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ORIGINAL PAPER

Students' distributive reasoning with fractions and unknowns

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Abstract To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning, a clinical interview study was conducted with 18 middle and high school students. The study included six students with each of three different multiplicative concepts, which are based on how students create and coordinate composite units (units of units). Students participated in two 45-min semi-structured interviews and completed a written fraction assessment. This paper reports on how 12 students operating with the second and third multiplicative concepts demonstrated distributive reasoning in equal sharing problems and in taking fractions of unknowns. Students operating with the second multiplicative concept who demonstrated distributive reasoning appeared to lack awareness of the results of their reasoning, while students operating with the third multiplicative concept demonstrated this awareness and the construction of more advanced distributive reasoning when they worked with unknowns. Implications for relationships between students' fractional knowledge and algebraic reasoning are explored.

Keywords Fractional knowledge · Algebraic reasoning · Distributive reasoning · Distributive property · Distributive partitioning scheme · Multiplicative concepts · Unknowns

Fractional knowledge is regarded as important for algebraic reasoning (Kilpatrick & Izsák, 2008; National Mathematics Advisory Panel [NMAP], 2008), in part because such knowledge is a basis for typical algebra topics such as rates and slopes of lines (Ellis, 2007), and because reasoning with fractions can promote use and awareness of mathematical properties important

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in algebra (Empson, Levi, & Carpenter, 2011). For example, generating strategies for multiplying fractions can help students develop distributive reasoning (Hackenberg & Tillema, 2009): To determine one-fifth of three-fourths of a yard, a student can take one-fifth of each of the three one-fourths of the yard. So, 1/5 of 3/4 is 1/5 of (¼ + ¼ + ¼). Students who reason this way may not be aware that they are using the Distributive Property of Multiplication over Addition, a fundamental mathematical property in both numerical and algebraic reasoning. However, thinking of one-fifth of three-fourths in this way can be considered to have an algebraic character because students who do so may develop awareness of the structure and functioning of numerical systems, one aspect of algebra (Bass, 1998; Empson et al., 2011).[1] Despite this potential for fruitful connections, little research has focused on how students' ways of operating with fractions may influence their algebraic reasoning (Lamon, 2007).

The purpose of this paper is to examine the distributive reasoning with fractions and unknowns of 18 middle and high school students who participated in a clinical interview study about relationships between students' fractional knowledge and algebraic reasoning. The study was designed to include six students operating with each of three different whole number multiplicative concepts. These concepts are based on how students coordinate units of units (composite units) (Hackenberg & Tillema, 2009; Steffe, 1994), and they have been found to influence students' fractional knowledge (Hackenberg, 2010; Steffe & Olive, 2010) and algebraic reasoning (Hackenberg & Lee, 2015; Olive & Caglayan, 2008). In our study, students operating with the first multiplicative concept showed little evidence of distributive reasoning, so in this paper, we focus on the 12 other students. The research questions addressed are the following:

1. How do students in the study demonstrate distributive reasoning in solving equal sharing problems (e.g., share five identical candy bars equally among seven people)?
2. How do students in the study demonstrate distributive reasoning in solving problems that involve taking fractions of multiple unknowns (e.g., determine the weight of one-seventh of five identical candy bars, each of weight h ounces)?
3. What differences are there in the distributive reasoning of students operating with the second and third multiplicative concepts (to be defined later)?

1 Students' distributive reasoning with whole numbers, fractions, and unknowns

Although there are multiple perspectives on the relationship between arithmetical and algebraic knowledge (Linchevski & Livneh, 1999), a popular view across countries is that early algebraic reasoning is about making abstractions and generalizations of arithmetical and quantitative reasoning and systematically representing those abstractions and generalizations, not necessarily initially with standard algebraic notation (e.g., Kaput, 2008; Kieran, 2006). This view manifests in recent research on distributive reasoning with whole numbers and fractions (e.g., Empson, et al., 2011; Russell,

[1] Multiplying with whole numbers can also help students construct and generate awareness of a distributive operation; we are not claiming that only multiplying with fractions can do so.

Bastable, & Schifter, 2011). For example, Russell and colleagues articulate a case of fifth grade students learning to adjust whole number products after rounding, which requires the use of the distributive property. Specifically, a student who was to compute 17×36 rounded the numbers to compute 20×40; he then adjusted by subtracting 3 and 4 from 800. Whole class discussion led students to adjust by subtracting three products, 17×4, 36×3, and 3×4, because 17×36 can be seen as one of four partial products that result from multiple uses of the distributive property when multiplying 20×40, i.e., $20 \times 40 = (17 + 3)\ (36 + 4) = 17 \times 36 + 17 \times 4 + 3 \times 36 + 3 \times 4$. Russell and colleagues argue that this reasoning is itself algebraic, because students can use similar reasoning to determine that $(a + b)^2$ is not equal to $a^2 + b^2$.

Some researchers have investigated students' distributive reasoning with fractions by exploring how students equally share multiple items (e.g., Empson, Junk, Dominguez, & Turner, 2006; Lamon, 1996; Steffe & Olive, 2010; Streefland, 1991). For example, when children share two identical cakes equally among three people, they may share each cake equally among three people to arrive at a share for one person consisting of one third of a cake plus another one third of a cake. This reasoning can be notated as follows, even if students do not yet use such notation: $1/3 \times 2 = 1/3\ (1 + 1) = 1/3 \times 1 + 1/3 \times 1 = 1/3 + 1/3 = 2/3$. This notation indicates a distributed multiplicative process to achieve a goal: The student is taking a fraction (one-third) of a composite unit (two cakes) by taking that fraction of each of the single units that make up that composite unit (each single cake). Researchers argue that this reasoning is itself algebraic because it demonstrates the use of a fundamental mathematical property and can be a basis for similar reasoning with unknowns or variables, e.g., for reasoning that one-third of $2x$ is two-thirds of $1x$ (Empson et al., 2011). However, students who reason distributively with whole numbers and fractions, as portrayed in this research, are not yet operating explicitly on unknowns or variables.

