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Putting the unit in pre-service secondary teachers' unit circle

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Abstract We discuss a teaching experiment that explored two pre-service secondary teachers' meanings for the unit circle. Our analyses suggest that the participants' initial unit circle meanings predominantly consisted of calculational strategies for relating a given circle to what they called "the unit circle." These strategies did not entail conceiving a circle's radius as a unit of measure. In response, we implemented tasks designed to focus the participants' attention on various measurement ideas including conceiving a circle's radius as a unit magnitude. Against the backdrop of the participants' actions on these tasks, we characterize shifts in the participants' unit circle meanings and we briefly describe how these shifts influenced their ability to use the unit circle in trigonometric situations.

Keywords Unit circle · Trigonometry · Pre-service secondary teachers · Measurement · Teaching experiment · Quantitative reasoning

1 Introduction

The unit circle is central to the study of trigonometric functions, with many historical developments in and applications of trigonometry occurring in circle settings (Bressoud, 2010). Yet, researchers (e.g., Akkoc, 2008; Moore, 2013, 2014; Thompson, 2008; Weber, 2005) have argued that students' and teachers' difficulties in trigonometry partially stem from impoverished connections between trigonometric functions and the unit circle. In this study, we seek to better understand individuals' meanings for the unit circle by characterizing two pre-service secondary teachers' (henceforth referred to as students) thinking during a teaching experiment on the unit circle. We extend previous work in this area by describing relationships between the students' unit circle meanings and their reasoning about measurement. After providing relevant background knowledge and a *conceptual analysis* (Thompson, 2008) of the unit circle that incorporates *quantitative reasoning* (Thompson, 1990), we illustrate the students' meanings for the unit circle

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upon entering the study. We then discuss shifts in the students' unit circle meanings including a discussion on how understanding a circle's radius as a unit magnitude enabled the students to understand the unit circle as representative of quantities' measures (or values) on all circles. We close with general observations drawn from our analyses.

2 Research on unit circle meanings

In summarizing 31 undergraduate students' understandings of trigonometric functions, Weber (2005) noted the students' meanings primarily consisted of step-by-step procedures and calculations that required given cues. For example, when asked to approximate the sine function for specified input values, some students could not proceed without a given triangle or circle labeled with numbers necessary for performing calculations. Furthermore, when the students provided approximations, they had difficulty creating triangles or circles to justify their approximations. Weber (2005) concluded:

> The first limitation in students' understanding concerns the role that geometric figures played in their understanding of these functions. Clearly relating trigonometric functions to appropriate geometric models is important for understanding these functions…What these students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric situations. (p. 103)

Reiterating Weber's findings, a pervasive finding across the literature base on trigonometric functions is that students' and teachers' difficulties stem from their developing only superficial connections among circle contexts and trigonometric functions (Akkoc, 2008; Brown, 2005; Thompson, Carlson, & Silverman, 2007; Topçu, Kertil, Akkoç, Yilmaz, & Önder, 2006). As a specific example, Akkoc (2008) argued that impoverished meanings for the unit circle and radian measures inhibit students from using circle contexts to define trigonometric functions on the real numbers.

Despite the apparent connection between students' and teachers' difficulties in trigonometry and their lacking sophisticated meanings for the unit circle, we are not aware of a study in which the primary purpose was a detailed examination of individuals' unit circle meanings. Rather, authors of the aforementioned work have foregrounded characterizing individuals' meanings for trigonometric functions while providing underdeveloped implications for students' unit circle meanings. Most relevant to our study, Hertel and Cullen (2011) and Moore (2014) recently illustrated the potential benefits of introducing trigonometric functions through an approach incorporating quantitative reasoning (Thompson, 1990, 2011). An approach incorporating quantitative reasoning emphasizes students' construction of measurable attributes (e.g., angle measures and directed lengths in the context of circular motion) and relationships between these attributes (e.g., modeling how these attributes vary in tandem) (Moore, 2014). Collectively, Hertel and Cullen's and Moore's findings imply that engaging in quantitative reasoning supported the students' spontaneous use of circles (and triangles) in trigonometric situations.

Because Hertel and Cullen (2011) and Moore (2014) did not intrinsically focus on their students' unit circle meanings, we extend their work (and the broader literature base) in two novel ways. First, we detail a perspective of the unit circle with foundations in quantitative reasoning. Second, we provide an examination of students' unit circle meanings when exposed to an approach that attempts to develop the posed perspective.

3 Understanding and meaning

Our efforts to characterize student thinking are rooted in Thompson and Harel's description of *understanding* and *meaning* (Thompson, Carlson, Byerley, & Hatfield, 2014; Thompson & Harel, in preparation; Thompson, Harel, & Thomas, 2015), which has foundations in Piagetian notions of actions, schemes, assimilation, and accommodation (Piaget, 2001). *Understanding* is an in-the-moment cognitive state of equilibrium that results from (successful) assimilation to a scheme. *Meaning* refers to the actions and schemes that an individual anticipates or enacts in the moment of understanding. For instance, when presented with the phrase *the unit circle*, a student might anticipate drawing a circle of arbitrary size and taking the magnitude of the radius as a unit of measure for other quantities, thus obtaining a radius with a measure of 1 unit. Or, the student might recall the unit circle chart labeled with specific coordinates and (often called *special*) angles (Fig. 1) and seek to execute calculations (e.g., multiplication or division by the radius measure) to convert between measures on the unit circle and measures on a circle with a radius length not equal to 1. In either case, the student created an understanding by assimilating the phrase *the unit circle* to an organization–the meaning–of actions and schemes.

