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Sense making in the context of algebraic activities

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Abstract This article concerns student sense making in the context of algebraic activities. We present a case in which a pair of middle-school students attempts to make sense of a previously obtained by them position formula for a particular numerical sequence. The exploration of the sequence occurred in the context of two-month-long student research project. The data were collected from the students' drafts, audiotaped meetings of the students with the teacher and a follow-up interview. The data analysis was aimed at identification and characterization of the algebraic activities in which the students were engaged and the processes involved in the students' sense-making quest. We found that sense-making process consisted of a sequence of generational and transformational algebraic activities in the overarching context of a global, meta-level activity, long-term problem solving. In this sense-making process, the students: (1) formulated and justified claims; (2) made generalizations, (3) found the mechanisms behind the algebraic objects (i.e., answered why-questions); and (4) established coherence among the explored objects. The findings are summarized as a suggestion for a four component decomposition of algebraic sense making.

Keywords Sense-making · Algebraic activities · Project-based learning · Integer sequences · Coherence

1 Introduction

Sense making has long been a focal concern of the mathematics education research community (e.g., Kieran, 2007; NCTM 2009). NCTM (2009) recognized sense making as a means to know mathematics as well as an important outcome of mathematics instruction. To review,

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NCTM (2009) refers to sense making in mathematics as "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (p. 4). Nevertheless, NCTM (2009), as well as many additional mathematics education publications, is rather inexplicit as to what sense making comprises of and how it occurs. Moreover, it has been broadly acknowledged (e.g., Schoenfeld, 2013) that empirically-based knowledge about the processes involved in sense making, is insufficient.

The case of interest presented in this article occurred with two 9th graders, Ron and Arik (pseudonyms) who participated in the Open-Ended Mathematical Problems project, which was conducted by the authors of this paper in a school in Israel. The initial part of Ron and Arik's project is analysed elsewhere (Palatnik & Koichu, 2014, 2015). It lasted for three weeks and resulted in an insight solution to the problem of finding a position formula (i.e., a formula specifying the value of the n^{th} term of a sequence in terms of n, its position, without a reference to any of the previous terms) for a particular sequence. The discovery of the formula was a highlight of the project. The students told us, however, that they found the formula "by chance" and that it did not make sense for them. As a result, making sense of the obtained formula became an explicitly chosen goal and the main theme of the second part of the students' project. This part had lasted for four weeks and ended when the students succeeded, in quite an idiosyncratic way as will be evident shortly, to make sense of the formula.

Accordingly, the second part of the student project became for us a valuable opportunity to explore the students' sense making. The goal of our exploration was to discern the activities and processes involved in their sense making effort. Specifically, we pursued the following research questions:

While attempting to make sense of a position formula for a particular numerical sequence in the context of a long-term problem-driven exploration,

- In which events and algebraic activities were the students engaged?
- What were some of the processes involved in the students' explicitly expressed conviction, by the end of the exploration, that the formula "makes sense"?

Based on the answers to these questions, we introduce our proposal for a decomposition of the "sense making" notion in the context of school algebra.

2 Theoretical background

In mathematics education publications, sense making appears as a name of a broadly understood process involved in the desirable ways of studying mathematics. For Schoenfeld (1992, 2013), who consistently argues for teaching mathematics as a sense-making activity, sense making in mathematics is something that occurs through the dialectic of conjecture and argumentation, as well as through problem solving and exploration of mathematical objects, relationships or phenomena. NCTM (2009) defines sense making in rather a broad way (see above), and points out that sense making should underlie all mathematical activity in a classroom. In this publication as well as in a subsequent publication dedicated to instruction of high school algebra (Graham, Cuoco, & Zimmermann, 2010) the sense making notion is consistently used in conjunction with the reasoning notion. The main argument (NCTM 2009, pp. 9–10) is that the desirable reasoning habits (e.g., seeking patterns and relationships, making purposeful use of procedures, applying previously learned concepts to new problem situations, making logical deductions based on current progress, justifying or validating a solution, considering the reasonableness of a solution, generalizing a solution to a broader class of problems) develop in the processes of mathematical inquiry and sense making. NCTM (2009) also asserts that refocus of mathematical instruction on reasoning and sense making can increase understanding and foster meaning.

In empirical studies, the notion of sense making frequently denotes ways by which learners of mathematics act upon a particular entity in the context of particular mathematical activity. The expression "to make sense of…" is attributed in different studies to such entities as proofs, instructional devices, concepts, solution methods and problem situations (e.g., Smith, 2006; Rojano, Filloy, & Puig, 2014). For instance, in the study of Rojano et al. (2014) an intertextual analytical perspective of algebraic activity was proposed. The context for sense production was provided by algebraic activities focusing on solution of word problems and on solution of systems of two linear equations. The theoretical framework of Rojano et al. (2014) study was based on the broadly defined notion of a *mathematical text*. According to Rojano et al. (2014) the sense of a new mathematical text (e.g., of a new solution method) is produced when students are able to incorporate a "new" text that they read/transform into their personal networks of the related texts. Rojano et al. (2014) present algebraic equations and solutions as open *mathematical texts* with embedded intentionality to introduce students with activities that require production of new (for them) knowledge and sense by relation "to the student's experience –to texts that have been previously built" (p. 17). As one can see, the above approach to sense making is quite in line with the abovementioned NCTM's (2009) definition of sense making as understanding by means of constructing connections to the past knowledge and experience.