Indeed, a second facet of learning to reason algebraically involves learning to reason with standard algebraic notation in lieu of reasoning with numbers and quantities (Kaput, 2008). However, research has shown that treating letters as representations of unknowns, and then arithmetically operating on those unknowns, are both significant challenges for students (Kieran, 2006). Secondary school students tend to ignore letters, substitute specific values for letters, treat letters as labels of objects, use letters as an alphabetical code, or treat each letter as having the value 1 (e.g., Booth, 1984; Küchemann, 1981; MacGregor & Stacey, 1997). Researchers have also distinguished between arithmetic and algebraic uses of letters (e.g., Küchemann 1981; Slavit, 1999; Vlassis, 2002). For example, an arithmetic use of letters means a student may use them but only because she thinks of them as temporary placeholders for numbers, not as mathematical objects on which to operate. In contrast, an algebraic use of letters means the student does not focus on concrete referents but can sustain operating on the letters as unknowns.

This body of research would support exercising caution in assuming a simple transition from reasoning distributively with whole numbers and fractions to reasoning distributively with unknowns. At the same time, based on the view of algebra we have put forward, it would be sensible to investigate what students abstract from their distributive reasoning with whole numbers and fractions when working with and on unknowns. In this paper, we focus on distributive reasoning with fractions because, to our knowledge, little research has been done to understand how the kind of distributive reasoning described above with fractions may be drawn upon when students write algebraic expressions or equations with explicit unknowns. That is, if students solve equal sharing problems, such as sharing five loaves of bread equally among eight people, to what extent is that useful for them in determining one-eighth of $5h$ in an algebraic problem?

Researchers have found that the strategy of sharing each single unit into the number of equal shares one is trying to make (in the above example, partitioning each of the five loaves of bread into eight equal parts) to be common among elementary and middle school students (Empson et al., 2006; Lamon, 1996). In a study with fourth through eighth grade students, Lamon concluded this strategy occurs fairly naturally, but composing the resulting parts into units in relation to various other units in the situation does not. For example, eighth grade student Robert shared four identical pizzas fairly among three people and determined that each person got 1 and 1/3 pizzas, so he created the share in relation to 1 pizza as a unit. However, he also concluded that each person got 1 and 1/3 of the total amount of pizza, which is incorrect.

Steffe and Olive (2010) call the mental action that underlies the above equal sharing strategy a *distributive partitioning operation*. A distributive partitioning operation involves taking a fraction of a composite unit by taking a fraction of each unit of that composite unit, such as taking one-eighth of five loaves by taking one-eighth of each of the loaves that make up the five loaves. However, a student who uses this operation will not necessarily have worked out the results of the operation, i.e., that one-eighth of five loaves is five-eighths of one loaf.

A distributive partitioning operation is one component of a *distributive partitioning scheme*, the highest level of seven-level framework for fragmenting (Steffe & Olive, 2010). A situation of this scheme includes a goal to share multiple (identical or non-identical) items into some number of equal shares, e.g., share five loaves of bread represented by rectangles (bars) equally among eight people. The activity of the scheme involves implementing distributive partitioning operations: partitioning each of the items into the same number of equal parts and distributing one part from each of the items to make the shares. In the case of identical loaves, students view one share as both one-eighth of five loaves and five-eighths of one loaf—these two views are identical to them. If students also reverse this scheme (Liss, 2015), then they can justify that one share is an equal share by iterating that share the requisite number of times to produce the original composite unit. In the bread example, students who have constructed a *reversible distributive partitioning scheme* justify five-eighths of one loaf is one-eighth of five loaves by iterating the share eight times to produce the five loaves.

A reversible distributive partitioning scheme goes beyond what Lamon (1996) could attribute to the fourth through eighth grade students in her study. In addition, Steffe & Olive (2010) found that only two out of 10 third through fifth grade students in their 3-year teaching experiment were on the verge of constructing a distributive partitioning scheme by the end of the study. Making the coordinations among units represented by a distributive partitioning scheme may be an important feature in distributive reasoning with unknowns because developing quantitative meaning for one-eighth of $5h$, where h is, say, the weight of a loaf of bread, requires operating on both $5h$ and h and determining the result in relation to these two different unknown quantities.

2 A quantitative and operational approach

2.1 Quantitative reasoning

Following Thompson (1993, 2010), we conceive of quantitative reasoning as a basis for helping students build fractional knowledge and algebraic reasoning. A *quantity* is a property of one's concept of an object or phenomenon. To conceive of a quantity requires a person to conceive of a measurement unit, of the property as subdivided into these units, and of a process for enumerating these units to find a value of the quantity. Approaching fractions from a quantitative perspective means opening possibilities for students to conceive of fractions as measureable extents, or lengths; these lengths may represent other quantities as well (e.g., weight). Approaching algebraic reasoning from a quantitative perspective means that unknowns are quantities for which a value is not known, but for which a value could be determined.

2.2 Operations and schemes

As indicated in the literature review, our work is also based on conceiving of mathematical thinking in terms of people's mental actions, or *operations* (von Glasersfeld, 1995). Critical operations for fractional knowledge include *partitioning*, or marking a quantity into some number of equal parts, as well as *iterating*, or repeatedly instantiating a part to make a larger fraction (Steffe & Olive, 2010). Operations are the components of *schemes*, goal-directed ways of operating that consist of an assimilated situation, activity, and a result (von Glasersfeld, 1995). We take students' *reasoning* to be the functioning of their schemes and operations in ongoing interaction in their experiential worlds. So, we define *distributive reasoning with fractions* as the functioning of students' schemes and operations that include at least a distributive partitioning operation, but not necessarily a distributive partitioning scheme.