In addition to our interest in characterizing students' meanings, we seek to characterize shifts in students' meanings. Shifts in students' meanings can occur in the event that a person experiences a state of disequilibrium (i.e., when assimilation to a scheme produces an unexpected result, which might stem from the lack of schemes or operations necessary to reach an anticipated or successful result) that is reconciled through a cognitive reorganization or construction. This cognitive reorganization or construction is referred to as an *accommodation* (Piaget, 2001; von Glasersfeld, 1995). Based on the pervasiveness of student difficulties with the unit circle, we conjectured that the students involved in the study would experience states of disequilibrium when encountering instruction focused on the unit circle in ways described in the following section. Thus, we were interested in the accommodations made by the students.

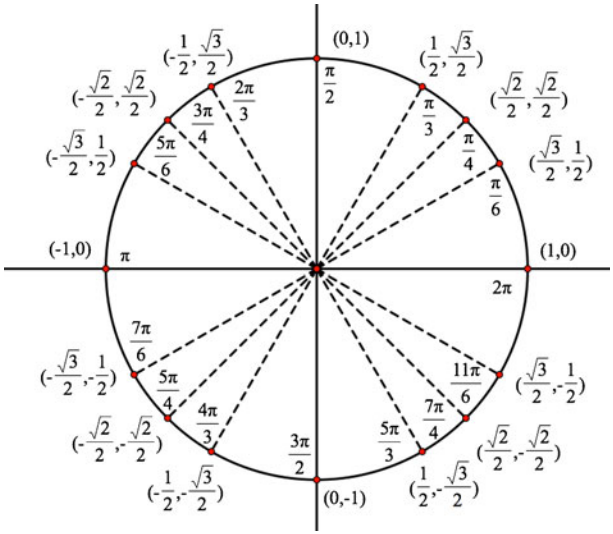


Fig. 1 A common representation of the unit circle

We note that there are numerous perspectives on mathematical understanding (and meaning), and a review of these perspectives is beyond the scope of this paper. We adopt Thompson and Harel's system for a few primary reasons. Most prominently, their system seeks to be non-judgmental with respect to the correctness or normativity of an individual's understandings (Thompson et al., 2014), thus avoiding an emphasis on judgments of depth or levels of understanding. Relatedly, their system does not approach understanding as representing (or involving interpretations of) features of an objective reality or mathematics (Thompson, 1994, 2013), which differs from those approaches to understanding or knowing that make explicit distinctions between internal and external representations (e.g., Sierpinska (2003)). Researchers have found approaching so called external representations and their interpretations as irreducible to be particularly useful when characterizing students' measurement schemes (Steffe & Olive, 2010; Thompson et al., 2014), which forms a partial focus of this study. Lastly, we interpret Thompson and Harel's system to be most suitable to our goal of characterizing students' in-the-moment understandings without attempting to make more general claims about students' understandings or imposing pre-defined stages or levels (e.g., Breidenbach, Dubinsky, Hawks, and Nichols (1992) and other APOS work).

4 A measurement and equivalence class perspective on the unit circle

It is common for textbook authors and educators, both internationally and in the United States, to define the unit circle as a circle with a *radius of one*. Attention is rarely given to what the number '1' represents, how '1' might be related to a presented circle's radius that is not equal to '1' (e.g., a circle with a radius of 57 ft), or how a radius of '1' relates to measuring arcs in radii (Moore, 2013, 2014). In the case that the unit circle and a radius of one are connected with trigonometric ratios, a quantitative meaning is often not provided for these ratios (cf. measuring a length relative to another length). A likely consequence of this common approach is that students do not interpret '1' as the result of some measurement process with an associated unit (Moore, 2013; Thompson, 2008).

Ambiguity of values of coordinate pairs on the unit circle (Fig. 1) in textbooks likely stems from what these values are intended to represent. Values on the unit circle can be thought of as multiplicative relationships between lengths (e.g., arcs and directed lengths) and the corresponding circle's radius. Furthermore, these values are equivalent across all circles. In other words, just as a radian measure can be thought of as an equivalence class of arcs that stems from abstracting quantitative relationships (Moore, 2013), values typically labeled on the unit circle can be thought of as equivalence classes involving both arcs and directed lengths. Because equivalence classes are dimensionless, a linear unit is not associated with the values on the unit circle.