In our study we adapt NCTM's (2009) perspective on sense making, and elaborate on it in a particular context. Our theoretical framework is built upon the idea of algebra as an activity (Kieran, 1996, 2007) and on analytical apparatus of Mason's (1989) model of mathematical thinking known as Manipulating – Getting-a-sense-of – Articulating (MGA).

2.1 Algebraic activities

Kieran (1996, 2007) puts forward the idea of algebra as an activity. She identifies three types of activities in school algebra: *generational*, *transformational*, and *global/meta-level* activities and argues that each activity has affordances for meaningful learning of algebra and development of students understanding of algebraic concepts.

The *generational activity* involves the forming and interpreting of the objects of algebra (e.g., algebraic expressions or formulas) including objects expressing generality arising from geometric patterns or numerical sequences. Kieran (2007) argues, with reference to Radford (2001), that the generational activity involves recognition of the role of algebra as a language (cf. also Rojano et al., 2014, for the related argument). The *transformational activity* includes various types of algebraic manipulations. Kieran (2007) points out that transformational activity can contribute to students' manipulative fluency with symbols, but also to development of students' notions of equivalence as a core idea of algebra. A related point is highlighted by Hoch and Dreyfus (2006). They propose the notion of *structure sense*, which is related to algebraic manipulations but includes much more than manipulative, procedural skills. Structure sense is related to the learner's capabilities to recognize a familiar structure in its simplest form, manipulate a composite term as a single entity, and choose between appropriate manipulations. Further, Arcavi (1994, 2005) coins the notion of *symbol sense* and points out its main attributes in relation to: a friendliness with symbols as tools to uncover relationships, an ability to switch between attachment and detachment of meaning, a capacity to express graphical and verbal information, a search for the best symbolic representations, an examination of the meaning of symbols, and an attention to the varying roles of symbols in different contexts.

Finally (here we come back to Kieran's model), algebra as a *global/meta-level activity* refers to mathematical activity for which algebra is used as a tool. Kieran (2007) argues that "these activities provide the context, sense of purpose, and motivation for engaging in the previously described generational and transformation activities" (p. 714). Examples of such activities include problem solving, modeling, working with generalizable patterns, conjecturing, justifying and proving. It is essential for the forthcoming analysis that when the learners are engaged in a global/meta-level activity such as problem solving or proving, they can carry out the activity in a variety of ways, and the decision to use the algebraic apparatus arises as a choice of the learners.

2.2 MGA model

Manipulating – Getting-a-sense-of – Articulating (MGA) model (Mason, 1989; Mason & Johnston-Wilder, 2004) considers a sense-making act to be an inseparable part of mathematical thinking. The model postulates that manipulating familiar mathematical objects (M) leads to the formation of a sense of generality or regularity based on properties of these objects (G), and then to the articulation of that general property or regularity (A), which in turn forms new objects for further manipulations. Thus, MGA model elaborates a helix of activity during mathematical thinking, in which each cycle includes its own, local, sense-making act. Mason (1989) suggests that the driving force behind this process is the gap between expected and actual results of manipulations.

MGA was a core component of the analytical framework used by Sandefur, Mason, Stylianides, and Watson (2013) in their study of the role of example generation in the proving process. As Sandefur et al. (2013) pointed out, "getting-a-sense-of" cannot itself be observed, it is implied in the connection between M and A. In their analysis, Sandefur et al. (2013) inferred about the "getting-a-sense-of" stage, based on observed objects the students manipulated with and on the associated "articulations".

The MGA model has an important role in the forthcoming analysis of the case of interest: it enabled us to characterize at a fine-grained level the students' struggle at some (though not all) parts of their exploration. As will be explained in the section 3, we also focus on objects of students' manipulation, and on articulated features as indirect indicators of "getting-a-sense-of" stage. In addition we can infer from students' choices of activities to be engaged with and their explicitly stated intensions to explore certain features of the sequence.

To summarize, we approach the given students exploration as a sequence of algebraic activities (Kieran, 1996, 2007), chosen and enacted by students. Each activity in turn is characterized in terms of MGA model (Mason, 1989) with particular attention to manipulated objects and local sense making acts.

3 Method

3.1 Research settings and participants

The Open-ended Mathematical Problems project, in the context of which the case of Ron and Arik took place, has been conducted, since 2010, in 9th grade classes for mathematically promising students in Israel. The students studied mathematics at the A-stream level (the highest level of middle-school mathematics in Israel, cf. Leikin & Berman, 2016, for details), and, in addition, has two enrichment classes a week. The learning goal of the project is to create, for all students, a long-term opportunity for developing algebraic reasoning in the context of exploring numerical sequences. It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge of sequences; this topic is taught in the 10th grade.

The project is designed in accordance with the principles of the Project-Based Learning (PBL) instructional approach. Blumenfeld et al. (1991) define PBL as

> …a comprehensive perspective focused on teaching by engaging students in investigation. Within this framework, students pursue solutions to nontrivial problems by asking and refining questions, debating ideas, making predict, designing plans and/or experiments, collecting and analyzing data, drawing conclusions, communicating their ideas and findings to others, asking new questions, and creating artifacts (p. 371).

