particularities of the three processes related to validation that emerged from the analysis of the corpus: justifying, proving, and formal proving. For the three definitions, the word *searching* has been included to emphasize the process-nature of the activity. It involves searching for discursive information (data, warrant, backing), which allows for a change in epistemic value.

Justifying Justifying is viewed by Yackel and Hanna (2003) as a social process, that is, more than one individual can be involved and the process is founded on public knowledge. Likewise, for Duval (1995), Stylianides (2008), and Cabassut (2005), this process is linked to a change in the epistemic value of a narrative by searching for data, warrant, or backing to support this change. However, in the literature, it is mainly the passage from likely to true that is addressed. Nevertheless, the process of justifying is associated with two types of epistemic passage. The first is related to the justification of a conjecture that arises from the process of conjecturing. This passage allows for changing the epistemic value from likely to more likely, as stressed by Cabassut (2005) in his discussions of plausible validation. The second type of epistemic passage is related to a validation that changes the epistemic value from likely to true or false, without being considered necessarily as constituting the process of proving. Thus, justifying is defined as follows:

> **Justifying**: A MR process that, by searching for data, warrant, and backing, allows for modifying the epistemic value of a narrative.

The change of epistemic value is, as just mentioned, not necessarily from likely to true. The elements supporting the process are constrained by meta-discursive rules within a certain community. For example, the change from likely to true has to be based on a deductive structure. On the other hand, in changing from likely to more likely, some meta-rules constrain the process, but a deductive structure is not necessary.

Proving The literature on proving can be divided into two groups: the texts that deal with proving and the texts that deal with what will be named formal proving.[4] The next section deals with formal proving. Proving is, as was the case for justifying, a social process. According to Balacheff (1988), for example, proving is a type of explanation that is socially acceptable. In addition, proving is linked to changing the epistemic value of a narrative: "Proving is the process employed by an individual (or a community) to remove doubts about the truth of an assertion" (Harel & Sowder, 2007, p. 807). However, the process of proving is associated more with deductive reasoning than is the process of justifying. For Maher (2009): "Proof making is a special type of mathematical activity in which children attempt to justify their claims by deductive argumentation" (p. 121). Thus, we define the process of proving as follows:

> **Proving**: A MR process that, by searching for data, warrant, and backing, modifies the epistemic value of a narrative from likely to true. This process is constrained by:

i) the narratives that are accepted by the class community (the set of accepted narratives) that are true (from the viewpoint of the expert mathematician) and available without additional justification;
ii) a final restructuring that is deductive in nature;

[4] Formal proving is referred to in French as *démontrer*.

iii) the *realizations* (in the sense of Sfard, 2008, p. 301) that are appropriate and known, or accessible, to the class.

In mathematics, the deductive structure is associated with rigor. While the meta-discursive rules of mathematical discourse dictate that the validating process has to be restructured in a deductive way at some point, we emphasize that proving as a process does not have to be deductively structured at every moment. The notion of theorization underscored by Mariotti (2005) is that proving relies on a set of narratives that are accepted as true. Moreover, by accepting non-formalized realizations, the proving process can be developed from the primary school onward.

Proving is differentiated from justifying by its potential for theorization. It is also more constrained than justifying in that it has to be restructured deductively and bear on a set of accepted narratives that are coherent with the mathematical discourse of the expert (e.g., the teacher), even if realized differently (informally).

Formal proving For Hanna and Jahnke (1996), "formal proof arose as a response to a persistent concern for justification" (p. 889) among mathematicians. Formal proving[5] is thus strongly associated with change of epistemic value. For Balacheff (1988), formal proving is constrained by a strict structure and meta-rules. Arsac (1996) highlights the social nature of those meta-rules with theory brought to the forefront. Formal proving can be differentiated from proving essentially by its rigor and formalism. While the process of proving is founded upon narratives that are mathematically true, the process of formal proving goes farther in that the narratives must be integrated explicitly into some mathematical theory. Formal proving is thus defined as follows:

Formal proving: A MR process that, by searching for data, warrant, and backing, modifies the epistemic value of a narrative from likely to true. This process is constrained by:

i) the narratives that are accepted by the class community (the set of accepted narratives) that are true (from the viewpoint of the expert mathematician) and systematized in a mathematical theory;.
ii) a final deductive restructuring;
iii) realizations that are formalized and accepted by the class and mathematical communities.