We conceive of a *distributive partitioning operation with an unknown* to involve taking a fraction of multiple unknowns by taking a fraction of each unknown, such as taking one-seventh of three identical weights h by taking one-seventh of each weight h. However, a student who demonstrates this reasoning will not necessarily have worked out that the result is both three-sevenths of $1h$ and one-seventh of $3h$. Because the results are not necessarily worked out for the student, the use of a distributive partitioning operation with unknowns is *not* equivalent to the formal distributive property. If a student did have these results worked out and could justify them via iteration, we would attribute a reversible distributive partitioning scheme with unknowns to the student. We view a reversible distributive partitioning scheme to underlie the formal distributive property with fractions.

2.3 Multiplicative concepts

For us, a *concept* is the result of a scheme that people have *interiorized*—i.e., reprocessed so that the result is available prior to operating. We use students' whole number multiplicative concepts as one tool (not the only tool) for understanding differences in how students build fractional knowledge (e.g., Hackenberg, 2010). These multiplicative concepts are based on the number of levels of units a student has interiorized, and to progress from one concept to the next requires a major reorganization of operations (Steffe & Cobb, 1988).[2] Here, we discuss the second and third of three multiplicative concepts that have been identified in prior research (Hackenberg & Tillema, 2009; Steffe, 1994), because students operating with these concepts are the focus of this paper.

Students operating with the second multiplicative concept (MC2 students) have interiorized two levels of units, which means that prior to operating, they see whole numbers as units of units, or composite units. For example, 7 is a unit of 7 ones. Furthermore, these students can coordinate three levels of units in activity (Steffe, 1994). For example, they can distribute the units of a composite unit of 7 across each unit of a composite unit of 5, creating 35 as a unit of 5 units, each containing 7 units. Although they have created 35 as a three-levels-of-units structure, in further operating 35 becomes for them just a unit of 35 ones (a composite unit). These students have the potential to treat a length as a unit containing some number of equal units, and they can do so prior to operating in a situation. For example, these students can treat a length that represents 1 m as a unit containing 7 units—a unit of units structure—without having to actually make the partitions. These students can also make three levels of units in activity in this context. For example, they can insert 3 parts into each of the 7 parts in the 7/7-m segment and determine that they have made 21 parts in all. However, once they do so, the 21/21-m segment is not likely to retain its structure as unit of 7 units, each containing 3 units (Hackenberg & Tillema, 2009).

Students operating with the third multiplicative concept (MC3 students) have interiorized three levels of units, which means they can view whole numbers as units of units, each of which contain some number of units—a unit of units of units structure. For example, prior to activity, MC3 students can structure 35 as a unit of 5 units, each containing 7 units. In addition, prior to activity, these students can treat lengths as a unit containing some number of units, each of which contains some number of units. So, in the length example above, MC3 students can do what MC2 students do, but they can also retain views of the 21/21-m segment as a unit of 7 units each containing 3 units. Furthermore, they can switch to viewing the segment as a unit of 3 units each containing 7 units. Being able to flexibly switch between such unit structures is critical for constructing a variety of fraction schemes (Hackenberg, 2010; Steffe & Olive, 2010).

Prior research has indicated that a distributive partitioning scheme requires the interiorization of three levels of units (Izsák, Jacobson, de Araujo, & Orrill, 2012; Steffe & Olive, 2010). For example, let us say a student who has constructed a distributive partitioning scheme is sharing three identical candy bars equally among seven people. This student will take one-seventh of each of the three bars by partitioning each bar into seven equal parts (Fig. 1, top bar). Pulling out one-seventh of each bar results in a three-part bar (Fig. 1, middle bar). To determine that this result is identically three-sevenths of one bar and one-seventh of three bars requires viewing the three bars as a unit of 3 units each containing 7 units, so that 1 bar is a unit of 7 units to which the three-part bar is compared. However, it also requires switching unit structures to view the three bars as a unit of 7 units each containing 3 units, so that the three-part bar is compared with all of the candy (Fig. 1, bottom bar). Thus, the student has to operate with and switch between two different three-levels-of-units structures, which is a hallmark of the third multiplicative concept.

[2] Social interactions of many kinds, including classroom interactions, are essential in these transitions, although generally speaking, interiorization of units cannot be directly taught (Steffe & Olive, 2010).

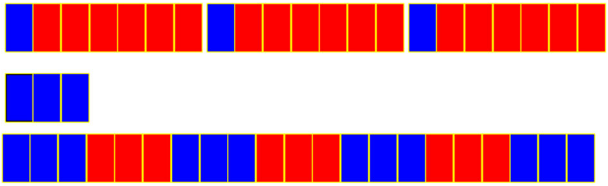


Fig. 1 Two three-levels-of-units structures involved in taking one-seventh of three bars

3 Method

A clinical interview study was chosen as appropriate methodology to explore the research questions because a strength of clinical interviewing is "the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, 2000, p. 547). Although interview studies with small numbers of students cannot provide responses to our research questions based on statistical inference, in-depth analysis of interview data can provide important insights into the questions that can guide future research into assessment and instructional design.

3.1 Participants and data collection

Seven seventh grade students, 10 eighth grade students, and one 10th grade student participated in the study.[3] Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts. Six students with each multiplicative concept were invited to participate. Table 1 shows the course enrollment of the MC2 and MC3 students, the focus of this paper. All students had experienced some instruction in their mathematics classes on unknowns and on use of the distributive property. We chose to work with students from different classes for two reasons. First, we aimed to work with both MC2 and MC3 students, and we did not find MC2 students in the algebra classes. Second, none of the school mathematics classes took a quantitative approach to algebraic reasoning. All classes used traditional materials, and algebra instruction focused on developing procedures with algebraic notation. So, it was not clear that students' quantitative operations were going to be well developed in their school mathematics classes or that learning from these classes would be a strong resource for students in our interviews. We acknowledge that students' experiences with multiplication and fractions in their mathematics classes, beginning in elementary school, influenced their activity in the interviews. However, our focus was on their activity with our quantitative approach, which we expected to be unfamiliar to the students. Indeed, we intended to capitalize on making the familiar unfamiliar in order to prompt new insights (Mason, 2011).