The organizational framework of the project is as follows. At the beginning of a yearly cycle of the project, a class is exposed to 8–10 challenging problems. When introduced to the problems, the students are briefly explained the terms “recursive formula” and “position formula”. The students choose one problem and work on it in teams of two or three. They work on the problem almost daily during leisure hours at home and during their enrichment classes. Weekly 20-min meetings of each team with the instructor take place during the enrichment classes.[1] After the initial problem is solved, students are encouraged to pose and solve follow-up problems. At the end of the project, all teams present their results to their peers. Then 4–6 teams, chosen by their classmates, present their work at a workshop at the Technion – Israel Institute of Technology, attended by students and lecturers of the Faculty of Education in Science and Technology.

In terms of grades, Ron and Arik were not among the most mathematically advanced students of their class, but they were eager to succeed and present their project at the Technion. Similarly to many teams (though not all; see Palatnik, 2016, for an overview of all 23 student projects), Ron and Arik retained an interest in the project from the beginning to its end; the students appeared to us as enjoying the PBL approach. They never missed their meetings with the instructor, always came prepared, and were open to discussing new ideas. Additionally, they initiated many between-meetings conversations and email exchanges.

3.2 The problem

Ron and Arik chose to pursue the Pizza Problem (see Fig. 1).

This problem is a variation of the problem of partitioning the plane by n lines (e.g., Pólya, 1954). Pólya used this problem to demonstrate inductive mathematical research. He took the

[1] The first author was the instructor and also the regular mathematics teacher for the class, in which the case of interest occurred.

> Every straight cut divides a pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts?
>
> A. Solve for $n = 1, 2, 3, 4, 5, 6$.
>
> B. Find a recursive formula for the n^{th} term of the sequence.
>
> C. Find a position formula for the n^{th} term of the sequence.

Fig. 1 The Pizza Problem

reader into a multi-stage journey shaped by a series of questions, including questions about partitioning a straight line by n points, partitioning a plane by n lines, and partitioning a space by n planes. At no stage Pólya did provide rigorous proofs for the given formulas.[2] Instead, he helped the reader to make sense of the discovered regularities by showing how different conjectures come to complement each other. Using Kieran's (2007) terminology, we expected the Pizza Problem to provide students with rich opportunities to engage with generational and transformational activities in the context of a global/meta-level activity.

3.3 Data sources and analysis

Ron and Arik worked on the project for two months. We audiotaped and transcribed protocols of the weekly meetings with Ron and Arik (eight 20-min meetings), collected written reports and authentic drafts that the students prepared for and updated during the meetings (more than 40 pages) and interviewed the students by the end of the project. These data were used to create a description of the students' exploration and for dividing it into events/episodes.

We identified episodes in due course of the project where the students were engaged in activities related to: proving, generalizing, pattern-seeking, question-generating etc. We looked at the outcomes of these activities (e.g., patterns, questions, generalizations, proofs) and at the moments where the students' switched between the activities.

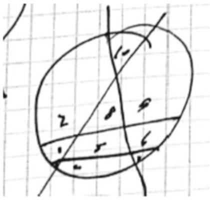
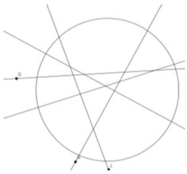
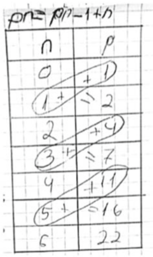
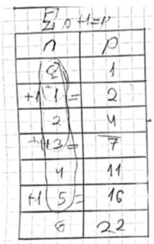
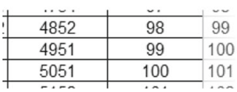
We also looked, by means of the MGA model, for how the students used the mathematical objects they considered in the course of their exploration. Namely, in each episode/event we discerned which objects the students considered, how they were manipulated, and how and which articulated properties served as a basis for the next MGA cycle. These helped us to infer about the local sense making acts (cf. Sandefur et al., 2013).

4 Prelude: Ron and Arik solve the Pizza Problem

The initial part of Ron and Arik's project, up to the point at which the students experienced insight and discovered the position formula $\left(P_n = \frac{n^2+n}{2} + 1\right)$, is analyzed in detail in Palatnik and Koichu (2015). An element of that analysis is presented in the left-hand column of Table 1, which consists of paradigmatic examples of the objects the students attended to when solving the problem (e.g., the drawing in the first row is one of about 30 drawings produced by the students during the first week). As a whole, Table 1 presents the MGA analysis of five main steps in solving the Pizza Problem and precedes the analysis of the sense-making part of the students' project.

[2] The proofs are provided, for instance, in Golovina and Yaglom (1963).

