Educ Stud Math (2017) 95:123–142
DOI 10.1007/s10649-016-9740-5

Values and norms of proof for mathematicians and students

Paul Christian Dawkins[1] **· Keith Weber**[2]

Published online: 16 December 2016
© Springer Science+Business Media Dordrecht 2016

Abstract In this theoretical paper, we present a framework for conceptualizing proof in terms of mathematical values, as well as the norms that uphold those values. In particular, proofs adhere to the values of establishing a priori truth, employing decontextualized reasoning, increasing mathematical understanding, and maintaining consistent standards for acceptable reasoning across domains. We further argue that students' acceptance of these values may be integral to their apprenticeship into proving practice; students who do not perceive or accept these values will likely have difficulty adhering to the norms that uphold them and hence will find proof confusing and problematic. We discuss the implications of mathematical values and norms with respect to proof for investigating mathematical practice, conducting research in mathematics education, and teaching proof in mathematics classrooms.

Keywords Proof · Values · Norms · Mathematical culture

1 Introduction

Many mathematics educators consider proving to be one of the cornerstones of mathematical practice and believe an important goal of mathematics education is to apprentice students into the mathematical practices associated with proof (e.g., Harel & Sowder, 1998; Stylianides, Bieda, & Morselli, 2016). The broad aim of this theoretical paper is to analyze particular reasons why such apprenticeship is difficult and to propose a framework for proof that could inform future research and instruction. In particular, we will argue that more attention is needed on mathematicians' *values*—that is, what features mathematicians believe to be

✉ Paul Christian Dawkins
pdawkins@niu.edu

Keith Weber
keith.weber@gse.rutgers.edu

[1] Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL, USA

[2] Graduate School of Education, Rutgers University, New Brunswick, NJ, USA

necessary or desirable in the production of mathematical knowledge. Before doing so, we clarify our use of key vocabulary for what follows.

By *proofs*, we are referring to the written artifacts that mathematicians call proofs, which they produce and disseminate to sanction mathematical assertions as theorems.[1] By *proving*, we are referring to the activity that a mathematician engages in to produce proofs, which, as we will explain, involves the simultaneous processes of advancing a mathematician's own personal values and satisfying communal norms so that these proofs will be acceptable to her colleagues. By *proving practice*, we are referring to the constellation of activities that a mathematician engages in with respect to proof, including producing proofs, presenting proofs to her colleagues, evaluating the proofs that her colleagues present to her, and appreciating, understanding, and learning from the proofs that she and her colleagues produce.

We define *written classroom proofs* to be the written artifacts that students produce to affirm mathematical claims and demonstrate learning to their teacher (and sometimes their peers). *Classroom proving* refers to the activity of writing proofs to simultaneously express students' understanding and chains of inference as well as satisfying teacher (and sometimes peer) expectations. *Classroom proving practice* is the constellation of activities that students and teachers in mathematics classrooms engage in with respect to proof, including the production of written classroom proofs. We elaborate on these terms later in this paper.

While there has been a sustained push among mathematics educators for proof to play a central role in mathematics classrooms (see Stylianides et al., 2016), numerous research studies have shown that both teachers and researchers find it challenging to create classroom environments in which students and teachers engage in target classroom proving practices (see Stylianides, Stylianides, & Weber, in press, for a review). A central goal of this paper is to provide new insights into why this might be the case.[2] We contribute to the broad issue of why mathematicians' proving practice is so problematic to communicate to students and why classroom proving practice is difficult to foster in three ways. First, we proffer a framework for conceptualizing proof in terms of *values* and *norms.* This entails identifying particular values that mathematicians seek to achieve and the norms adopted to uphold these values. Second, we identify both specific and general ways in which these norms and values pose pedagogical difficulties when they are incorporated into classroom proving practice. Third, we discuss implications for investigating mathematical practice, conducting research in mathematics education, and teaching proof in mathematics classrooms.

To accomplish these goals, we structure what follows into four sections. In the next section, we frame what we mean by values and norms with respect to proof by integrating Laudan's (1984) philosophical work on values in science with recent sociocultural work in mathematics education. In section 3, we propose specific epistemic aims that the community of mathematicians values and discuss how some norms with respect to proof are in place to uphold these values. In section 4, we discuss the difficulties inherent in communicating norms associated with proof to those who do not share mathematicians' values. In the final section, we conclude

[1] Other scholars have taken a different perspective and have not treated the written artifacts that mathematicians have produced as being proofs, but rather serving as a pointer to the true proof which must be enacted by the reader (e.g., Livingston, 2006).

[2] We do not mean to dismiss the numerous efforts to improve proof instruction or the lessons learned therefrom (e.g., Brown, 2014; Douek, 2009; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Stylianides & Stylianides, 2009; Stylianides et al., in press). We expect that our framework of norms and values could help explicate many such instructional successes. For the sake of space, we reserve our analysis here to explicating some learning challenges and not the various solutions currently provided by the literature.

by discussing implications for researchers of mathematical practice, researchers of mathematics education, and teachers of proof-oriented mathematics.

2 Theoretical framing of values and norms

In any scientific community, there is a dialectic relationship between three levels of scientific commitment: the community's axiology, the community's methodology, and the theories that this community produces (e.g., Laudan, 1984). By axiology, Laudan referred to the shared values, goals, and principles that the discipline is trying to achieve in the theories that it produces. The methodologies of a community specify acceptable means for developing and justifying theories. By the theories that mathematicians produce, we are speaking of the results of successful mathematical investigations represented by coherent collections of mathematical concepts and relationships between these concepts, such as group theory. These theories include definitions, theorems, proofs, examples, and algorithms, in addition to the shared conceptual tools that mathematicians use to understand, discuss, and reason about these concepts. Laudan posited that each level of commitment influences the others. The most basic examples of these influences are that axiology helps justify choices of methodology and methodology helps justify claims within a theory.

To coordinate Laudan's (1984) framework for scientific practice with cultural theory, we investigate the *values* of mathematicians as part of their axiology and the *norms* of proof as part of their methodology. In what follows, we outline our use of the terms *values* and *norms* drawing on literature from mathematics education and sociology. Values represent a community's shared orientations and goals that underlie shared activity. We concur with Herbst, Nachlieli, and Chazan's (2011) notion that values are what "members of a practice use to justify or otherwise discard possible actions" (p. 219) in a given situation. While values may justify actions, values themselves are generally assumed without justification. By saying that a community has adopted a value, we do not suggest that the value will necessarily be achieved or even that the community will always act to uphold it. The community may fail to achieve a value for many reasons, such as human fallibility, practical limitations, the value being an unattainable ideal, or multiple values being in conflict. All we mean is that values describe features in scientific theories that the community finds desirable and that the community will take active steps to increase the likelihood or extent to which that value is achieved.