Not associating a linear unit with values on the unit circle may provide an explanation for textbook authors' and educators' vague treatments of these values, but, from a development standpoint, understanding the unit circle as conveying equivalence classes entails conceiving a circle's radius as a unit magnitude and anticipating that such a unit leads to numerically equivalent measures for all circles (Fig. 2). By assimilating a presented circle with a specific radius to such a meaning, one can use (or anticipate the use of) the radius length as the unit magnitude for other quantities, thus creating an instantiation of the unit circle. In this case, values of coordinate pairs represent linear measures in radius lengths (or in radii). When no particular circle is specified–as is intended by the unit circle–the unit circle can be thought of as

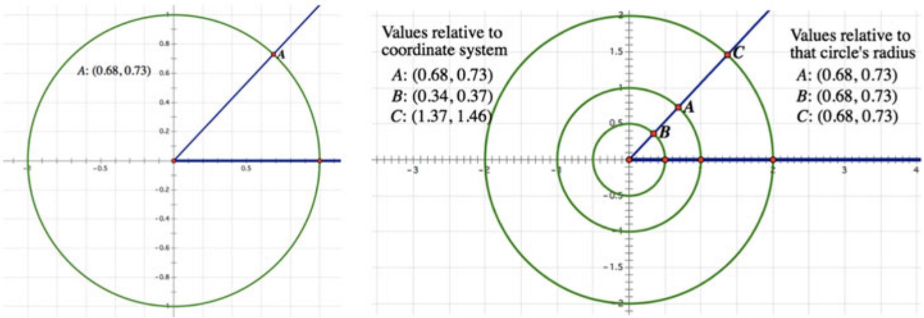


Fig. 2 The unit circle as an equivalence class and measuring coordinates in radii

the result of picking an arbitrary circle and using that circle's radius as the unit magnitude.[1] In this case, values of coordinate pairs represent equivalence classes. Both cases involve meanings that entail conceiving the circle's radius as a unit magnitude.

We interpret a meaning for the unit circle that builds on using a circle's radius as a unit magnitude and the abstraction of multiplicative relationships to have several benefits. The meaning supports: (a) defining the unit circle in a way such that coordinates and radian angle measures on the unit circle are associated with a unit magnitude (i.e., the radius); (b) connecting the outputs of trigonometric functions to ratios and giving meaning to said ratios (i.e., measuring in radii); and (c) constructing a meaning for the unit circle that encompasses a circle whose radius length is given in any unit other than radii. To illustrate, consider a circle with a radius of 4.2 ft. One can reason that the radius is a magnitude that is 4.2 times as large as the magnitude of a foot. If the radius is now thought of as the unit magnitude, it follows that measures in radii will be 1/4.2 times as large as corresponding measures in feet. Hence, to convert a measure in feet to a measure in radii, one divides the measure in feet by 4.2 (or multiply by 1/4.2), yielding a radius length of 1 radii and numerical values equivalent to those on the unit circle. Figure 3 shows a general situation with 1, y/r, and x/r emerging as measures in radii.

With the above meanings in place, one can understand the sine and cosine functions as relating quantities' measures in radii or, more generally, the equivalence classes conveyed by the unit circle. To illustrate, one can think of the equation $\sin(\pi/6) = 0.5$ as conveying that for a counterclockwise arc $\pi/6$ radii from the 3 o'clock position on *any* circle, the endpoint of that arc is 0.5 radii above that circle's horizontal diameter. With the radius conceived as a unit, the size of the circle does not influence the input–output pairs represented by the sine and cosine functions. In the case that quantities' measures on a circle are given in some unit other than radii (Fig. 4), the sine function can still be used to relate the two quantities' measures in radii (e.g., $\sin\left(\frac{12}{6}\right) \approx \frac{5.456}{6}$).

The above perspective on the unit circle, which emphasizes quantitative reasoning through a focus on measurement and relating quantities, informed our work with the students during the study. Despite our conjecture that the aforementioned analysis provides a productive meaning for the unit circle, we were not sure how such meanings might develop when

[1] This presents one of the difficulties in the teaching and learning of the unit circle. As soon as a circle is put on paper, a circle with a particular radius length has, in a way, been specified.