Table 1 Synopsis of initial part of Ron and Arik's project

Object of attention	Manipulating...	Getting a sense of…	The articulated properties
1. Handmade drawings of pizzas 	…different position of the cutting lines for the cases of 1, 2, 3, and 4 lines.	…which numbers can be obtained for n=1, 2, 3 and 4 … limitations of the handmade sketches.	The maximum number of pieces is achieved when exactly two lines intersect within the circle. For the cases of 1, 2, 3, and 4, the maximum numbers of pieces are 2, 4, 7, and 11.
2. GeoGebra-made drawing 	…different positions of the cutting lines for the case of 5 lines.	…which number can be obtained in case of 5 cuts; … limitations of GeoGebra-made sketches.	For the case of 5, the maximum number of pieces is 16. Increasing the number of lines makes it difficult to count the pieces even when GeoGebra sketch is used.
3. Two-column tables 	…the numbers in the tables.	…how the observed relationships between the numbers can be expressed using algebraic notation.	The formulas representing the observed patterns: $P_n = P_{n-1} + n$, $P_n = (P_{n-1} + n - 1) + 1$, $P = \sum n + 1$
4. Left column of the below table 	…with the numbers of the left column of the table.	…which arithmetic manipulations with the numbers of the left column can result in a number from the right column of the table.	The following relationships: $(4 \div 2) \times 5 + 1 = 11$ $(5 \div 2) \times 6 + 1 = 16$ etc. Mapping the relationship into the formula $P_n = \frac{n}{2}(n+1) + 1$.
5. Excel spreadsheet ...  4852 | 98 | 99 4951 | 99 | 100 5051 | 100 | 101	… with n, a row number in a spreadsheet; n=100 is chosen as a generic example.	… the result can be generalized for "big" values.	The explicit formula is verified as it returned the number matching the number obtained by means of the spreadsheet.

The MGA analysis reveals the nature of the local sense-making steps on the way to discovering the position formula. In particular, it suggests that all the recursive formulas (see row 3) and the position formula (row 4) were generated as symbolic expressions for the discovered pictorial or numerical patterns. Furthermore, Table 1 implies that at each stage, the students succeeded in "getting a sense of" the pictorial (rows 1 and 2) and numerical (rows 3–5) objects, but not of the algebraic/symbolic objects, as these had not been further manipulated or used. Consequently, it should not be surprising that though the students solved the problem, they later (see Episode 1) expressed the feeling that the obtained formula did not make sense to them. Indeed, solving the Pizza Problem was a global/meta-level activity for the students' generational and transformational activities, but transformations were of a geometric or arithmetic nature.

5 Ron and Arik make sense of the obtained formula

According to the timetable of the Open-ended Mathematical Problems project, Ron and Arik had about four more weeks to progress their work. Six milestone events that occurred during these four weeks are presented and discussed below. Each event is concluded with analysis in terms of algebraic activities and MGA cycles.

Event 1: Choosing goals

When the students presented their solution to the Pizza Problem at the third weekly meeting with the instructor, they received very positive feedback from him (see Palatnik & Koichu, 2015, for details). Then, the following conversation took place.

Instructor: Now you have a lot of work to do, and this is great. First of all, you see that the formula works. Now we have to think why it works, and try proving that it works.
Ron (to Arik): Write it down. "Why it works, and prove that it works" (laughs), it is interesting!
Instructor: To see, to understand the reason why it works.
Ron: The reason why it works… Interesting!
Instructor: In addition, there can be other interesting series that you can explore in a similar manner.
Arik: We have thought to do so.

Ron gladly accepted the idea to keep thinking on the position formula. In his words: "When we have a formula, but don't know its meaning, it is not interesting. If we knew how the formula is constructed, we would know it 100%. We got it by chance. So we do not know what it means."

This theme of meaning was not further discussed, and the conversation turned to "other interesting series". Both students expressed interest in exploring the problem in the case where there is no pizza to cut, but an entire plane to divide (see Fig. 2). In Fig. 2, they distinguished between two types of pieces: open and closed.

In addition, the students suggested that they could explore the points of intersection of the cutting lines amongst themselves and with a circle representing a pizza. They also planned to find a formula expressing the number of mutually exclusive segments (i.e., segments that did not intersect at their inner points) on the cutting lines within the circle representing a pizza.

Comments on Event 1

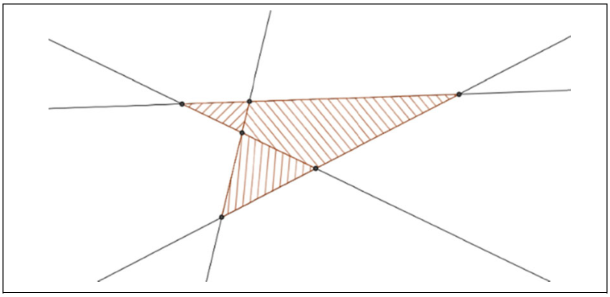


Fig. 2 The closed pieces are hatched; there are three closed and eight open pieces

The instructor's suggestion (to understand why the formula worked and how to prove it) seemed to resonate well with Ron's own intentions. This is evidenced in Ron's laughter, repeated comment of "interesting!", and his explicit request for Arik to write down the instructor's words (a unique event in all of the students' meetings with the instructor). Ron accepted the goal of proving the formula as a worthwhile one, in spite of the fact that the students had verified the formula empirically by testing it with big numbers (see row 5 in Table 1). This observation implies that the students realized that empirical justification was not enough, at least in the context of their project. Using Hewitt's (1992) metaphor of a train-spotter ("[train spotters] are left with numbers… not a train in sight," p. 8), Ron was not content with the role of a train-spotter. He grasped that the lucky discovery of the position formula by manipulating numbers could not be a self-contained goal of the project. However, he did not know how to begin the quest for meaning and proving. Thus the instructor's suggestion, supported by Arik, to explore additional sequences appeared operational and timely.

In this event the students have chosen a new goal. To recall, according to Kieran (2007) meta-level activity provide students the goal, overarching context, and motivation to be engaged in various activities where algebra is used as a tool.