To date, mathematics education research on the axiology of mathematics has been limited. Researchers generally have described values as the subset of personal beliefs that are strongly cherished, highly ingrained, or contain a moral dimension (DeBellis & Goldin, 2006). For instance, mathematicians are said to hold the value of mathematical integrity, which entails being open and honest about what one does not understand and taking active measures to try to resolve gaps in understanding (Carlson & Bloom, 2005; DeBellis & Goldin, 2006). Our use of the term values differs in that the values we identify are not only used to appraise an individual's behavior but also to evaluate the mathematical theories that an individual or community produces; our use of value is therefore akin to the criteria that Tao (2007) discussed for what constitutes good mathematics. Our investigation of values addresses Chinn, Buckland, and Samarapungavan's (2011) observation that educational research tends to ignore the relationship between epistemic aims (which express values) and the epistemic actions (which are guided and constrained by norms). It also addresses Solomon's (2006) recommendation that enculturating students into classroom proving practice involves an enculturation into mathematicians' values, a topic of research that she noted had received scant attention.

We use the term *norm* to refer to expectations on practice accepted by the scientific community to uphold a value. By *upholding* a value, we mean that the community believes that satisfying this norm will generally increase the likelihood or extent to which the value is achieved. In particular, we say a community has adopted a norm if (i) the default assumption of the community is that its individual members will conform to the norm, (ii) members of the community are attuned to breaches of the expectation, and (iii) a community member breaching the norm can be held to account for this violation (c.f. Herbst et al., 2011; Yackel & Cobb, 1996). In this sense, scientific norms constrain the types of knowledge claims practitioners may produce and how they justify those claims.

Several important clarifications are in order regarding norms, values, and their relationship. First, values are often implicit, to the extent that they are invisible to some practitioners in their daily work. As Taylor (1993) argued, much cultural "knowledge" is embedded in "know-how" about the rules of behavior within a community (what we call norms). Practitioners often do not attend to the values that these rules were put in place to uphold lest everyday activities become untenably cumbersome. For instance, a pure mathematician would not constantly be comparing the merits of deductive justification versus empirical justification; in her ordinary practice, she would instead simply accept deductive justification as a normative requirement for making knowledge claims. Norms, as default expectations, facilitate human activity. They allow practitioners the opportunity to practice their craft without constantly evaluating the nature of their craft. Practitioners may be able to articulate how the norms that they follow uphold their values, but these practitioners frequently enact rules without consciously attending to values (Taylor, 1993).

A second clarification regarding norms and values is methodological. The emergent perspective (e.g., Yackel & Cobb, 1996) and the work of Herbst and his colleagues (e.g., Herbst & Brach, 2006; Herbst et al., 2011) represent two primary traditions of norm research in mathematics education. What they share with much prior sociological work is the stance that norms are often most noticeable when they are breached. By observing how people identify and try to repair such breaches, one can identify the norms people treat as operative in a situation and the values by which they legitimize or delegitimize behavior. We contend that this occurs in mathematical practice. For instance, the publication of the computer-assisted four-color theorem proof violated the norms that proofs are transparent and do not rely on the author's experience that a reader cannot independently verify. The publication of this non-normative proof served as the impetus for a discussion on mathematicians' values about mathematical knowledge and the norms that uphold these values, especially with relation to proof (e.g., Hanna, 1995; Swart, 1980; Tymoczko, 1979).

A third clarification regards the extent to which norms and values determine behavior. We agree with Herbst et al.'s (2011) claim that norms are "neither ineluctable like physical laws nor compulsory like rules of a game. They are rather defaults, or tacit expectations about behavior that, if done, go without saying" (p. 226). Systems of related norms and values afford human activity by simultaneously facilitating routine activity while creating space for individual agency (e.g., Holland, Lachicotte, Skinner, & Cain, 1998). An individual member of a community may reject a value and so regularly deviate from a norm, but that individual will generally be compelled to acknowledge and justify the breach. This consistent act of positioning one's actions in relation to a perceived expectation is what marks the influence of the values and norms on individual behavior, rather than mere obedience to a rule. Thus, one can distinguish an individual's *personal* beliefs and expectations from what she *perceives* to be the values and norms of the community in which they participate. Even if an individual personally rejects a value or norm that she perceives their community to hold, that value or norm will still influence her behavior. She will either act in accordance to the community's norms so that the community accepts her intellectual contribution or she will feel obligated to explain why she is breaching this norm.

Much of the explanatory power of norms comes from this insight that social expectations often influence individual behavior not through direct enforcement of such expectations but rather through the individual's curtailment of their own activity in light of their expectation of others' judgments. This closely links proof and proving inasmuch as proofs are obviously the result of some proving process and the proving process takes into account the anticipated expectations to be placed upon the proof artifact that is produced. As our definitions above reflect, mathematicians and students enact proving not only to reflect their personal understandings and values but also to satisfy the expectations by which their work will be judged. While much mathematics education literature has shifted toward focusing on the proving process, we acknowledge that the proof is the interface between the prover's activity and anticipations and the reader's judgments. As such, we shall frequently discuss proof in this paper with a clear understanding that a proof is not and cannot be fully distinct from the process by which it was produced. Furthermore, disparities in teacher and student understandings of proving often become manifest in the student proof that serves as the interface between student anticipations of the norms of proving and their teachers' actual judgments.

3 Values of professional mathematicians

Although classroom practices are not, and cannot be, exact replications of mathematicians' proving practice, mathematicians' proving practice informs and constrains what classroom proving practice should be (e.g., Harel & Sowder, 1998; Herbst & Balacheff, 2009; Weber, Inglis, & Mejia-Ramos, 2014). In this section, we analyze mathematicians' proving practice by discussing mathematicians' values and the norms of proof in place to uphold these values. In the next section, we describe how the norms and values in mathematicians' practice inform and pose challenges for the creation of classroom proving practice.

The values and norms that we discuss below should be viewed as the conceptualization we use to make sense of proving practice, which we have generated based on other researchers' analyses of proofs (e.g., Burton & Morgan, 2000; Konoir, 1993; Selden & Selden, 2013), mathematicians' reflections on their own practice (e.g., Davis & Hersh, 1981; Devlin, 2003; Halmos, 1970; Knuth, Larrabee, & Roberts, 1989; Thurston, 1994), and philosophers' analyses of mathematical practice (e.g., Buldt, Löwe, & Müller, 2008; Fallis, 2002; Paseau, 2011; Steiner, 1978). We also incorporated the observations of mathematics educators (often not made on the basis of systematic investigation; see, e.g., Alcock & Simpson, 2002; Balacheff, 1988; Selden & Selden, 2003) to highlight beliefs about mathematical practice that we believe are common in the mathematics education community. We reiterate that what we report here is only our conjectured framework and we urge systematic research to assess the accuracy and utility of this framework for proof. We make recommendations for how this could be done in the concluding section of this paper.