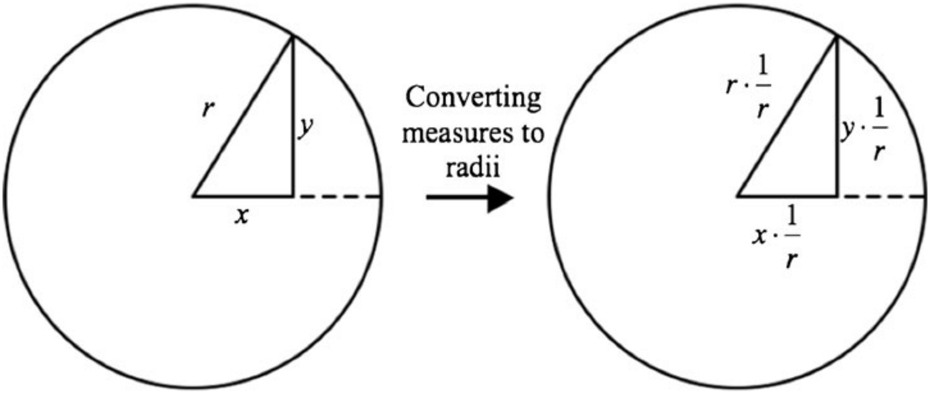


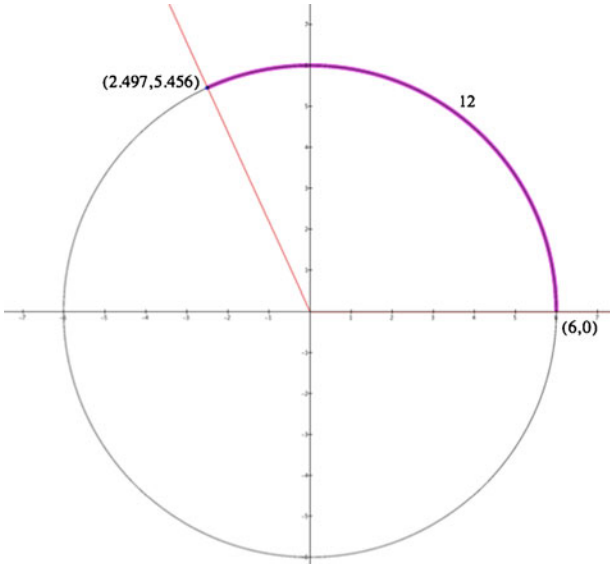
Fig. 3 Illustration of connections between the unit circle, ratios, and units of measure

working with students. We were also uncertain how the meanings that the students held upon entering the study would relate to these ideas.

5 Data collection and analysis

Our interest in characterizing students' unit circle meanings and shifts in these meanings is well-served by a methodology that offers the flexibility of developing and pursuing hypotheses (both during and across sessions) over a series of direct interactions with students. For this reason, and consistent with our prior work in this area (Moore, 2013, 2014), we sought to build viable characterizations of students' unit circle meanings using a teaching experiment methodology (Steffe & Thompson, 2000).

Fig. 4 Connections between the trigonometric functions and measuring in radii



5.1 Participants and setting

The study's participants (Dexter and Deb) were second-year undergraduates enrolled in a pre-service secondary mathematics education program at a large university in the southeastern United States. We chose the students on a voluntary basis from their first secondary mathematics content course in the mathematics education program. Prior to the study, they had completed at least two courses past an undergraduate calculus sequence. Dexter and Deb were the only students (out of 10 in the class) to volunteer for the study. We note that Dexter and Deb did not represent atypical students relative to course performance (both in grades and in the nature of classroom interactions). Our decision to work with pre-service teachers was part strategic and part pragmatic. From a strategic standpoint, we hoped to gain insights into their meanings upon entering the study in order to better understand students' meanings for the unit circle. Pragmatically, trigonometry was a topic in their content course and relevant to their future teaching.

5.2 Data collection and analysis

Each student participated in five 60- to 90-min teaching sessions that occurred within a span of 18 days at the beginning of the content course. The primary focus of this paper is on the first two sessions with each student. During the study, the two students did not attend the content course with their peers. Instead, each student met individually with the research team. The lead author acted as the teacher-researcher for each individualized teaching session, with the co-authors acting as observers.

We chose a teaching experiment due to our goal of building models of students' mathematics including shifts in students' meanings. Through a series of teaching sessions, ongoing analysis, and retrospective analysis, teaching experiments enable researchers to study students' current ways of operating, as well as the reorganizations and constructions that the students experience over the course of a teaching experiment. Namely, teaching experiments enable researchers to experience constraints when working with students on a sequence of tasks, and it is through a researcher experiencing, reflecting upon, and attempting to eliminate these constraints across a series of teaching sessions that she can develop, test, and refine her models of students' mathematics (Steffe & Thompson, 2000).[2]

We videotaped and digitized all student work and interactions during the teaching sessions. As part of an ongoing analysis effort, we met between the teaching sessions to discuss our observations and form hypotheses about the students' meanings, which in turn informed modifications to subsequent teaching sequences in order to test our hypotheses. Analyses of the data after the teaching sessions involved an open and axial approach (Strauss & Corbin, 1998) in combination with a conceptual analysis (Thompson, 2008). After transcribing the data in order to capture the students' utterances and observable behaviors, we first identified instances in which each student reasoned about the unit circle and characterized each student's meanings during such instances. We then compared and contrasted our characterizations of a student's meanings across the data in order to test our characterizations and determine how his or her thinking evolved. After conducting analyses of both students, we juxtaposed the two students' progress in order to draw deeper insights into their meanings.

[2] We point the reader to Steffe and Thompson (2000) and Thompson (1979) for thorough descriptions of teaching experiments in mathematics education.

6 Results

We first discuss observations made during each student's initial teaching session. Against the backdrop of the findings from the first sessions, we describe the design and results of the subsequent teaching sessions. We draw attention to how the students' capacity to coordinate unit magnitudes (including the radius) influenced the students' abilities to relate given circles to the unit circle.

6.1 Teaching session one – given circles and the unit circle

During the first session, the students' unit circle meanings emerged as consisting of calculational strategies that did not stem from quantitative relationships or using the radius as a unit magnitude. An important implication was that both students understood the unit circle as distinct from given circles and relatable through executing their calculational strategies–measures on the unit circle were not representative of measures on the given circles. To illustrate, the students first attempted *the arc length problem* (Fig. 5), which we designed to offer insights into how the students related an angle measure to a collection of circles and arc lengths.

Dexter first converted the given angle measure to 0.611 rad and multiplied each given radius length by 0.611 to determine each arc length. When asked to explain his solution, Dexter claimed, "0.611 would be the number of radius lengths within, on the unit circle." He then drew a new circle called "the unit circle," marked the radius as "1", drew an angle, and labeled the subtended arc on his new circle with a measure of 0.611.