Event 2: Simultaneous exploration of several sequences

Having chosen the above goals, the students returned to making sketches and counting: segments within the circle, closed and open parts of the plane and points of intersection of the cutting lines, for different numbers of lines. They observed:

– The numbers of the intersections (both between the cutting lines and between the cutting lines and the circle) were 2, 5, 9 and 14 for 1, 2, 3 and 4 cuts, respectively (Fig. 3a–c).
– The number of mutually exclusive segments on n cutting lines is represented by the formula $Z_n = n^2$ (see Fig. 3d).
– As a by-product, the students noticed that the sum of the first n odd numbers also equals n^2.

The students obtained each of these results using the same strategy as in solving the Pizza Problem (cf. Table 1 and Fig. 3). When several observations were made and articulated, the students tried something new: they began exploring the connections between different sequences (see Fig. 3d–e). In particular, Ron noticed the connection between the sequence 2, 4, 7, 11… (number of pieces in the Pizza Problem) and the sequence 2, 5, 9, 14… (number of intersections), as follows: the differences between the corresponding terms of the sequences form a sequence 0, 1, 2, 3…

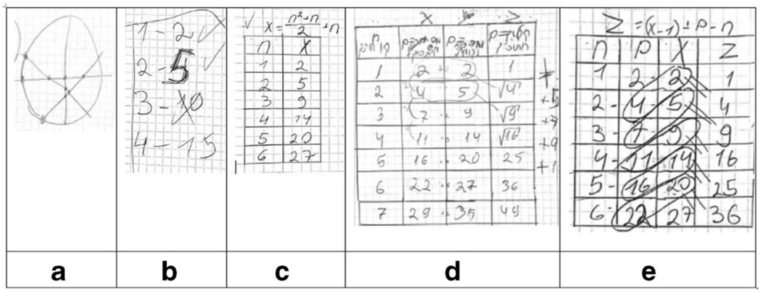


Fig. 3 The strategy employed in Events 1 and 2

Comments on Episode 2

In the context of a meta-level exploration activity, the students were engaged in new *generational activities*. They used the exploration strategy of counting objects on the sketches, organizing the results in tables, exploring the tables, and mapping the results as algebraic formulas, in a way similar to what was done while discovering the position formula $P_n = \frac{n}{2}(n+1)$ +1. An analysis in terms of the MGA model of the generation of each formula led us to the same conclusion: the students succeeded in getting a sense of the sketches and numerical patterns in the tables, but not necessarily of the resulting formulas *when taken separately.*

In this episode, however, the students' exploration included an additional MGA cycle. The objects of manipulation were the separately obtained numerical sequences which were used in a new way; the manipulation consisted of comparing the terms of the sequences; the articulated property was related to the differences between the corresponding terms of two sequences. The explored sequences represented different objects that appeared on the same drawings. In this way, geometric mechanisms behind the numerical sequences became clearer to the students.

Event 3: The first manipulation with the formula (adjustment)

To obtain an explicit formula for the sequence 2, 5, 9, 14 … (the numbers of intersections of the cutting lines with each other and with the circle), Ron used the above connection (see Event 2) and *adjusted* the formula $P_n = \frac{n^2+n}{2} + 1$ into the explicit formula $X = \frac{n^2+n}{2} + n$ (Fig. 3c).

In his words: "I thought it would be like the previous formula $\left(P_n = \frac{n^2+n}{2} + 1\right)$, but it did not fit. So I got rid of 1 and added n [to the right side of the formula], and it was right."

Comments on Event 3

Our interpretation of Ron's actions in this *transformational activity* is as follows. For the first time, an algebraic formula was object of Ron's *manipulations*, it was used in a new, though a very informal way. In this MGA cycle, M (manipulating with) was related to the formula $P_n = \frac{n^2+n}{2} + 1$ and A (articulation of a new property) – to the formula $X = \frac{n^2+n}{2} + n$. We infer that G (getting a sense of) was the realization that that "similar" sequences can be represented by formulas of similar structure and, more importantly, that one formula can be transformed into another, instead of being developed through an exploration of numerical regularities in a corresponding numerical table.

Event 4: Producing an explanation of why the target formulas worked

The wish to understand why the formula $P_n = \frac{n^2+n}{2} + 1$ returns the maximum number of pieces was a repeated theme in weekly meetings with the instructor. The students eventually answered this query in the following way. After obtaining the formula $X = \frac{n^2+n}{2} + n$ by adjustment (see Event 3), Ron and Arik came back to exploring the drawings and slightly reformulated one of their prior observations. They realized and articulated that the maximal number of pieces is obtained when a new cutting line crosses *all* the previous lines in new points. As a result, the students observed that a new cutting line added n new intersection points to the existing configuration of lines. For the students, there was an explanation of why the formula $X = \frac{n^2+n}{2} + n$ returned the *maximum* number of the intersection points. They further asserted that this idea also explained, for them, why the target formula $P_n = \frac{n^2+n}{2} + 1$ returned the *maximum* number of pieces.

Comments on Event 4

The students succeeded to combine their prior discoveries made in the course of generational activities and to connect geometrical and algebraic aspects of the process of adding a new line to the system of the existing lines (cf. Radford, 2004, for the idea that meaning is produced at the crossroad of different semiotic systems).