At a broad level, the mathematical community aims to increase its list of statements that it believes to be true while minimizing the likelihood that this list contains a statement that is false. However, Fallis (2002) remarked that the goals of seeking truth and minimizing error are hardly unique to mathematics; most sciences share these goals.[3] What is more specific to mathematics is the desire for *a particular way of knowing* why mathematical assertions are true. As Paseau (2015) demonstrated, physicists and mathematicians would permit and desire different justifications of the *same* mathematical claim. In this paper, we consider four specific values we perceive as being held by the mathematical community[4]:

(1) Mathematical knowledge is justified by a priori arguments.
(2) Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
(3) Mathematicians desire to increase their understanding of mathematics.
(4) Mathematicians desire a set of consistent proof standards.

This list is non-exhaustive, but we argue that many of the norms that students are expected to adhere to when producing classroom proofs serve to uphold these values. As an important caveat, these values are often not actually achieved in mathematical practice for a variety of reasons. For the sake of brevity, we do not elaborate on this point here, but more detailed arguments can be found in Ernest (1991) and Fallis (2002). In the following four sub-sections, we discuss the nature of each value listed above and—for all but the last—identify norms that we claim are intended to uphold it.

3.1 Value 1: mathematical truth is a priori

Buldt et al. (2008) observed that mathematicians take pride in the fact that mathematics is regarded as a "very special science" whose results carry "the characteristic objectivity that other sciences lack" (p. 310). Buldt et al. attributed the current privileged status of mathematics to the work of the philosopher Immanuel Kant, who distinguished between a priori knowledge (knowledge or justification that is independent of experience) and *a posteriori* knowledge (knowledge or justification that is dependent on experience or empirical evidence). To Kant, only mathematics could produce new knowledge that was independent of experience, providing mathematical knowledge with a more secure footing. Weber and Mejia-Ramos (2013) elaborated on this common viewpoint. When scientists read a publication in their field, they sometimes have no recourse but to rely on the testimony of the authors of the paper regarding the empirical data presented. With mathematical proof, the situation is different. If a mathematician reads the result of a calculation in a paper, she need not trust the author but could, in principle, perform the calculation herself.

We emphasize that mathematicians' desire for a priori knowledge extends beyond the desires to obtain absolute certainty and to minimize the risk of error. As Fallis (2002) and Paseau (2011) illustrated, there are some statements in mathematics when empirical evidence

[3] Fallis (2002) acknowledged that mathematics might have a lower tolerance for error than other sciences, although some (e.g., Grcar, 2013) question whether this is actually true in practice.

[4] While subgroups like experimental mathematicians (e.g., Borwein, 2008) reject some of these values, the need to classify them as "experimental" and their publication of philosophical articles defending their consistent breaches of community norms both justify that these values are part of proving practice. One would not call someone a "deductive mathematician" as it would be redundant.

could provide more conviction than a proof because of the possibility that the purported proof contained an error that was not detected. Nonetheless, mathematicians will continue to seek proofs of such statements. Multiple norms about proof are in place to uphold the value a priori knowledge; we discuss two such norms below.

3.1.1 Justification in proof must be based upon stipulated definitions

In contemporary mathematics, a concept is defined by a stipulated constructive definition; that is, a well-defined criterion that unambiguously determines concept membership (Edwards & Ward, 2008). In mathematical proof, deducing properties about a concept must proceed from this definition or from other statements that had previously been deduced from this definition, rather than from unstated or assumed properties of the concept, a concept prototype, or other common representations associated with that concept (e.g., Alcock & Simpson, 2002). Such a viewpoint is consistent with Kant's view that a priori knowledge can be obtained from analytic statements, which Quine (1951) defined as "true by virtue of meanings and independent of fact" (p. 21). For mathematical theorems to have a similar claim to be a priori, the concepts within the theorem would need to be explicitly defined and the justification of the theorem would need to be based solely upon these definitions.[5]

3.1.2 Justification in a proof should be deductive and not admit rebuttals

The need for deduction follows immediately from the desire for a priori truth. Proofs are presented sequentially where the order entails logical or epistemic dependence (each claim may depend upon what comes before, but not on what comes later). Every warrant applied in a mathematical proof must be truth preserving. Mathematical statements and arguments are brittle to single counterexamples (Durand-Guerrier, 2008).

3.2 Value 2: mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author

This value is a corollary of the previous value—if mathematical truth is to be established by an a priori argument, it should not depend on the experiences of the author or the author's community (c.f., Selden & Selden, 2003). This provides some proofs with the feeling of being objective and atemporal (c.f., Balacheff, 1988). We identify two norms intended to uphold this value.

3.2.1 Mathematical proof is written without reference to author or reader's agency

Because the purpose of a mathematical proof is to describe objective properties of mathematical objects, how an individual comes to understand these objects is not relevant. Likewise, although some creativity and ingenuity may have been needed by the author to create (or "discover") a proof, the correctness of the proof can be judged independent of the process used

[5] This highlights Mariotti's (2006) point that a proof of a theorem is dependent upon a reference theory. The correctness of a proof is necessarily dependent upon the axioms, definitions, and rules of inference within that reference theory.

to create it. Davis and Hersh (1981) described the writing of the ideal mathematician as follows:

> His writing follows an unbreakable convention: to conceal any sign that the author or the reader is a human being. It gives the impression that, from the stated definitions, the desired result follows infallibly by a purely mechanical procedure. (p. 36)

In particular, in mathematical writing, the use of the pronoun “I” is avoided, unless the author’s identity had relevance (e.g., Knuth et al., 1989). Burton and Morgan (2000) inferred that, “the unwritten assumption here is that, in most cases, the author’s persona is not relevant” (p. 437). Burton and Morgan (2000) noted that the presence of human agency is often lacking in mathematical papers, although this convention is sometimes broken as a way of positioning oneself within the mathematical community.

3.2.2 A proof is an autonomous object, not a description of a problem-solving process

In a proof, the author does not usually describe the processes that she used to create the proof (e.g., Selden & Selden, 2013). A report of the problem-solving process is necessarily expendable; if it were not, the validity of the proof could not be judged a priori as it would depend upon the experiences of the author.

3.3 Value 3: proofs should increase mathematicians’ understanding

Some mathematicians argue that what they desire is not only a list of true mathematical theorems but also (and perhaps primarily) an understanding of the mathematical theory that they are studying (e.g., Thurston, 1994; Rav, 1999). Thurston (1994) specifically made this point as follows: “what we [as mathematicians] are doing is finding ways for *people* to understand and think about mathematics” (p. 162, italics were the author’s emphasis). Articulating exactly what constitutes mathematical understanding is an elusive theoretical problem that has vexed philosophers and mathematics educators for decades; the issue of how this understanding is constructed also remains open. Nonetheless, there is consensus that some mathematical proofs can help convey mathematical understanding, both for mathematicians (e.g., Steiner, 1978; Rav, 1999) and for students (De Villiers, 1990; Hanna, 1990).