We took Dexter's initial calculations to indicate that he understood 0.611 as the number of radius lengths along each subtended arc length on the *given* circles. However, as the interaction proceeded, Dexter only referred to 0.611 as a measure in the context of the drawn unit circle. His actions left us unsure whether he understood that the subtended arc lengths on the given circles are 0.611 radius lengths when measured in a magnitude of the corresponding circle's radius. To further probe Dexter's thinking, we asked him to determine an angle measure when

Given that the following angle measurement θ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches (*figure not to scale*).

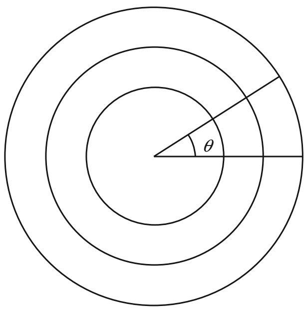


Fig. 5 The arc length problem

given a subtended arc length (1.2 in.) and the radius (3.1 in.) of a circle (Fig. 6).[3] Dexter immediately drew "the unit circle" to "convert to radians" (Excerpt 1).

Excerpt 1: *Dexter converting to radians.*

Dexter: I usually convert to radians first…So I would divide this [*the radius*] by 3.1 to get 1. So like if this were here [*drawing a new circle*], that's our angle again, copied, dividing by 3.1 gives that this is one radian [*labeling the radius as 1*]. And I'll divide this [*the arc length*] by 3.1 also [*using calculator*].

Int.: Ok, so why would you divide that by 3.1 also?

Dexter: So 1.2 divided by 3.1 is 0.387. And then I could say, well you wanna know theta [*labeling theta in the new circle*], right. So I know there is a relationship between the arc length, *s* is equal to *r* times theta [*writing corresponding formula*]. So our arc length here is 0.387, is equal to, our *r* is one, is equal to theta.

Dexter's explanation suggests that he based his conversion between circles on the scaling of a number: dividing 3.1 by 3.1 yields 1, so divide other measures by this number. His subsequent use of the formula $s = r\theta$ with $r = 1$ and $s = 0.387$ provides additional evidence that he did not interpret the quotient 1.2/3.1 or 0.387 as the measure of an arc in radii. To Dexter, the division of the given measures by the radius length did not yield an angle measure in radians or the measure of a given arc in radii.

We interpreted Dexter's scaling strategy to provide an explanation for his actions on *the arc length problem* not including him describing 0.611 as a measure of each arc length on the *given* circles. Just as his solution in Excerpt 1 entailed identifying and carrying out a calculation to obtain a particular result (i.e., a radius of 1), his solution to *the arc length problem* likely entailed identifying the number to multiply 1 (the labeled radius on the unit circle) by to obtain the particular radius lengths (i.e., multiplying 1 by 2.4 yields 2.4). Then, with calculations that scale 1 to the appropriate numerical results identified, Dexter generalized these calculations to convert the number of radians associated with the unit circle to arc lengths on the given circles. Importantly, this action did not stem from his conceptualizing each given arc length as 0.611 times as large as the respective circle's radius; Dexter understood the unit circle as a distinct circle from the given circles in that values associated with the unit circle *were not* simultaneously measures in some unit on the given circles. Instead, "the unit circle" was a distinct circle to be related to particular circles through a calculation that yielded a particular result (i.e., scaling a given number to 1 or vice versa).

Like Dexter, Deb began *the arc length problem* by determining a radian measure. She then calculated the arc length *a* along the smallest circle using the equation $\frac{0.61087\,\mathrm{rad}}{2\pi\,\mathrm{rad}} = \frac{a}{4\pi\,\mathrm{rad}}$, noting units for all but one value. Deb explained, "I don't like to do something unless I can see the units perfectly dividing out." Deb used the same solution strategy (i.e., setting up ratios by matching units and quantities across the equality) to determine the correct arc length for each circle. When asked to describe a meaning of the determined radian angle measure, Deb explained, "Radians are just talking about the radius…the theta is going to be cutting off 0.61087 times our radius because that is what radians mean." Deb's explanation left us perplexed as to why she did not use a calculation that reflected her stated relationship (i.e.,

[3] We produced the original diagram with given measures.

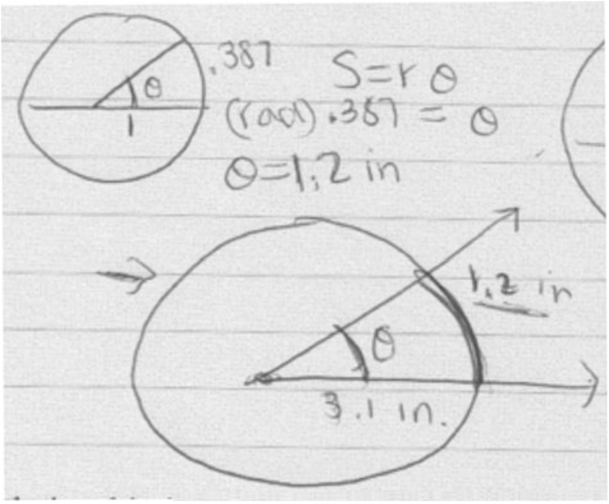


Fig. 6 Dexter's work on an angle measure problem

0.61087 times each radius length), and thus we further pursued her understanding of how radian angle measures related to specified circles.