Students participated in two 45-min semi-structured interviews: a fractions interview and an algebra interview. Thirty-two of the 36 interviews were conducted by the first author; the second author conducted 2 interviews for each of 2 MC3 students. When not acting as interviewer, the authors assisted with and observed the interviews in order to provide another perspective on the interaction. All students completed the fractions interview prior to the

[3] The 10th grade student was selected because the original intent of the study was to include middle and high school students. However, due to scheduling constraints, we could not interview other high school students. So, we interviewed more middle school students to achieve the targeted number of participants, 18.

Table 1 Course enrollment of the MC2 and MC3 students

Course	MC2	MC3
Seventh grade class for struggling students	1	
Seventh grade class for advanced students	1	2
Eighth grade pre-algebra	4	
Eighth grade algebra		3
High school algebra (10th grade student)		1
Total	Two seventh and four eighth	Two seventh, three eighth, one 10th

algebra interview, but the time between interviews varied from about 3 weeks to 4 months, with an average time of just less than 2 months.[4]

The interview protocols were refined in a prior pilot study, and quantitative situations were used as a basis for all problems. The protocols were designed so that the reasoning involved in the fractions interview could be drawn upon for solving problems in the algebra interview. For example, in the fractions interview, students were posed this problem:

F2. *Equal Sharing of Multiple Identical Bars Problem:* Here are five identical candy bars (congruent rectangles). Can you show how to share them equally among seven people and determine the size of the share for one person? Draw the amount one person gets.

In the algebra interview, students were posed a similar problem but with each bar assigned a weight of h ounces:

A1. *Weight of Multiple Identical Bars Problem:* Here are five identical candy bars (congruent rectangles). Each candy bar weighs some number of ounces. Let's say that $h=$the weight of one bar. How much does one-seventh of all the candies weigh? Write an expression for this result.

In working on A1, students were also encouraged to draw a picture to demonstrate how they determined or viewed their expression.

In addition to the two interviews, students completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge. This paper focuses on the interview data; results from the written assessments confirm interpretations of students' fractional knowledge from the interviews (Hackenberg & Lee, 2015).

3.2 Data analysis

Each interview was video-recorded with two cameras, one focused on the interaction between the researcher and student, and one focused on the student's written work. The videos were mixed into one file for analysis, which occurred in two overlapping phases. The aim of the first phase was to formulate a model of each student's fraction schemes, as well as each student's expression and equation writing and solving. Toward this end, the researchers viewed video

[4] Four months between interviews was unusual and occurred for only one student who switched schools in the middle of the year. It took some time to locate this student and schedule a second interview. The median length of time between interviews was a little under 2 months, and the modes were 3 weeks and 1 month.

files and took detailed notes (Cobb & Gravemeijer, 2008), which included transcriptions of major portions of each interview; summaries of each student's work on each problem in each interview; and memos (Corbin & Strauss, 2008, p. 117) consisting of interpretations and conjectures about a student's work on a particular problem. Then, the researchers wrote a narrative summary of the model for each student, shared the summaries, and reconciled discrepancies in interpretations and conjectures. These models provided the basis for responding to the first two research questions for this paper.

In the second phase of analysis, the researchers looked across the students to articulate differences in how students with different multiplicative concepts solved the problems in each interview. This phase included assessing differences in students' distributive reasoning with fractions and unknowns. The first author wrote syntheses across students with each multiplicative concept, drawn from the narrative summaries, which she discussed with the second author. These syntheses allowed us to address the third research question in this paper.

4 Analysis and findings

Four MC2 students and only one MC3 student demonstrated evidence of a distributive partitioning operation in the fractions interview. Based on prior research (e.g., Empson et al., 2011; Lamon, 1996), we expected that MC2 students would demonstrate distributive reasoning of some kind, but we were surprised that only one MC3 student did so. We present data and analysis to support this finding. Then, we use students' work from the algebra interview to help illuminate it.

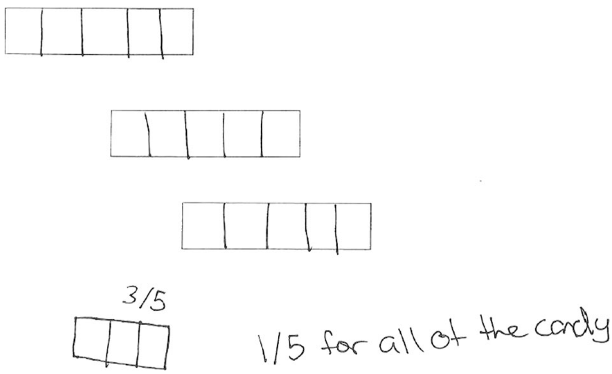
4.1 Distributive reasoning in the fractions interview

MC2 students Of the four MC2 students who demonstrated evidence of a distributive partitioning operation in the fractions interview, two demonstrated it immediately upon sharing three identical candy bars equally among five people, which we will call problem F1, as well as in working on F2 (sharing five identical bars equally among seven people, stated in Sect. 3.1). The other two students showed evidence of a distributive partitioning operation over time in working on F1 and F2. We present an example of each of these two types of evidence.

On F1, eighth grade student Lisa immediately divided each of the three bars into five equal parts and said that each person would get one-fifth from each bar. She seemed certain, and she called the share that she drew "three-fifths" (Fig. 2). When asked whether the share was three-fifths of all the candy or three-fifths of one bar, she said it was three-fifths of all the candy. Upon further questioning over 1.5 min, she seemed uncertain about the size of the three parts in relation to all the candy. Finally, she said three-fifths referred to "kind of both," because the share was three-fifths in relation to one bar and one-fifth from each of the three bars. Twenty seconds later, when asked if each person got three-fifths of all the candy, she said, "uh, no!" She noted that there were five people, so one share was one-fifth of all the candy. Directly following F1, she solved F2 similarly, but with fewer conflations of units.