Unfortunately, the quest of a priori truth and mathematical rigor is sometimes at odds with gaining understanding. Thomas Hales, a mathematician, observed:

> Traditional mathematical proofs are written in a way to make them easily understood by mathematicians. Routine logical steps are omitted. An enormous amount of context is assumed on the part of the reader. Proofs, especially in topology and geometry, rely on intuitive arguments in situations where a trained mathematician would be capable of translating those intuitive arguments into a more rigorous argument. In a formal proof, all the intermediate logical steps are supplied. No appeal is made to intuition, even if the translation from intuition to logic is routine. Thus, a formal proof is less intuitive, and yet less susceptible to logical errors. (Hales, 2003, as cited in Devlin, 2003, para. 37–38)

Hales lists several norms of proof that are in place to increase comprehensibility, including the omission of routine logical steps and the allowance of intuitive arguments, provided a knowledgeable mathematician could provide a more rigorous argument. To Hales, these norms do not affect the proof’s validity, or as the above excerpt indicates, pose only minor threats to the validity of the proof. Proofs that violate these norms might still be judged as "correct" by the community, but as Burton and Morgan (2000) documented, mathematicians may find these proofs to be insufficiently comprehensible to warrant publication or acceptability. Below, we discuss four examples of norms meant to increase comprehensibility.

3.3.1 Routine calculations and obvious justifications are omitted from a proof

As noted above, these are not included in a proof (see also Selden & Selden, 2013). One rationale for this is that a reader bogged down in extensive calculations would be less able to perceive the overarching arguments contained in the proof. Furthermore, because a trained mathematician can carry out such calculations or identify implicit warrants, the omitted details are understood as being available without being articulated.

3.3.2 Irrelevant statements are not presented in a proof

There is the expectation that when an inference or assumption is introduced in a proof, it will be built upon at a later point. As Selden and Selden (2003) noted, stating a true statement or adding an irrelevant assumption cannot render the proof invalid, but it is not ordinarily done. One reason is that the reader of a proof will assume all statements in the proof play some role in the final chain of argumentation. Adding irrelevant statements or assumptions will confuse the reader as she struggles to find out how this is relevant (Lai, Weber, & Mejía-Ramos, 2012).

3.3.3 Published proofs are typeset to reveal their mathematical structure

Konoir (1993) noted that mathematicians employ a number of typesetting conventions to reveal the structure of their proofs. For instance, portions of a proof that are indented indicate to the reader that a sub-proof is being discussed and that its contents are independent of the rest of the proof (implicitly suggesting that if the reader accepted the claim being proven, the reader can simply omit this part of the proof). The mathematicians in Lai et al. (2012) study indicated that they would center particularly important statements in their proof for emphasis.

3.3.4 Symbol choice follows conventions

In mathematical writing, close attention is paid to symbol choice (Halmos, 1970). Mathematicians avoid using the same symbol to refer to different objects in the same proof (Selden & Selden, 2013) and some symbols are typically used to refer to the same class of objects. For instance, "+" usually denotes a binary commutative operation, f usually refers to a function, and x usually refers to a variable. As Halmos (1970) humorously noted, you would not expect to read "let 6 be a group" even though such a statement would be syntactically permitted.

3.4 Value 4: mathematicians desire a consistent set of norms and practices

To the extent possible, it is desirable to have a consistent set of norms and practices across mathematical disciplines. In this sense, mathematicians have a *paradigm* (in the sense of Kuhn, 1962) for what constitutes an appropriate problem (provable theorem), method of solution (proof), and standards for evaluating proposed solutions (the aforementioned cluster of norms). While there is some variety in the standards of proof held by various subdisciplines, there is a general consensus among the mathematical community that conjectures become theorems when they are proven and the proofs should (to the extent possible) be a priori, verified independently of the author's experience, and written so that reasonable attempts are made to foster comprehension. Manin (1998) argued as much when he said:

> Epistemologically, all of us who have bothered to think about it know what a rigorous proof is. It has an ideal representation which was worked out by mathematical logicians in this century, but is now only more explicit and not fundamentally different from the notion Euclid had. (p. 154)

His claim that the original notion of proof is "not fundamentally different" but is now "more explicit" could be reframed to say that Euclid's *Elements* has always served as the paradigm; it was based on the same axiology and broad methodological principles as contemporary mathematics. Modern mathematicians have only changed in how they codified explicit norms. Having this shared paradigm permits *normal science* to occur, allowing mathematicians to make steady and shared progress without having to continually engage in methodological or philosophical debate (Kuhn, 1962).

4 Challenges with establishing classroom proving practice

Numerous mathematics educators and influential organizations desire that proof play a central role in mathematics classrooms at all levels (e.g., Common Core State Standards for Mathematics, 2012; Department of Education, 2013; NCTM, 2000; Stylianides et al., 2016). What should these classroom proving practices entail? One cannot expect the practices of mathematics classroom communities to perfectly mimic the practices of professional mathematical communities. Classroom communities are, at best, approximations of professional communities (e.g., Dowling, 1998). This is necessarily the case with proving practice, both because mathematicians have access to representational systems and conceptual tools that students lack (Weber et al., 2014) and because the needs of the classroom community sometimes differ from those of the mathematical community (Staples, Bartlo, & Thanheiser, 2012). Nonetheless, mathematics educators' pedagogical aims with respect to proof are informed by mathematicians' proving practice (Harel & Sowder, 1998; Herbst & Balacheff, 2009; Weber et al., 2014). In particular, one obligation that mathematics educators and teachers have is to avoid presenting proof to students in a way that *distorts* the proving practice of mathematicians (Herbst & Balacheff, 2009).

There is not a consensus among mathematics educators as to what classroom proving practice should be (e.g., Cirillo, Kosko, Newton, Staples, & Weber, 2015). Nonetheless, it appears that there is general agreement that (i) classroom proofs should be based on deductive reasoning and not on empirical generalizations (e.g., NCTM, 2000; Stylianides, 2007); (ii) at least for older students, classroom proofs about a concept should be based on the stipulated definition of that concept or other facts and theorems that were established as consequences of those definitions (e.g., Alcock & Simpson, 2002; Edwards & Ward, 2008); (iii) classroom proof should be more than a description of the proving process and some effort is required for students to transform intuitive arguments into deductive arguments that show how acceptable premises necessitate the conclusions that are drawn (e.g., Mamona-Downs & Downs, 2010; Zazkis, Weber, & Mejia-Ramos, 2016); and (iv) the methods of inference in a classroom proof should be methods of inference that mathematicians believe to be generally valid (e.g., Stylianides, 2007). Thus, many of the mathematician proof norms that we described above are analogous to the norms to which classroom proofs should also adhere.