As with Dexter, we asked Deb to determine a radian measure when given an arc length (6.6 in.) subtended by an angle and the radius (2.4 in.) of that circle.[4] In response, she drew a new circle (Fig. 7) called "the unit circle" and then determined an angle measure (Excerpt 2).

Excerpt 2: *Deb determining an angle measure using the unit circle.*

Deb: Maybe I can simplify this by creating a unit circle and converting these measurements to what they would be. So this is going to be our original circle [*writing this phrase by the given circle*], and then this is going to be the unit circle [*writing this phrase by the new circle*]. So, here's your center. So we know that, just, by nature the unit circle is going to have a radius of one, and because we're already given the unit, we can go ahead and say one inch [*writing 1 in. below the radius*]. So if, um, so we can say, using ratios again, we're trying to find what the equivalent length the arc length would be.

We took Deb's actions, which included drawing a second circle and assigning this circle a radius of one inch (choosing 'inches' based on the given measure being in 'inches'), to suggest that her drawn unit circle did not stem from conceiving the *given* circle's radius as a unit magnitude. Deb then used an equation between two ratios that stemmed from matching quantities between circles and comparing units (1 in. to x in.–measures on the unit circle– and 2.4 in. to 6.6 in.–measures on the given circle) to determine the unit circle arc length. We also note that throughout her solution to the task, Deb tracked the units associated with each value and she claimed that her solution is correct because the units matched in the original ratio and the answer had the correct units (2.75 in.). She then switched the unit from inches to radians because the problem asked for an angle measure.

[4] We produced the original diagram with given measures.

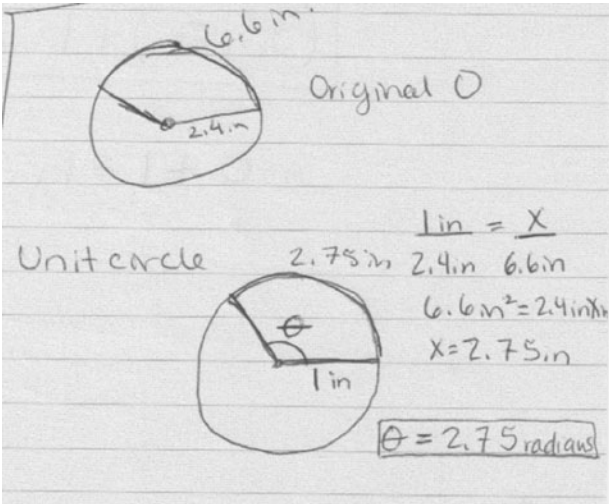


Fig. 7 Deb's work on an angle measure problem

At this point in Deb's solution, her thinking appeared similar to Dexter's in that quantities and measures on her unit circle were understood as distinct from those on the given circle (cf. understanding values associated with unit circle as measures of like quantities but in a unit magnitude different than that given). However, immediately after Deb obtained the answer to the problem, she noted, "We see that…we're just dividing [the given] arc length by the [given] radius, which would be putting [the arc length] into terms of radians." Her statement indicates that she did connect radian measures to using a *given* circle's radius as a unit magnitude. When asked why she had not used this line of reasoning at the onset of the problem, she responded, "Once again units, just understanding. But if you know that you're trying to…find out what this is [*pointing to original arc length*] in terms of the radii, we would be dividing by what the given radius is." Apparently, Deb considered a solution that involves a setting up ratios and tracking units as representative of a better "understanding" than using calculations that stemmed from reasoning about measuring quantities in radii. We also note that, when pressed further, Deb did not understand her drawn unit circle to convey measures in radii. She instead maintained that her drawn unit circle entails measures in a unit (i.e., inches) defined by the problem.

6.2 Summarizing the first session – given circles and the unit circle

Although Dexter and Deb both focused on relating the unit circle to given circles through a series of calculations, there were differences in their meanings for this activity. As described above, Dexter's strategy relied on identifying a calculation that scaled a given circle's radius to the number 1 (the radius of the unit circle) or vice versa, which he understood as resulting in radii or radian measures on "the unit circle" but not on the given circles. Whereas Dexter considered measures on the unit circle as measures in radii, Deb understood the unit circle as a circle with measures in the same unit as given circles but with a radius of one (in that unit). She then relied on using two ratios and cross-multiplication to determine various measures, where she created the ratios by matching quantities and units (i.e., unit circle values are in the numerator and the given circle values are in the denominator with units cancelling on both sides of the equation). Despite these differences in the students' actions, we inferred from those interactions described above a common feature to their unit circle meanings: neither student understood values on the unit circle to convey radii measures *on the given circles*.

We note that the students' meanings were not, for the most part, problematic when solving the angle measure problems presented above; the students' solutions typically included correct resulting values. However, when attempting problems that were more quantitatively complex–requiring function evaluations and longer sequences of calculations (see Fig. 12)–both students had difficulty keeping track of their progress. Specifically, they often lost track of the meaning for values when evaluating trigonometric functions and executing their calculational strategies to convert between the unit circle (as they understood it) and given circles.