This data excerpt demonstrates that Lisa had constructed a distributive partitioning operation: To take one-fifth or one-seventh of multiple identical items, she took one-fifth or one-seventh of each of those items. However, for some time in working on F1, she named the fair share three-fifths of *all* the candy. Our interpretation is that to determine a share, Lisa made a unit of units of units in activity: All of the candy was a unit of three units (bars), and she

Fig. 2 Lisa's solution of F1

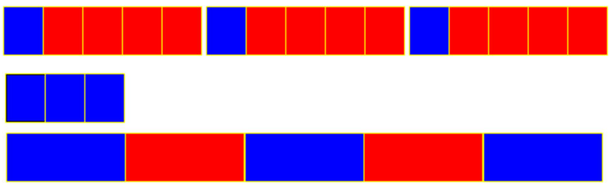


partitioned each of those units into five units (parts) (Fig. 3, top). Once she determined a share was three parts, she named the result three-fifths, which from our perspective means she also viewed each bar as a unit of five units and named the three parts in relation to a single bar. However, a single bar as a referent unit seemed to be out of Lisa's awareness, since she maintained that the share was three-fifths in relation to all of the candy as the referent unit.

Lisa said that one person's share was one-fifth of all the candy only after some time, and after she thought again about five people sharing all of the bars. Then, she viewed the share as one-fifth, consistent with viewing all of the bars together as a quantity partitioned into five equal parts (a unit of units). She did not produce this view easily, and it seemed to require her to re-initialize her thinking about the situation. So, Lisa appeared to view the three-part share in relation to two different organizations of two levels of units: Implicitly, she named the share in relation to one bar as a unit of five units even though she continued to view all of the candy as the referent. Then, she named it in relation to all of the candy as a unit of five units in which she did not focus on further partitioning of shares (Fig. 3, bottom). Lisa did not demonstrate a view of the result in relation to all of the candy structured as a unit of five units, each containing three units (contrast Figs. 1 and 3). Another eighth grade MC2 student operated similarly. The work of these two students is evidence that MC2 students can produce two different views of the result of using a distributive partitioning operation, but they demonstrate conflations in the process that indicate their structuring of the situation is different from a student who has constructed a distributive partitioning scheme.

In contrast with Lisa, eighth grade student Sheila was one of two MC2 students who developed a distributive partitioning operation during the interview. Sheila did not reason distributively on F1. On F2, she first partitioned each of the five bars into fourths, and she numbered the fourths from one to seven sequentially, as if trying to determine whether there would be an equal number of parts for each of seven people (Fig. 4, left). Then she tried again but partitioned each bar into fifths, again numbering sequentially. On her third try, she partitioned each bar into sevenths, numbered them sequentially, and said that each person

Fig. 3 Analysis of Lisa's solution of F1



would get "five-sevenths." That is, each person would get all the pieces with the same number on them (person 1 gets all the 1's, Fig. 4, right).

Sheila's work on F2 indicates the possible emergence of a distributive partitioning operation in the interview. She introduced the idea of numbering the pieces to try to establish an equal amount for each of the seven people, and she adjusted the number of pieces in each bar until she had a total number of parts in each bar that exhausted all the candy and allowed her to give an equal number of same-size parts to each of seven people. Thus, at least in solving this specific problem, Sheila determined that she could make equal shares for seven people by taking one part out of each bar. During the algebra interview, Sheila used this way of operating to solve other problems, confirming an emergence of a distributive partitioning operation that did not apply only to solving F2. We return to this point later.

MC3 students In contrast, the only MC3 student to demonstrate a distributive partitioning operation on F1 and F2 was eighth grade student Liam. On F1, Liam immediately said that you could divide each bar into five parts, but "that would be kind of cumbersome." So, he drew his three bars stacked vertically and then partitioned them vertically, all at once, into fifths (Fig. 5). He drew out one column as the share for one person and called one share 3/15 and then one-fifth of all the candy. When asked for the size of the share in relation to one bar, he paused and then said "three-fifths." His justification was that one column was the same as three parts horizontally, which was three-fifths of one bar. He solved F2 similarly (Fig. 6).

Although Liam's process was different from Lisa's and Sheila's, it still implies the construction of a distributive partitioning operation because one column consists of one part from each bar. However, Liam may not have been aware of the distributive pattern in his thinking. For example, when asked in F2 whether the result of his calculation to determine the fraction of all the candy, one-seventh, made sense, he pointed out that the seven shares in the five bars could be seen in five horizontal rows (five-sevenths of each bar) and then two more columns, each containing five parts. He did *not* point to each of the seven columns to show the seven shares (Fig. 6). So, although Liam produced two views of his results, we conservatively attribute only a distributive partitioning operation to him at this point.

The other five MC3 students did not reason distributively in the fractions interview. Four students made a total number of parts that was a multiple of the number of shares they were trying to make and divided by the number of shares. For example, in F2, they partitioned each bar into 7 parts to make 35 parts and then divided by 7 to get 5 parts as 1 share. The fifth MC3 student, eighth grader Suzanne, tried a fractional share and reasoned with what was leftover. In F2, she gave a share of two-thirds to each of five people because there were five bars, but five-thirds was leftover, which was too much for two more people. Then, she tried shares of three-fourths, four-fifths, and five-sevenths. This solution represents sophisticated reasoning, but it is not distributive reasoning as we have characterized it.