There is ample evidence that students at all levels have difficulty producing these classroom proofs (e.g., Healy & Hoyles, 2000; Ko & Knuth, 2009; Moore, 1994; Weber, 2001; Weber & Alcock, 2004) and identifying which arguments should be classified as classroom proofs (e.g., Alcock & Weber, 2005; Healy & Hoyles, 2000; Selden & Selden, 2003; Weber, 2010). What is more revealing is that students have difficulties with the norms themselves. For instance, many students do not appreciate the role of definitions in the proofs that they write (Alcock & Simpson, 2002). They think that empirical or perceptual arguments should constitute an acceptable form of proof (e.g., Harel & Sowder, 1998). In general, many students are utterly perplexed by what constitutes a proof (e.g., Mamona-Downs & Downs, 2005). Even when students do follow these norms, they sometimes wonder why they are required to do so (e.g., Segal, 1999).

We argue that our analysis of proof with respect to norms and values can shed light on why this is the case. In classroom proving practice, students are being asked to adopt mathematicians' proof norms, but students may not perceive the mathematicians' values that those norms are intended to uphold. To the mathematical community, the norms that are in place represent an intellectual achievement. These norms are the mathematical community's solution, obtained through communal negotiation, to the difficult question of how the community can increase the likelihood or extent to which their shared values can be achieved. To students who do not share mathematicians' values, classroom proof norms represent arbitrary solutions, transmitted via imposition, to questions the students never asked and might not even consider meaningful. As Solomon (2006) emphasized, asking students to adapt norms without corresponding values can leave students feeling passive and marginalized with respect to their learning of proof.

The transposition of proof and proving into the classroom also must respect the unique values imposed by the pedagogical context. For instance, many researchers on teaching proving have sensibly advocated loosening various norms for the purpose of encouraging students' genuine insights (such as acknowledging generic proofs that rely on examples, but often display generalizable lines of inference, e.g., Ball & Bass, 2000). We find that attending to values and their related norms helps frame and understand this shift in classroom proving. We also think it could guide a much-needed discussion on how the transposition of proof and proving to the classroom may vary at different levels of instruction.

In what follows, we explore how analyzing values and norms helps address four questions relating to students' misunderstanding of or resistance to mathematical norms (sections 4.1–4.2), mathematicians' deficit interpretations of some students' proving abilities (section 4.3), and an overarching view of what it means to enculturate students into proving practice (section 4.4).

4.1 Do students prove to attain reliable knowledge or a priori knowledge?

The distinctions between the value of highly reliable knowledge and a priori knowledge has been discussed by some philosophers (e.g., Fallis, 2002; Paseau, 2011), but generally has been ignored by practitioners and mathematics education researchers. Mamona-Downs and Downs (2010) acknowledged this difference when they noted that "the point [of proof] is not so much about conviction, but how we can clarify the bases of the reasoning employed" (p. 2338). Elsewhere, we have argued that mathematics educators often conflate a priori knowledge and reliable knowledge, treating a convincing argument as the fundamental metaphor for what a proof is (Stylianides et al., in press; Weber & Mejia-Ramos, 2015).

Prior research demonstrates that many students struggle when they are asked to adhere to the norms upholding a priori and impersonal knowledge, particularly avoiding the use of empirical evidence and relying strictly on deductive evidence (Healy & Hoyles, 2000; Harel & Sowder, 1998). Prior studies indicate that some students may come to accept and act in accordance with these norms, particularly with avoiding empirical arguments (e.g., Brown, 2014; Healy & Hoyles, 2000; Segal, 1999) and visual arguments (Weber, 2010), even while these students think these norms serve no purpose.

In mathematics education, many researchers desire that students justify mathematical assertions with deductive reasoning and not empirical reasoning on the grounds that deductive reasoning alone is how mathematicians gain certainty in mathematical assertions (Harel & Sowder, 1998). If students are told that the justification for using deductive reasoning is that it alone eliminates the possibility of error, then one can see why students would be perplexed. There are situations in which empirical and authoritative evidence provide legitimate grounds for believing mathematical claims (and some mathematicians indeed believe mathematical claims on these grounds, as documented in Weber et al., 2014). Demonstrations on dynamic geometry software are common classroom occurrences where empirical evidence provides adequate grounds for believing a claim for both students and mathematicians (De Villiers, 2004).

It becomes especially problematic to say that in order to increase the reliability of our knowledge, proofs must be deductive while recognizing that students at all levels, even mathematics majors, have difficulty detecting errors in proofs (e.g., Healy & Hoyles, 2000; Selden & Selden, 2003; Weber, 2010). Hence, it is entirely sensible for students to question whether norms of deductive reasoning fully and uniquely uphold the value of having reliable knowledge.

4.2 Why should students desire methodological consistency?

Instructors of proof-oriented courses frequently present false proofs or deceptive diagrams intended to teach students the "dangers" of empirical induction, non-generalizable deduction, and diagrams. These spurious arguments are used to justify the norm that students should only employ inferential techniques that are generally valid (or explicitly justify why a technique is valid in this situation). For instance, a popular example in the mathematics education literature is to present students with the claim that "$1141n^2 + 1$ is not a perfect square for any natural number n," a claim that is only true for the first 10^{25} natural numbers. This is done so that students will cease arguing that a claim is true for all natural numbers by verifying the claim holds for a finite set of natural numbers. Harel and Sowder (1998) claimed that their students were unreceptive to this argument, although Stylianides and Stylianides (2009) and Brown (2014) found some success with this, if the activities for students are structured in a particular way.

Using false proofs to teach the dangers of non-deductive reasoning implicitly relies on the norm that methods that yield a false claim in one situation must be avoided or locally justified in *every* situation. This notion of methodological generalizability is quite common in mathematical practice. Logical validity, for instance, can be understood as designating whether a form of inference is generalizable across semantic content.

However, one can understand why a student would not accept the value of methodological consistency. In most situations, students are not obligated to use a method of inference in one situation to all future situations that they encounter. For example, if a student said, "the train was late for the last three days, so it will probably be late today," she should not feel compelled to say that anything that occurs three consecutive times will probably occur in another circumstance. If a student did not have the value of methodological consistency, the student may rightly question whether the "analogous false proof" arguments described above are compelling.

While there are clearly times that students violate norms in ways that render their arguments open to mathematical rebuttal, in many of these cases, students violate norms in amendable ways. For instance, when students attempt to prove a trigonometric identity by manipulating the equality to be proven with reversible algebraic steps, it is hard for students to accept a justification for why they must reorganize those steps to begin with one expression and manipulate it into the other. A teacher might choose to show spurious lines of algebraic inference that involve squaring both sides of an equation. However, for a student who is aware that each of their algebraic manipulations is indeed reversible, what further justification can be given for revising the derivation? We have frequently heard students make an objection such as, "but I never tried to justify that $-2 = 2$, so how is the bad argument that you are showing me even relevant?" The notion that the implicit warrant in an argument needs to be generalizable across semantic content only makes sense when looking for context-independent rules that will always work. Ultimately, mathematics teachers are asking students to adopt generalizable forms of inference and argumentation, which is a methodological value mathematicians maintain. It cannot itself fully be justified, especially in contexts where arguments of invalid form are nevertheless sound (see Durand-Guerrier, 2008, for a textbook example).