6.3 Designing session two – coordinating magnitudes and measures

In response to the students' (a) propensity to reason about the unit circle as distinct from given circles and (b) lack of associating values on the unit circle with measures relative to a given circle's radius, we designed the second teaching sessions to develop using any given circle's radius as a unit magnitude. To accomplish this goal, our instructional decisions drew on Wildi's (1991) description of magnitude reasoning. Magnitude reasoning builds on the notion that the magnitude, or amount, of a quantity is not dependent on the unit used to measure the quantity (Wildi, 1991). For instance, the radius of a circle is the same magnitude regardless of the unit used to measure the radius. To say a radius is 21 in. implies that the radius is a magnitude that is 21 times as large as the magnitude of 1 in. That same radius is 1.75 ft, a magnitude that is 1.75 times as large as the magnitude of 1 foot. Or, when measured in a magnitude equivalent to the radius, the radius has a measure of 1 radius. Despite numerical differences, each measure conveys the same magnitude. More generally, and borrowing an example from Thompson (2011) that compares two *unit-measure pairs*, "if the measure of a quantity is M_u in units of u, then its measure is $12M_u$ in units of magnitude $(1/12)\|u\|$ and its measure is $(1/12)\,M_u$ in units of magnitude of $12\|u\|$" (p. 21). In short, the magnitude of a quantity is the invariant amount of a measurable attribute that is conveyed by all unit-measure pairs (Thompson, 2011).[5]

A specific goal of the second session was to have the students compare unit magnitudes and reason about length as a quantity that can take on measures in different units simultaneously, with each measure representing the same magnitude. We conjectured that such reasoning would support their understanding lengths and a circle's radius as taking on measures in radii and more common length units (e.g., inches) all at once. Based on our previous work with precalculus students (Moore, 2013), we also speculated that drawing the students' attention to using a circle's radius as a unit magnitude would support them in coming to view the unit circle as stemming from measuring in radii, with all circles having a *radius of one radius length* (versus a *radius of one*, where *one* is unit-less or associated with a more common length unit).[6]

As an example task, we gave each student *the stick problem* (Fig. 8), which we designed to have the students determine various measures for a stick and then identify an underlying invariance in these measures. In our attempt to promote reasoning about relative magnitudes, we added the stipulation that the students were not to use formulas or dimensional analysis (i.e., a focus on unit tracking and cancellation based on operations like division and

[5] This assumes that the quantity's magnitude is not varying. In the case that the quantity's magnitude is varying, an invariant relationship exists between the unit-measure pairs for any instantiation of the quantity's magnitude.
[6] The *radius* is often defined as a distance, and thus the phrase *a radius of one radius length* might seem redundant. We point out that the *one* in the phrase *a radius of one*, to students, does not necessarily entail thinking about the radius as a unit magnitude.

a. What does it mean for the stick to have a length of 3.4 feet?

b. Given that there are 12 inches in 1 foot, how long is the stick when measured in inches? Given that there are 300 feet in a football field, how long is the stick when measured in football field lengths?

c. Given that a *fraggle* is a unit of measure that is 221 times as large as 2 feet, what is the length of the stick when measured in fraggles?

d. When answering the above questions, did the length of the stick change?

Fig. 8 The stick problem

multiplication; see Deb's ratio strategy above). These stipulations caused both students difficulties.

6.4 Teaching session two – putting the unit in the unit circle

The students exhibited compatible initial attempts on *the stick problem* (Fig. 8), which included identifying equivalent measures (e.g., 12 in. is equal to 1 foot) and multiplying by ratios involving these measures (e.g., $3.4\,\frac{12}{1}$). However, their actions did not imply that the ratios represented multiplicative comparisons. To illustrate, when prompted to explain a meaning for their ratios (e.g., 12/1) and why they multiplied by these ratios to convert measures, both students alluded to units cancelling. When pressed to describe a meaning for the ratio itself, they explained that the ratio is "essentially" one, as opposed to reasoning about the ratio in a way that entails a multiplicative relationship between two equivalent measures or two magnitudes (i.e., ratio as a quantity). For instance, an interpretation of the ratio 12/1 is that a measure in inches is 12/1 times as large (numerically) as the equivalent measure in feet, a relationship that can be deduced from understanding that the unit foot is 12 times as large as the unit inch. Such a ratio meaning involves envisioning multiplicative comparisons between magnitudes and measures and the partitioning activities this might entail, as opposed to solely matching equivalent measures without envisioning how the associated unit magnitudes might be related (Fig. 9).

As an illustrative example of the students' tendency to focus on equivalent measures without attention to relative magnitudes, consider Deb's approach to question (c) of *the stick problem* (Excerpt 3).