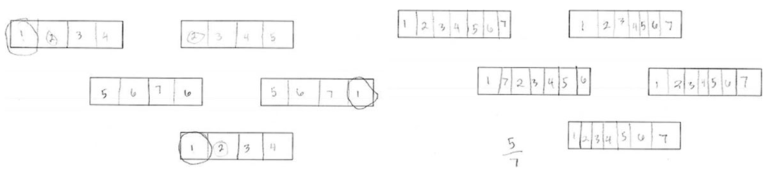
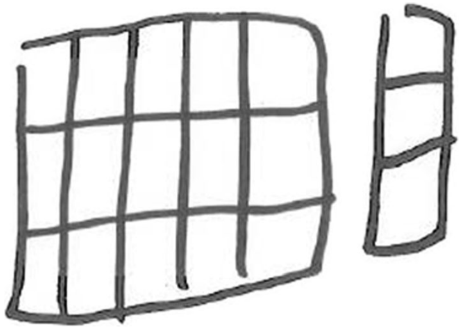


Fig. 4 Sheila's initial and final solutions of F2

Fig. 5 Liam's solution of F1



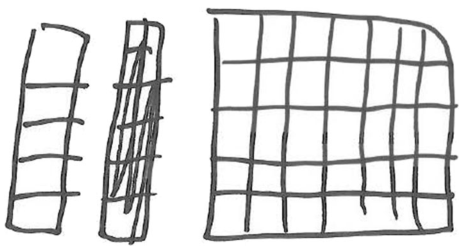
4.2 Distributive reasoning in the algebra interview

In contrast with their work in the fractions interview, on A1 in the algebra interview only two MC2 students, Lisa and Sheila, demonstrated distributive reasoning with unknowns. However, their work with unknowns revealed limitations in their distributive reasoning. Also in contrast, in the algebra interview, four MC3 students demonstrated distributive reasoning with unknowns. First, we present Lisa's response to A1, followed by Sheila's response. Then, we discuss the ways of operating of the MC3 students on A1, again highlighting the work of Liam and Suzanne.

MC2 students Lisa solved A1 by partitioning each of the five bars into seven equal parts and saying that she would take "one from each." However, when asked to draw out that amount, she believed that it would be a whole bar. After she had insisted on this result for over 2 min, the interviewer counted the pieces when taking one from each bar, and Lisa said that the result would only be five-sevenths of a bar. When asked how she knew that amount was one-seventh of all of the candy, Lisa repeated her solution process but also indicated she was not sure. The interviewer asked, "Is there any other fraction name you would give to that [5-part bar], if you were talking about the amount from all the candy?" After a 5-s pause, Lisa said, "five-sevenths." Then, the interviewer asked whether the result would be more or less than the weight of one bar, *h*. Lisa said less because it was less than one bar. When asked for an algebraic expression, she said she did not know how to write it, and she wrote "H<."

This piece of data sheds more light on the nature of Lisa's distributive reasoning. Her initial response to A1 was similar to her initial response to F1 and F2, allowing us to corroborate that Lisa had constructed a distributive partitioning operation. However, on A1, she did not appear to realize for some time that her action would have to produce *five* pieces; she thought it would produce a whole bar. So, she appeared to lack awareness of the result of her operation. In

Fig. 6 Liam's solution of F2



addition, once she had made the result, a five-part bar, she did not appear to have any way outside of repeating her initial action to justify that the share she produced was one-seventh of all the candy. In fact, on A1, it is not clear that Lisa viewed the result as one-seventh of all the candy. It was as if the initial problem of finding one-seventh of all of the bars became lost to her once she arrived at her result, five-sevenths, and this time, we infer that she did not re-initialize the situation to view all of the bars as a unit of 7 units. We account for her activity as we did with her work on F1 and F2, by appealing to her multiplicative concept.

The analysis across the two interviews means that Lisa's distributive reasoning with fractions and unknowns was limited to an initial use of a distributive partitioning operation but a lack of awareness of the results of using this operation. In addition, Lisa's work on A1 indicates that she did not know how to represent her idea of five-sevenths of the weight of one bar (or all bars) with algebraic notation. Using a fraction as a multiplier of a known or unknown is challenging (Behr, Harel, Post, & Lesh, 1993; Hackenberg & Lee, 2015), so this finding is not a surprise. It does not necessarily indicate a lack of distributive reasoning with unknowns, because unknowns can be reasoned about without the use of algebraic notation. However, it does indicate that Lisa did not have many ways to represent taking fractions of an unknown, which countermands that she was explicitly operating on unknowns.

On A1, Sheila operated similarly to Lisa in that she immediately divided each bar into seven equal parts and colored the first part of each bar. When asked for the weight of that, she said "There are five, so five-sevenths of one candy bar." The interviewer asked her to write an expression using h to show the weight of this amount. Sheila wrote $5/7h$. In explanation, she said that she had multiplied 5/7 and h. So, in contrast with Lisa, Sheila demonstrated knowledge of how to use algebraic notation to represent five-sevenths of an unknown.[5]

Because Sheila had worked so swiftly on A1, the interviewer posed A2:

A2. *Weight of Three Different Bars Problem:* There are three candy bars on the table, each of different weight (three non-congruent rectangles). The first bar weighs a ounces, the second bar weighs b ounces, and the third bar weighs c ounces. How much does three-fifths of all the candy weigh? Write an expression to show your result.

Sheila partitioned each of the three bars into five equal parts, colored the first piece of each bar, and wrote "$3/5b$" (Fig. 7, lowest inscription, left). Then, she tried to fit the bars together in some way. The interviewer clarified that the bars had no particular relationship to each other and decided to ask Sheila to find just one-fifth of the weight of all of the candy.

Sheila then wrote that the weight was "$3/15b$" (Fig. 7, middle left inscription). When asked what one-fifth of the first bar weighed, Sheila answered "$1/5a$" (Fig. 7, upper left inscription). The interviewer asked similar questions for the other two bars, and Sheila answered similarly. When asked about the weight of all three parts together, Sheila pointed to her expression "$3/5b$." The interviewer then asked her to write an expression that included a, b, and c, using her ideas about the weights of one-fifth of each bar. Sheila wrote "$3/5\ c\ b\ a$" (Fig. 7, lowest right inscription) and said, "multiplying 3/5 by the third bar, by the second bar, and by the first bar."