4.3 Why might some mathematicians who teach proofs believe students incapable of understanding proof?

When students do not recognize the mathematical values that mathematicians hold, they may not understand the norms imposed to uphold those values. One would thus expect to find mathematicians and students interacting according to different values to experience dissonance in their cross-cultural communication. Indeed, in many mathematicians' reflections on student learning, one finds just such evidence. When mathematics professors are interviewed about teaching proof-oriented mathematics, a number of them question whether all students were capable of learning the material (e.g., Alcock, 2010; Harel & Sowder, 2009; Weber, 2012). A common analogy that some mathematicians use is to compare learning proving practice to learning music, suggesting that some students are "tone deaf" when it comes to learning proving practice (e.g., Harel & Sowder, 2009; Weber, 2012). Consistent with this view, Halmos (1970) remarked (without citing evidence) that "to understand syllogism is not something that you can learn; you are either born with the ability or you are not" (p. 124). Hence, in this view, teaching proof-oriented mathematics to the unlucky students who are not born with that ability will not be possible.[6]

We strongly disagree with the view that there are students who are innately "tone deaf" with respect to proof. We also note that some mathematicians disagree with this viewpoint as well, such as Epp (2003) who has worked hard to develop instruction to help undergraduates learn logic and proof. Nonetheless, we think it is important to understand why it seems plausible to

[6] The work of Bieda (2010) suggested that this belief may be held by some elementary and secondary teachers as well. Bieda found that some middle school teachers in the USA devoted little attention to proving activities because they felt that only the most exceptional students were capable of writing a proof.

the mathematicians that we described above. We offer a hypothesized account for how these mathematicians' beliefs can develop. Students who are not cognizant of mathematicians' values may nonetheless learn to adhere to some of the norms with respect to proof. However, we hypothesize that when students do not recognize how these norms support *their* mathematical values, the norms will appear arbitrary. Therefore, they will not use the norms flexibly. They might misapply the norm in a novel situation or over-apply the norm in other situations. They also might not appreciate a mathematician's rationale for breaching a norm.[7] Using the musical analogy above, we might figuratively say that these students may be able to play the "right notes" but they will not hear "the mathematical music." However, we stress that this is not due to students lacking the innate ability to do advanced mathematics, but rather due to their operating with a different value system from mathematicians (c.f., Solomon, 2006). Further, as we argue below, it is not that the students' value system is necessarily deficient. This accords well with the pattern of cross-cultural miscommunication. Because mathematicians simply act within their value system, they may truly perceive it impossible to teach someone who fails to share those values. As with many cultural interchanges, it is common to attribute failed communication or understanding to qualities about the other person (e.g., cognitive ability) rather than to their cultural background (Solomon, 2006; Tannen, 1984).

4.4 Why should students adopt mathematicians' values?

Our preceding analyses highlight the importance of students adopting mathematicians' values if their classroom proof norms are going to be meaningful to them. However, this is fundamentally difficult as mathematicians' values are often held without justification. Consider the following passage from Paseau (2011) in which he argued that mathematicians' desire for proof does not mean that one cannot gain legitimate conviction from inductive evidence.

> Mathematicians prefer deductive evidence and actively look for it even in the presence of overwhelming inductive evidence. The reason for this is that *they are mathematicians and as such value deduction* […] Mathematicians go on seeking proofs of [conjectures] in the presence of overwhelming inductive evidence. (p. 144, italics are our emphasis)

In this passage, Paseau claimed that mathematicians value deduction (a hallmark of a priori justification) not because it offered more compelling accounts that theorems were true *but because they were mathematicians*. To us, this is a key point: part of enculturation into the community of mathematicians involves adopting the value of a priori justification via the deductive paradigm. Training students to adopt this value ultimately entails training students in an epistemology. We discuss how teachers might meet this challenge in the last section of this paper.

5 Implications

In this paper, we proposed a theoretical framework for the axiology and methodology of proving practice in terms of the values mathematicians seek to uphold and example norms to

[7] For instance, when lecturing, mathematicians' proofs violate standards of rigor that they use when evaluating students' proofs in an effort to increase student's understanding of the mathematical content (e.g., Lai & Weber, 2014). However, Lew (2016) demonstrated that students do not consider the role of context, specifically the differences between lecture, textbook, and student-generated proofs, when deciding how rigorous a proof should be.

support those values. Here, we discuss implications of our framework for investigating mathematical practice, conducting research in mathematics education, and teaching proof in mathematics classrooms.

5.1 Investigating mathematical practice

Philosophers of mathematical practice often rebut claims that proofs aim to achieve some epistemic value or that a norm for proof exists via counterexamples. For instance, the existence of computer-assisted proofs has been used to refute the idea that proofs aim to be a priori or that there is a firm rule that proofs must be strictly deductive (Fallis, 2002). It is telling that Laudan (1984) dismissed values that are not realistically achievable as "utopian" and therefore irrational; Laudan also described methodology in terms of "rules." Our framework challenges the legitimacy of dismissing values and norms via counterexamples. To us, norms are not rigid rules that guarantee that some epistemic value is achieved.[8] Instead, norms are default expectations that increase the likelihood of an epistemic value being achieved, but a mathematician can violate a proof norm provided that she supplies a good justification for doing so. Likewise, simply because mathematicians may sometimes act against a value in a particular situation does not imply that the value is not influencing their behavior more generally.

In this paper, our delineation of mathematical values and norms is admittedly conjectural. However, we believe that such claims can be placed on more secure empirical footing. First, to see if published proofs actually follow a norm (e.g., mathematicians avoid pronouns and references to personal agency), one can follow Selden and Selden (2013) in selecting a representative corpus of published proofs and see the proportion of time that this norm is in place. If the conjectured norm is followed most of the time, a researcher can better understand the norm by conducting breaching experiments as is advocated by Herbst et al. (2011). One can create proofs where this norm is violated and show this proof to focus groups of mathematicians. If the hypothetical norm were operative, we would expect the mathematicians to recognize the violation of the norm. In discussing how to repair the norm, we would expect them to reference their mathematical values. Alternatively, and more naturalistically, one could examine how papers that contain proofs in which a norm is violated are reviewed. If the norm were operative, we would expect the breach of the norm to be noted in the review.

5.2 Conducting research in mathematics education

At a broad level, we encourage the mathematics education community to devote more attention to the axiology of mathematics. We considered this in relation to proof, but researchers can do similar analyses with other activities. For instance, many recommend that students engage in mathematical modeling (e.g., CCSSM, 2012; Saxe & Braddy, 2015). Analyses of what constitutes a *good* mathematical model, *why* mathematicians create mathematical models, and *whether and how* mathematicians' models might differ from engineers' models would be an interesting research to undertake that could potentially have useful consequences for classroom design. Because mathematicians teach future engineers and scientists, how should the axiologies of those fields influence mathematics instruction?