It is also of note that the students manifested some fluency when shuttling between algebraic and geometric objects of manipulation, thus using a potential of the generational activity they were engaged in. Ron and Arik succeeded in reattaching themselves to the geometry context of the problem and thus exhibited developing symbol sense (cf. Arcavi, 2005, for the second component of symbol sense, namely, an ability to switch between attachment and detachment of meaning).

Event 5: Emergence of a plan for "proving" the target formula

As mentioned, the need to prove the position formula $P_n = \frac{n^2+n}{2} + 1$ for the Pizza Problem was an additional driving force for the students. This need remained unfulfilled even when the students' expressed an understanding of *why* the formula worked (see Event 4). Eventually, the students invented their own method of proving. In Ron's words:

Ron: We thought of a way to prove it [the position formula]…[in order to do so, we wanted] to connect all the formulas we had, every table we've made… may be it will give us the formula, then we will know that it is a true formula indeed. Then we'd have a proof.

The realization of this quite vague plan is outlined below. Ron and Arik built upon the following observation: for any number of cuts, the sum of the number of open and closed pieces (see Fig. 2) equals the overall number of pieces into which a plane is divided. They suggested that separate explorations of open and closed pieces might lead them to the target position formula, and would be a more feasible for them task. The number of open pieces for n cuts, $2n$, was easy for them to find and explain: adding a new cutting line adds exactly two open pieces to the drawing.

The story was different for the sequence of the numbers of closed pieces. The students empirically (i.e., by counting on the drawings) obtained a sequence 0, 0, 1, 3, 6 for 1, 2, 3, 4 and 5 cuts, respectively. They evaluated it as "quite close" to the target sequence (2, 4, 7, 11, 16…). They then continued their search for the connections between the sequences, and observed the resemblance between the sequence of closed pieces (0, 0, 1, 3, 6…) and the sequence representing the numbers of the points of intersections between the cutting lines, 0, 1, 3, 6… (see Fig. 4).

The students then began manipulating the target formula $\left(P_n = \frac{n^2+n}{2} + 1\right)$. They changed the right part of the formula to $\frac{n^2+n}{2}$, and then to $\frac{n^2+n}{2} - n$ (to recall, an adjustment to the expression $\frac{n^2+n}{2} + n$ was successful, see Event 3). The expression $\frac{n^2+n}{2} - n$ returned the numbers 0, 1, 3, 6…, and after several unsuccessful attempts to further adjust it, the students came back to the expression $\frac{n^2+n}{2}$. Now they observed that it gave "the same numbers, but with a two-step delay" (i.e., 1, 3, 6, 10…). The expression $(n-2)\,\frac{2+n}{2}$ was expected to take care of this two-step delay. It returned the numbers 1, 1, 2, 4, 7, …, which were different from the target numbers only by 1. From here, Ron and Arik obtained the desired expression $(n-2)\,\frac{2+n}{2-1}$.

Intermediate summary and comments on Event 5

The students' main achievements up to this point of the project are summarized in Table 2.

To recapitulate, the position formula $P_n = \frac{n^2+n}{2} + 1$ the students tried to make sense of was not an isolated object for them at this stage. Each formula or expression in Table 2 was a result of a particular combination of generational and transformational activities with corresponding MGA cycle that the students' went through. With each new activity the students' fluency in moving between representations and confidence in their exploration strategy grew. As demonstrated above, some of the MGA cycles were related to attempts to connect the formulas. In these attempts, some of the manipulated objects were composite algebraic objects, specifically, the other formulas (see Events 4 and 5). Here we note the students' growing structure sense, as defined by Hoch and Dreyfus (2006). Still, all the results were verified empirically by checking whether the formulas returned "the right numbers."

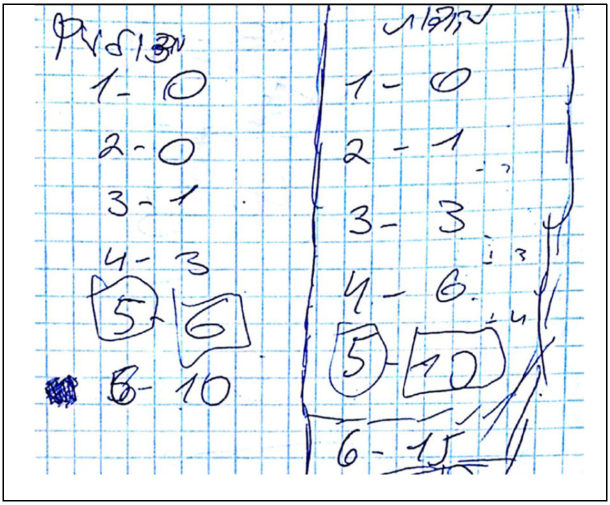


Fig. 4 Left table is for the closed pieces; right table is for the points of intersection

Event 6: "Proving" the target formula

The last piece of the puzzle came from one of Ron and Arik's classmates with whom they consulted. Ron and Arik presented him the chosen direction: to look separately at the numbers of open and closed pieces. The classmate's suggestion was as follows. Since the formula $P_n = \frac{n^2+n}{2} + 1$ represents the total number of pieces to which a pizza can be cut by n cuts, and since Ron and Arik have obtained the formulas for the numbers of closed and open pieces, the three formulas should match. After several unsuccessful attempts, Ron and Arik implemented this idea and algebraically connected the three formulas. In their final presentation, they showed a slide with the following transformations:

$$\begin{array}{lcl} \frac{(n-2)^2+n}{2} - 1 + 2n & = & \frac{n^2-4n+4+n-2+4n}{2} = \frac{n^2+n+2}{2} \\ \frac{n\cdot(n+1)}{2} + \quad 1 & = & \frac{n^2+n+2}{2} \end{array}$$

These two rows were proudly presented as "the proof" of the formula $P_n = \frac{n^2+n}{2} + 1$, in addition to justifications for all three formulas based on numerical tables.