Sheila's work on A2 helps illuminate the nature of her distributive reasoning that appeared so fluidly in her work on A1. In working on A2, like Lisa, Sheila did not appear to be aware of the results of her activity. Furthermore, on A2, Sheila's work does not indicate that she viewed one-fifth of the weight of all the candy either as the sum of one-fifth of the weight of each bar,

[5] Sheila and Lisa were taking the same mathematics class, eighth grade pre-algebra.

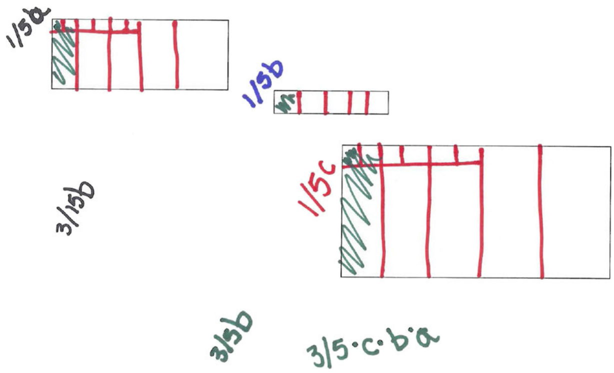


Fig. 7 Sheila's solution of A2

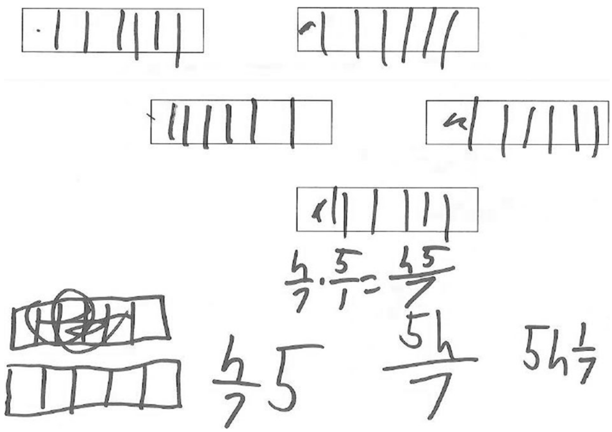
or as one-fifth of the sum of the three weights. We account for this response by appealing to Sheila's multiplicative concept. Once she had partitioned each bar in A2, the situation likely became a unit of 15 units: All of the candy was a unit partitioned into 15 parts, despite their different sizes. Since Sheila was trying to find the weight of one-fifth of all the candy, taking 3 parts out of 15 parts and producing a response such as "3/15*b*" was quite reasonable. Alternately, thinking of the result as "3/5*b*" likely stemmed from combining one-fifth of each bar without regard to size.

One difference between Sheila and Lisa was Sheila's use of 5/7 as a multiplier of an unknown in A1. However, what this notation meant to Sheila is debatable based on her work on other problems in the algebra interview. In A2 and in other problems Sheila concatenated unknowns and numbers into products that did not appear to have quantitative meaning for her. For example, in representing two unknown heights where one height was five times another, Sheila wrote equations such as $5x \cdot 5m = a$, where x represented one height, m represented the other height, and a represented the "answer." Given this characteristic of Sheila, we do not have strong evidence to conclude that her use of algebraic notation to represent distributive reasoning was necessarily more advanced than Lisa's.

MC3 students Liam's work on A1 demonstrated a distributive partitioning operation more clearly than his work on F2. He wrote the result as $(5h)/7$ and $5h \cdot 1/7$ (Fig. 8, lower right). When asked to show that in the picture, he partitioned each bar into sevenths and said to take one from each, gesturing to the first part from each bar. He drew out this amount (Fig. 8, lower left). When asked how much that was of h, he said "five-sevenths" and wrote $h/7 \cdot 5$ (Fig. 8, left). So, his notation indicates two views of the result of his operations: The result was one-seventh of $5h$ and also one seventh of h taken five times. Liam viewed these expressions as "the exact same thing, basically." We attribute his view that one-seventh of all five weights and five-sevenths of the weight of one bar were identical to his ability to switch between different three-levels-of-units structures as we have described.

However, in his work on A2, Liam only arrived at $(a+b+c) \times 3/5$ to express three-fifths of the weight of three bars of different weights. Despite several opportunities in which the interviewer probed how he would share the non-identical bars equally among five people, he said he could not make the fair shares because the bars were unequal in weight, and he did not ever articulate three-fifths of the weight of the bars as $3/5a + 3/5b + 3/5c$. So, to Liam, we attribute a distributive partitioning scheme with fractions and unknowns when operating with identical units only. Furthermore, we do not attribute a reversible distributive partitioning

Fig. 8 Liam's solution of A1

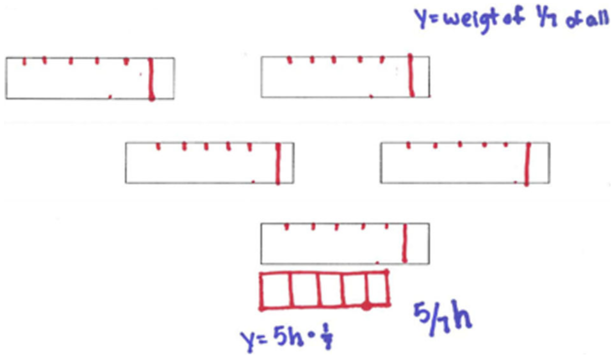


scheme to him because he did not explicitly justify the results of his distributive partitioning scheme using iteration. Two other MC3 students operated similarly to Liam.

On A1 and A2, Suzanne demonstrated distributive reasoning even more solidly than Liam did. On A1, she first wrote $5h \cdot 1/7$ as her expression (Fig. 9, lower left inscription). Then, she made a drawing in which she partitioned each bar into seven equal parts, marking the last part of each bar (Fig. 9). She said you could put all of those parts together to make one-seventh of all the candy. When asked, she drew out that amount and wrote an expression for its weight, $5/7h$ (Fig. 9, lower right inscription). The interviewer asked whether the two expressions were the same, and Suzanne noted that multiplying five times one-seventh produced five-sevenths. Unfortunately the interviewer did not press her on a justification using her picture, in part because Suzanne seemed convinced that the two expressions each represented one-seventh of the weight of all the candy.