[8] Indeed, we question whether in a realistic social scientific practice if there is any set of rules that *guarantee* an epistemic value is achieved.

In the specific case of proof, we challenge the assumption that students' violation of a proof norm is an indication of an innate deficit. For instance, if a student favors an empirical argument to a deductive one, many researchers attribute this to students failing to see the limitations of empirical arguments and the generalizability of deductive ones. This account might be accurate but other accounts are plausible as well. As a methodological principle, we suggest it is important to understand what the students are trying to accomplish with the arguments that they produce. Such students may have epistemic aims other than guaranteeing the truth of a claim or viewing the claim as a priori knowledge. For instance, an empirical argument might be sufficient if one only aims to show a claim is very likely (but not guaranteed) to be true (a point discussed in Weber & Mejia-Ramos, 2015). As Bieda and Lepak (2014) illustrated, eliciting such perspectives from students can make students appear far more rational than one might infer given only the products of their work.

5.3 Teaching proof in mathematics classrooms

At the heart of our cultural frame for mathematical practice is the view that students and teachers in proof-oriented classrooms are engaging in cross-cultural interactions according to distinct sets of values. The acknowledgement that values often remain implicit behind the norms intended to uphold them helps us make sense of the challenges inherent to proof-oriented instruction and students' apprenticeship into classroom proving practice. In line with the view that cultural understanding is generally embedded within cultural know-how, we maintain that proof instruction—as an apprenticeship in mathematical practice—should not shift away from proving practice itself. However, our framework of values and norms suggests that proof instruction must also seek to expose the underlying values that guide that practice. It is insufficient to train students in the norms of proof if students never develop some means of understanding and justifying their purpose. Alternatively, we do not advocate teaching the values of proof apart from the practice of proof.

Teaching at the level of axiology poses a particular challenge. We do not advocate explaining to students why they should adopt mathematicians' values, as this would involve recourse to philosophy or some extra-mathematical activity, which philosophers in the naturalistic tradition have abandoned as pointless (e.g., Maddy, 1997). Instead, we propose two mechanisms for aligning students' axiology with those of mathematicians. First, we anticipate that students are more likely to accept an epistemic aim if they experience that epistemic aim being achieved. Skemp (1976) made this point with relational understanding, describing relational understanding as organic. He postulated that once students experience genuine mathematical understanding in one context, they will value it and seek out similar understanding in other contexts. Experiencing understanding will endorse the mathematical value of understanding more than a logical argument for why this value is important. Having students work with norms initially should provide them with the opportunity to experience the values that the norms are in place to uphold (Pat Thompson, personal communication, September 18, 2006). In this case, the teacher can facilitate this process by encouraging students to reflect on the epistemic aims that they achieved in their mathematical work.

A second approach is to appeal to the values that were achieved in excellent instances of mathematics. This is based on the premise that although scientists, mathematicians, and philosophers often cannot explicate what great science is, they intuitively know it when they see it (Laudan, 1984; Worrall, 1988; although see Inglis & Aberdein, 2016, who question whether this is true with proof). Laudan (1984) noted that one way in which scientists adjust their axiologies is to align their axiologies with outstanding pieces of science. For instance, Laudan discussed how appeals to Newton's theory of motion, which relied on the process of gravitation that was not directly observable, convinced nineteenth century scientists to value scientific theories that relied on unobservable entities and theories that did not strictly explain complex phenomena in terms of simpler processes. This suggests that compelling instances of deductive proofs may be more valuable for conveying mathematical axiology than false proofs. Appealing to excellent proofs can help students appreciate the epistemic aims that proofs can achieve and the norms that support them. Mathematicians, of course, have access to a wide store of excellent proofs, which may be one reason why the values of proof seem so natural and intuitive to them. The challenge is for students to be in a position to witness and appreciate excellent pieces of mathematics themselves.

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. Thompson (Eds.), *Issues in mathematics education: Vol. 16. Research in collegiate mathematics education VII* (pp. 63–91). Providence, RI: American Mathematical Society.

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics, 22*(2), 28–34.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior, 24*(2), 125–134.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London, UK: Hodder & Stoughton.

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, constructivism in education* (pp. 193–224). Chicago, IL: University of Chicago Press.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education, 41*(4), 351–382.

Bieda, K. N., & Lepak, J. (2014). Are you convinced? Middle-grade students' evaluations of mathematical arguments. *School Science and Mathematics, 114*(4), 166–177.

Borwein, J. (2008). Implications of experimental mathematics for the philosophy of mathematics. In B. Gold & R. Simons (Eds.), *Proof & other dilemmas: Mathematics and philosophy* (pp. 33–60). Washington, DC: Mathematics Association of America.

Brown, S. A. (2014). On skepticism and its role in the development of proof in the classroom. *Educational Studies in Mathematics, 86*(3), 311–335.

Buldt, B., Löwe, B., & Müller, T. (2008). Towards a new epistemology of mathematics. *Erkenntnis, 68*(3), 309–329.

Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education, 31*(4), 429–453.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75.

Chinn, C. A., Buckland, L., & Samarapungavan, A. (2011). Expanding the dimensions of epistemic cognition: Arguments from philosophy and psychology. *Educational Psychologist, 46*(3), 141–167.

Cirillo, M., Kosko, K., Newton, J., Staples, M., & Weber, K. (2015). Conceptions and consequences of what we call argumentation, justification, and proof. In T. Bartell, K. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the Psychology of Mathematics Education* (pp. 1343–1351). East Lansing, MI: Michigan State University.

Common Core State Standards for Mathematics. (2012). *Standards for mathematical practice*. Retrieved from http://www.corestandards.org/Math/Practice/

Davis, P., & Hersh, R. (1981). *The mathematical experience*. Boston: Houghton-Mifflin.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*(1), 17–24.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology, 35*(5), 703–724.

DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*(2), 131–147.