Comments on Event 6

The students used the formulas representing the sequences for proving, for the first time in their project. In terms of Kieran (2007) the global algebraic activity (proving) provided the context and goal for transformational activity (working with equivalent expressions, expanding, simplifying). Each of the three formulas has previously been articulated as a result of a particular MGA cycle. The students "proof" can be seen as an algebraic test of a geometric connection between the formulas. Namely, they algebraically manipulated the formulas and articulated that they all match as parts of an algebraic equality. Obviously, the above "proof" contains a logical flaw: mathematically speaking,

Table 2 Intermediate summary of the students' exploration

Formula/expression	What it represents?	How was it found?	Why was it believed to be true?
$P_n = \frac{n^2+n}{2} + 1$	The sequence representing the maximum numbers of pieces to which a circle pizza can be cut by n cuts	By observing a numerical pattern and mapping it, step by step, into the formula.	It fits the empirically found sequence 2, 4, 7, 11, 16…It fits spreadsheet results.
$Z_n = n^2$	The number of mutually exclusive segments on n cutting lines	By generalizing a numerical pattern	It fits the empirically found sequence 1, 4, 9, 16…
$X = \frac{n^2+n}{2} + n$	The number of points of intersections between the cutting lines and between the cutting lines and the circle for n cuts	By adjusting the formula $P_n = \frac{n^2+n}{2} + 1$	It fits the empirically found sequence 2, 5, 9, 14, …
2n	The number of open pieces for n cuts	By generalizing a numerical pattern	It fits the empirically found sequence 2, 4, 6, …
$(n-2)\,\frac{2+n}{2-1}$	The number of closed pieces for n cuts	By a series of adjustments of the formula $P_n = \frac{n^2+n}{2} + 1$	It fits the empirically found sequence 0, 0, 1, 3, 6, …

the fact that three empirically found formulas match does not necessarily mean that one of them is correct. This observation, however, does not derogate the students' achievement. Let us remember that formal ways of proving were beyond their reach at the time of the project. To us, the described results indicated an important step in the development of Ron and Arik's algebraic reasoning: they got sense of the convincing power of symbolic manipulations (Arcavi, 1994, 2005; Hoch & Dreyfus, 2006; Kieran, 2007).

6 Discussion

The four-weeks-long exploration of two 9th grade students working on a particular project in the context of numerical sequences has been presented. We begin this section from discussing why Ron and Arik were so persistent when trying to make sense of their formula. We then discuss the algebraic activities in which the students were engaged (i.e., answer the first research question) and the processes involved in the students' sense-making quest (i.e., answer the second research question). This is followed by presentation of our proposal for decomposition of sense making in algebra, as it emerges from the presented case.

6.1 Why did Ron and Arik persist in making sense of their formula?

The students started their sense-making quest having three weeks of exploration already behind them. They justified the accuracy of the formula by checking that it returns "right numbers" for the first few terms of the sequence. In addition, the students addressed the generality of the formula by checking that it returned the "right number" in a generic case (recall the use of Excel in Table 1). In Lannin's (2005) study, 6th grade students who were engaged in a similar activity, got the feeling that their formula "makes sense" after making comparable progress. In that study, the students did not wish to continue the enquiry despite the attempts of the instructor to motivate them by raising why-questions. Similar situations have been reported in some additional studies (e.g., Tabach, Arcavi, & Hershkowitz, 2008). Thus, the natural question arises: why did Ron and Arik behave differently and persist to make sense of their formula? We suggest below an explanation that seems to us plausible.

The long-term sense-making quest of Ron and Arik was possibly triggered by what Sfard (2008) called the *sense of incomprehension*. Ron's manipulations with particular numbers resulted in the needed formula, however the students were unable to explain, first of all to themselves, the way of their serendipitous discovery (Palatnik & Koichu, 2015). This could be considered as one of the sources for incomprehension. However, according to Sfard, student's sense of incomprehension may cause an emotional reaction that can motivate but also can hinder learning.

We put forward two circumstances that may have converted Ron and Arik's sense of incomprehension into an incentive for learning. First, in the dialogue between the students and the instructor (see Event 1), the instructor made his dissatisfaction with some of routines used by the students explicit, and this dissatisfaction evidently resonated with the students' intentions. The sense-making questions about the meaning of the formula were treated by the students as legitimate and even interesting. Second circumstance is the organizational setting of the project. To recall, the project was organized in accordance with project-based learning instructional approach (Blumenfeld et al., 1991). In such an environment the students had a chance to get used to the long-term, open-ended explorations, to the high level of expectations and to having room and time to spend with a problem.