As opposed to proving, formal proving relies on mathematical theory built a priori and on formalized realizations (axioms and theorems). As a consequence, the generic example as elaborated by Balacheff (1988), which is acceptable within the previously described process of proving, cannot be used within formal proving.

4.2.3 Exemplifying: a support for the other mathematical reasoning processes

Despite the fact that a smaller number of texts have dealt with exemplifying, the analysis of the corpus yielded some highly relevant treatments. Mason (1982) defines exemplifying as a

[5] *Formal proving*, an action derivative of *formal proof* (a term widely used in the mathematics education research literature), should not be construed as reasoning that operates only syntactically.

process that allows for exploring a problem with the aim of conjecturing, or verifying the conjecture and refining it. For Pólya (1968), exemplifying can lead to generalizing. The generic type of exemplification (Balacheff, 1988) can be associated with the validating process. Thus, linked to both the search for similarities and differences and validating, exemplifying can support each of the processes of MR cited previously:

Exemplifying: A MR process that supports other MR processes by inferring examples that assist in:

i) the search for similarities and differences;
ii) the search for validation.

Exemplifying allows for inferring data about a problem (which can be linked to the abductive structure). Those data can then be recycled in the search for similarities or differences in patterns and relations, but also within the processes of validating. Exemplifying thus generates elements that will serve in generalizing, in conjecturing, and even in validating.

4.2.4 Concluding remarks on the process aspect of mathematical reasoning

Even if they have been treated separately, all the processes of MR are interrelated. They stimulate and influence each other, allowing for the development of an increasingly more complex mathematical discourse by the generation of new narratives on already existing discursive objects. In particular, conjecturing and proving play an essential role in mathematical theorization. Indeed, conjecturing infers narratives that can potentially enrich mathematical theories and proving allows for systematizing the discourse, with the idea of theorizing it. Even if school mathematics is not formalized in the same way as the mathematics of mathematicians, systematization of discourse, even if somewhat local, can be engaged in within the processes of conjecturing and proving. We have also seen, for instance, how exemplifying is tightly tied both to the processes related to the search for similarities and differences and to the processes related to validating.

5 A model of mathematical reasoning for school mathematics

The outcome of this research project is a conceptual model of MR for school mathematics. Framed by commognitive theory and the methodology of anasynthesis, the model unifies a previously unstructured domain according to two central aspects: the structural and the process aspects – two different ways of looking at a given discourse that are related dialectically. More specifically, the discursive framing of the research project provided a means of analyzing the various types of MR within the mathematics education literature and reformulating them according to their various discursive elements and the interrelationships among them. It is noted that while the model is general enough to be applied to various mathematical content areas, it remains a model of *mathematical* reasoning, as opposed to a model of reasoning per se, by the mere but important fact that it is rooted in a particular body of literature, that of *mathematical* reasoning, and is thus grounded within the discourse already built by members of the mathematics education community. By a similar argument, and also in line with the same commognitive stance, it is a model of MR *for* school mathematics – accounted for not only by the educational literature source from which the model arises but also by the manner in which its different elements (e.g., conjecturing and proving) have been characterized with the student of school mathematics in mind. In sum, rephrasing MR discourse according to commognitive theory can by its very nature help both researcher and teacher focus on what is visible and audible in the classroom. Furthermore, adopting a common discourse on MR carries with it the potential for members of the various mathematics education communities not only to better communicate with respect to MR but also to develop the learning resources needed for the improvement of MR in school. Nonetheless, because a narrative model is a living thing, this model of MR will surely evolve and grow in the hands of its users. Lastly, in the spirit of Balacheff (2008), the explicit formulation of the epistemological underpinnings of this model of MR can also be seen as a first step in eventually creating bridges between different epistemologies.

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