Department of Education (2013). *Mathematics: Programmes of study: Key Stages 1–2* (National Curriculum in England). Retrieved from https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239129/PRIMARY_national_curriculum_-_Mathematics.pdf
Devlin, K. (2003). The shame of it. *Devlin's Angle*. Retrieved from http://www.maa.org/external_archive/devlin/devlin_06_03.html
Douek, N. (2009). Approaching proof in school: From guided conjecturing and proving to a story of proof construction. In F. L. Lin, F. J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *ICMI study 19: Proof and proving in mathematics education* (Vol. 1, pp. 148–153). Taipei, Taiwan: National Taiwan Normal University.
Dowling, P. (1998). *The sociology of mathematics education: Mathematical myths/pedagogic texts*. London: Falmer Press.
Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof. *ZDM, 40*(3), 373–384.
Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Argumentation and proof in the mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 349–367). Dordrecht, The Netherlands: Springer.
Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education MAA notes #73* (pp. 223–232). Washington, DC: Mathematics Association of America.
Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886–899.
Ernest, P. (1991). *The philosophy of mathematics education*. London: Routledge.
Fallis, D. (2002). What do mathematicians want? Probabilistic proofs and the epistemic goals of mathematicians. *Logique et Analyse, 45*(179–180), 373–388.
Grcar, J. F. (2013). Errors and corrections in mathematics literature. *Notices of the American Mathematical Society, 60*(4), 418–425.
Hales, T. (2003). *The Flyspeck Project Fact Sheet*. Retrieved from https://github.com/flyspeck/flyspeck/wiki/Flyspeck-Project-Fact-Sheet
Halmos, P. R. (1970). How to write mathematics. *L'Enseignement Mathématique, 16*(2), 123–152.
Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.
Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42–49.
Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Issues in mathematics education: Vol. 7. Research in collegiate mathematics education III* (pp. 234–283). Providence, RI: American Mathematical Society.
Harel, G., & Sowder, L. (2009). College instructors' views of students vis-à-vis proof. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *Teaching and learning proof across the grades* (pp. 275–289). New York: Routledge.
Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428.
Herbst, P., & Balacheff, N. (2009). Proving and knowing in public: The nature of proof in a classroom. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades* (pp. 40–64). New York, NY: Routledge.
Herbst, P., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction, 24*(1), 73–122.
Herbst, P., Nachlieli, T., & Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry? *Cognition and Instruction, 29*(2), 218–255.
Holland, D., Lachicotte, W. J., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, MA: Harvard University Press.
Inglis, M., & Aberdein, A. (2016). Diversity in proof appraisals. In B. Larvor (Ed.), *Mathematics cultures* (pp. 163–179). Dordrecht: Springer.
Knuth, D., Larrabee, T., & Roberts, P. (1989). *Mathematical writing*. Cambridge, UK: Cambridge University Press.
Ko, Y. Y., & Knuth, E. (2009). Undergraduate mathematics majors' writing performance producing proofs and counterexamples about continuous functions. *The Journal of Mathematical Behavior, 28*(1), 68–77.
Konoir, J. (1993). Research into the construction of mathematical texts. *Educational Studies in Mathematics, 24*(3), 251–256.
Kuhn, T. S. (1962). *The structure of scientific revolutions*. Chicago, IL, USA: University of Chicago Press.
Lai, Y., & Weber, K. (2014). Factors mathematicians profess to consider when presenting pedagogical proofs. *Educational Studies in Mathematics, 85*(1), 93–108.
Lai, Y., Weber, K., & Mejía-Ramos, J. P. (2012). Mathematicians' perspectives on features of a good pedagogical proof. *Cognition and Instruction, 30*(2), 146–169.
Laudan, L. (1984). *Science and values*. Cambridge, UK: Cambridge University Press.

Lew, K. (2016). *Conventions of the language of mathematical proof writing at the undergraduate level*. (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses Database.
Livingston, E. (2006). The context of proving. *Social Studies of Science, 36*(1), 39–68.
Maddy, P. (1997). *Naturalism in mathematics*. Oxford: Clarendon.
Mamona-Downs, J., & Downs, M. (2005). The identity of problem solving. *The Journal of Mathematical Behavior, 24*(3), 385–401.
Mamona-Downs, J., & Downs, M. (2010). Necessary realignments from mental argumentation to proof presentation. In V. Durrand-Gurrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 2337–2345). Lyon: Institute National de Recherche Pedagogique.
Manin, Y. I. (1998). Truth, rigour, and common sense. In H. G. Dales & G. Oliveri (Eds.), *Truth in mathematics* (pp. 147–159). New York, NY: Oxford University Press.
Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Rotterdam, The Netherlands: Sense Publishers.
Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249–266.
National Council of Teacher of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Paseau, A. (2011). Mathematical instrumentalism, Gödel's theorem, and inductive evidence. *Studies in History and Philosophy of Science Part A, 42*(1), 140–149.
Paseau, A. (2015). Knowledge in mathematics without proof. *British Journal for the Philosophy of Science, 66*(4), 775–799.
Quine, W. V. O. (1951). Two dogmas of empiricism. *Philosophical Review, 60*(1), 20–43.
Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica, 7*(1), 5–41.
Saxe, K., & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025*. Washington, DC: Mathematical Association of America.
Segal, J. (1999). Learning about mathematical proof: Conviction and validity. *The Journal of Mathematical Behavior, 18*(2), 191–210.
Selden, A., & Selden, J. (2003). Validation of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*(1), 4–36.
Selden, A., & Selden, J. (2013). The genre of proof. In M. Fried & T. Dreyfus (Eds.), *Mathematicians and mathematics educators: Searching for common ground* (pp. 237–260). New York, NY: Springer.
Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.
Solomon, Y. (2006). Deficit or difference? The role of students' epistemologies of mathematics in their interactions with proof. *Educational Studies in Mathematics, 61*(3), 373–393.
Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior, 31*(4), 447–462.
Steiner, M. (1978). Mathematical explanations. *Philosophical Studies, 34*(2), 135–151.
Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321.
Stylianides, A., Bieda, K., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutierez, G. Leder, & P. Boero (Eds.), *2nd handbook on the psychology of mathematics education* (pp. 315–351). Rotterdam: Sense Publishers.
Stylianides, G., & Stylianides, A. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352.
Stylianides, G., Stylianides, A. & Weber, K. (in press). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.
Swart, E. R. (1980). The philosophical implications of the four-color problem. *American Mathematical Monthly, 87*(9), 697–707.
Tannen, D. (1984). *Conversational style: Analyzing talk among friends*. New York, NY: Oxford University Press.
Tao, T. (2007). What is good mathematics? *Bulletin of the American Mathematical Society, 44*(4), 623–634.
Taylor, C. (1993). To follow a rule…. In C. Calhoun, E. LiPuma, & M. Postone (Eds.), *Bourdieu: Critical perspectives* (pp. 45–60). Chicago, IL: University of Chicago Press.
Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 30*(2), 161–177.
Tymoczko, T. (1979). The four-color problem and its philosophical significance. *The Journal of Philosophy, 76*(2), 57–83.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.
Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306–336.
Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematical Education in Science and Technology, 43*(4), 463–482.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209–234.
Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist, 49*, 36–58.
Weber, K., & Mejia-Ramos, J. P. (2013). The influence of sources in the reading of mathematical text: A reply to Shanahan, Shanahan, and Misischia. *Journal of Literacy Research, 45*(1), 87–96.
Weber, K., & Mejia-Ramos, J. P. (2015). The contextual nature of conviction in mathematics. *For the Learning of Mathematics, 35*(2), 9–14.
Worrall, J. (1988). The value of a fixed methodology. *The British Journal for the Philosophy of Science, 39*(2), 263–275.
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.
Zazkis, D., Weber, K., & Mejia-Ramos, P. (2016). Bridging the gap between visual arguments and verbal-symbolic proofs in calculus. *Educational Studies in Mathematics, 93*(2), 155–173.