6.2 Algebraic activities and processes involved in student sense making

The answer to the first research question (about events and algebraic activities in which the students were engaged while attempting to make sense of the previously obtained position formula) straightforwardly follows from the exposition presented in Section 5. Briefly (here we use Kieran's, 2007 terminology), the students were engaged in generational and transformational activities in the context of the global/meta-level activities of explaining to themselves why the formula worked and of proving the formula. It is of note that the students' generational activity initially involved several separately explored sequences, and the search for connections between the sequences was framed by global activity.

The answer to the second research question (about the processes involved in the student sense-making quest) is presented below as a summative discussion of the ways by which the students addressed their self-imposed queries while being engaged in the aforementioned activities.

Briefly speaking, in order to answer the query "why the formula works" the students experimented with concrete drawings (i.e., drawings with 4–6 cutting lines) and then articulated properties of generalized drawings (i.e., drawings with n cutting lines). The concrete drawings apparently served as a visual tool to reveal a generic process that occurs when an additional line is added to a system of existing lines. Thus it looks like the processes of *generalizing* and *searching for geometric mechanism* behind the formula were involved in the students' sense making.

The query "how to prove the formula" turned to be the thorniest part of the project. Thus, the process of *justifying* played an important role in the students' sense making quest. The students' initial way of justifying formulas consisted of term-by-term comparing formula-generated numerical sequences with the empirically generated sequences. However, the students asserted that they fully addressed the how-to-prove query only when they had succeeded to show how the target formula came to *cohere* with two geometrically related formulas in a common algebraic structure (see Event 6). The related formulas were obtained as results of the students' attention to several sequences of their choice and by means of exploration of algebraic and geometric connections between the sequences. It is of note that the cloud of the formulas used by the students (see Events 5 and 6) did not exist when the students began. Thus, the process of generating a cloud of formulas and checking the cloud for coherence was a central process in the proving part of the students' sense-making effort (cf. Rojano et al., 2014, for sense production by means of connecting a new mathematical text to a system of other texts). However, the presented story suggests that the process of establishing coherence was an important one at different parts of the student sense-making quest, and not only in producing the final justification of the target formula. The coherence was pursued and achieved among various objects of exploration, such as diagrams, tables and formulas. Namely, at different stages of the global explorative activity the students employed the following strategy: they generated several objects of the same type and connected them, by means of manipulations in a particular mathematical register and then among the registers. In order to make sense of the target formula, the students established coherence first among the objects in geometrical and numerical registers and then constructed and connected corresponding algebraic objects.

The pivotal role of the process of establishing coherence in the students' sense making puts forwards the importance of divergent algebraic reasoning (i.e., simultaneous consideration of several related problems and search for connections) as a counterpart of convergent algebraic reasoning (i.e., moving from a less general to a more general problem on the way to obtaining a target result). From this perspective, Ron and Arik's simultaneous attention to several problems of their choice on the way to achieving their overarching goal – making sense of the target formula, makes sense.

6.3 A proposal for decomposition of sense making in algebra

As argued, Ron and Arik's sense-making process consisted of a sequence of generational and transformational algebraic activities in the overarching context of a global, meta-level activity, long-term problem solving. In this sense-making process, the students: (1) formulated and justified claims; (2) made generalizations, (3) found the mechanisms behind the algebraic objects (i.e., answered why-questions); and (4) established coherence among the explored objects. We now take the liberty of formulating this summary by using a more general language, and propose a four-component prismatic decomposition of sense making. It is schematically presented on Fig. 5.

Each of the aforementioned processes was considered in past research (cf. Arcavi, 2005, for the main attributes of symbol sense as well as Lannin, 2005, and Radford, 2010, for the role of generalizing and justifying in development of students understanding of algebra). However, consideration of all four processes as parts of the student sense-making quest in an algebraic context can be seen as a theoretical contribution of our study.

In particular, the above four-component decomposition presents sense making as a complex, multi-stage and multi-focus conjunction of processes. We deem that the decomposition elaborates on NCTM's (2009) conceptualization of sense making. Furthermore, we suggest that the prismatic decomposition in the context of generational, transformational and meta-level algebraic activities could be used as an analytic tool in future research on algebraic sense making, especially in complex learning environments such as PBL or inquiry-based environments.

Finally, the unusual characteristic of the documented case is that a sense-making goal was self-imposed by the students. As mentioned, Ron and Arik's determination was surprising to us. We have presented an explanation of this phenomenon that seems to us plausible, but, obviously, further research is needed in order to better understand factors involved in the emergence of this goal and in order to be able to intentionally design algebraic activities that can trigger and support sense making in students.

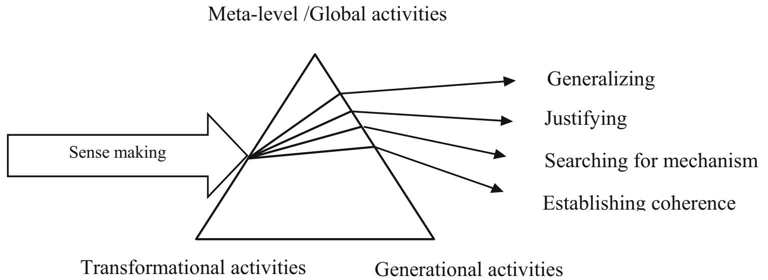


Fig. 5 Four processes involved in an algebraic sense making through algebraic activities

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