On A2, Suzanne wrote two expressions for the weight of the three different bars and explained that the first one, $3/5(a+b+c)$, meant that you added all the bars together and took three-fifths of it. The second one, $(3/5a+3/5b+3/5c)$, meant that you took three-fifths of each weight and then summed these weights. Again, she seemed convinced of this equivalence but did not justify it using her picture. So, to Suzanne we attribute a distributive partitioning scheme with unknowns and with both identical and non-identical units.

Fig. 9 Suzanne's solution of A1



5 Discussion and conjectures

In this section, we summarize responses to our first two research questions and respond to our third research question about differences in distributive reasoning between MC2 and MC3 students. Then, we discuss an implication and two conjectures from this study about relationships between students' fractional knowledge and algebraic reasoning.

Four out of six MC2 students demonstrated a distributive partitioning operation in solving equal sharing problems, although not a distributive partitioning scheme. In the algebra interview, these students clearly demonstrated that they did not distinguish the fractional meaning of the result in relation to a single bar versus multiple bars or have a way to justify their method. The main issue was that after partitioning each bar and making the result, MC2 students did not appear to view all of the bars as a unit consisting of units of units, where each unit of units was the size of the result (e.g., they did not structure five bars into a unit of 7 units each containing five parts). Doing so requires being able to switch between three-levels-of-units structures (Fig. 1), which is outside of MC2 students' ways of operating. Instead, the MC2 students appeared to view the result in relation to one two-levels-of-units structure at a time: in relation to just one bar as a unit of units or in relation to multiple bars as a unit of units without maintaining the smallest partitions (Fig. 3). We conclude that MC2 students may construct distributive partitioning operations but not distributive partitioning schemes with or without unknowns. This conclusion is consistent with other researchers who have found that students distribute partitioning across multiple items but are not aware of all the relationships that ensue (Empson et al., 2006; Lamon, 1996; Liss, 2015; Steffe & Olive, 2010).

In contrast, in our study only one of six MC3 students demonstrated a distributive partitioning operation in solving equal sharing problems, while four demonstrated distributive reasoning with unknowns. To these four students we can attribute a distributive partitioning scheme with unknowns and identical units, and an awareness of the results of their activity (with non-identical units as well, in the case of Suzanne). Our account for this claim is based on the students' abilities to switch between three levels of units in organizing parts, bars, and multiple bars. Unlike Steffe and Olive (2010), we found that four out of six MC3 students had constructed a distributive partitioning scheme, but we were working with older MC3 students. We conjecture that MC3 students can construct this scheme given instruction organized toward this end. Recent research has provided initial confirmation of this conjecture, while also determining other constructive resources involved in constructing this scheme (Liss, 2015).

With respect to how students in this study used algebraic notation to represent and operate on unknowns, we can say the following: MC2 students' use of letters to represent distributive reasoning with unknowns may not have been algebraic (Küchemann 1981; Slavit, 1999; Vlassis, 2002) even when the notation appeared correct from our perspective, as in the case of Sheila. One use for algebraic notation is to represent the results of operating, and so without awareness of these results, there may have been little for the MC2 students to represent with letters, from their perspective. In contrast, MC3 students appeared to use letters algebraically in the sense that they could relate them to their pictures and transformations of quantities, while also assessing their expressions as equivalent without direct reference to their pictures. An open question is the nature of the interaction between (a) students' construction of distributive partitioning operations and schemes with unknowns and (b) students' construction of ways to use algebraic notation to represent their distributive reasoning. This issue merits further investigation to understand how instruction toward (a) and (b) can be mutually supportive.

Our study is exploratory in the following sense: We do not expect the exact patterns we reported here to always emerge. For example, we do not expect MC2 students to always demonstrate distributive partitioning operations with knowns, and we do not expect MC3 students *not* to demonstrate distributive reasoning with knowns. However, what we found in this study points to an implication and two related conjectures that merit further examination. The implication is that student work on problems involving unknowns and fractions, such as A1 and A2, can shed more light on the nature of students' distributive reasoning with fractions. One reason is that working with quantitative unknowns involves nested units coordinations that cannot be simplified as they can sometimes be with knowns (Hackenberg & Lee, 2015). This implication can be helpful for teachers because it suggests that problems involving unknowns and fractions could be a tool for assessing students' distributive reasoning with fractions.

The first related conjecture is that MC2 students may show more clearly the lack of distributive partitioning schemes when they work with unknowns because of the nested unit coordinations involved. However, it is not yet known how work with unknowns and fractions might influence their progress. Might a focus on unknowns facilitate distributive reasoning across both domains for them? Another important question for MC2 students is what will promote the interiorization of three levels of units. That question is exceedingly complex because of the significant reorganizations involved (Steffe & Olive, 2010), but there is some evidence that work with multiple nested unit coordinations is helpful in promoting such progress (Norton & Boyce, 2015).

The second related conjecture is that MC3 students may show more clearly their construction of distributive partitioning schemes when they work with unknowns because they can rise to the challenge of nested unit coordinations. When working with relatively small knowns, it is possible that relevant numerical relationships are already structured multiplicatively for them. For example, for some middle school MC3 students, 35 is automatically five 7s and seven 5s. If they see this numerical structure automatically, the need to reason distributively with knowns may be lessened. Both conjectures require further research. If they prove robust, they support the close connection between students' fractional knowledge and algebraic reasoning that has been suggested by other researchers (e.g., Empson et al., 2011; Kilpatrick & Izsák, 2008) and policy makers (e.g., NMAP, 2008), while also underscoring that the connection can be two-way. That is, in reasoning algebraically with fractions and unknowns, students may construct new ways of operating with fractions, depending on how tasks and instructions are organized.

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