Excerpt 3: *Deb attempting to determine how many fraggles are in one foot.*

Deb: I'm going to compare my fraggles to, I guess feet, so I want to make a conversion from feet to fraggles. So I need to know what the conversion is from one foot, not two feet. I'm just going to divide two twenty one by two. So now I know that there's gonna be, for every fraggle it's 110.5 feet [*writing* '*1/110.5*']. Does that make sense? So, I'm just

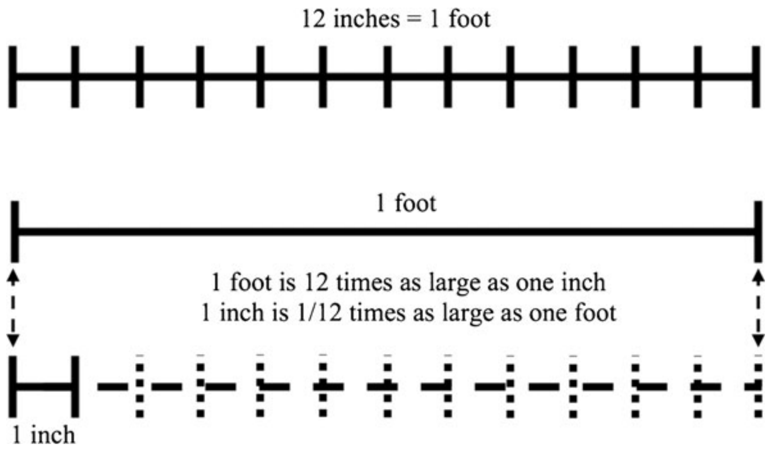


Fig. 9 Using equivalent measures (*top*) vs. comparing measures and magnitudes (*bottom*)

going to be dividing 3.4 by the number of feet in a fraggle. So now I know that there are 0.0307 fraggles here.

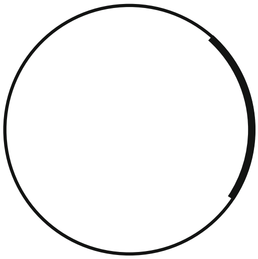
Int.: So could you explain that to me a little bit, how you determined that?

Deb: Ya, I felt that the, I guess the relationship given, fraggles to feet isn't very useful because I want to know one foot. Because that's what I'm given, the stick in feet. So that's the first thing I did. I need to know the conversion from feet to fraggles, and it needs to be from one foot to a certain number of fraggles. So that's the first thing I did.

After this interaction, we asked Deb to reread the problem numerous times and describe the meanings of the given numbers. Each time she maintained that 221 fraggles are equivalent to two ft. and that "110.5 fraggles are in one foot." With a focus on equivalent measures, Deb assimilated the 221 in the problem statement as describing a numerical measure equivalent to two feet. Likewise, Dexter did not coordinate unit magnitudes when attempting the problem, instead assimilating the given numbers as equivalent measures.

In response to both students' reliance on pairing equivalent measures and comparing units, after part (c) we directed them to describe key components of a measure. This question led to both students raising the idea of a unit magnitude. For instance, Deb responded, "We have to start off with an object of a particular length…This object is handy for comparing other objects to that one object…It's a comparison between two things." With Deb raising the idea of a unit magnitude and comparisons with this magnitude, we returned her to the second question and asked her how the "object[s] of a particular length" compared. She then claimed that an inch was 1/12 times as large as a foot. When prompted to identify how this might relate to equivalent feet and inches measures, she concluded that *any* measure in inches is numerically 12 times as large as the equivalent measure in feet. When returning to the fraggles question, she responded, "Oh noooo [*laughing*]. It's the opposite of what I did." At this time Deb reasoned that the *magnitude* of a fraggle as 221 times as large as 2 ft, or 442 times as large as one foot, leading her to conclude that *any* measure in fraggles is 1/442 times as large (numerically) as the equivalent measure in feet.

With both students concluding *the stick problem* by coordinating magnitudes and measures, we asked the students similar questions about attributes of a circle (Fig. 10). This task included referring to the radius as a unit magnitude. During the task, both students converted between measures by coordinating comparisons in unit magnitudes. Also, the students



a. What does it mean for the circle to have a radius length of 3.5 feet? What is the circumference of this circle?

b. Given that there are 12 inches in a foot, how long is the radius when measured in inches? Determine this answer without using a formula.

c. How long is the radius when measured in radii?

d. Given that the identified arc is 4.9 feet, how long is the arc when measured in radii?

e. How long is the circumference when measured in radii?

f. When answering the above tasks, did the length of any of these quantities change?

Fig. 10 The circle problem

identified that the circle's radius has a measure of 1 radius when measured in a magnitude equivalent to the radius. We note that both students spontaneously reflected on the two tasks (Figs. 8 and 10) during *the circle problem*, claiming that coordinating measures and unit magnitudes was unnatural to them and they could not recall instruction on such reasoning. In Dexter's case, he identified that such reasoning clarified how measures in multiple units can represent the same amount of something, claiming, "I feel like the measurement itself changes in units, but it's the same [magnitude] regardless, if that makes any sense…It's like when you measure things in radians and degrees, the angle is still the same."

Upon the completion of the aforementioned tasks, we presented the students with *the* "*Which circle*?" *problem* (Fig. 11). We designed this problem in an attempt to see if the students would connect the unit circle to the idea of anticipating the use of an arbitrary circle's radius as a unit magnitude.

Dexter responded, "I guess, in retrospect, or in theory, they could all be unit circles just by dividing by their corresponding [radius] lengths." Similar to the first session, he described "scaling [the radius] down" to obtain a different circle. However, he quickly changed his mind, claiming, "Well, not really scaling it down. I'm really just changing the unit again," and

Consider circles with a radius of 3 feet, 2.1 meters, 1 light-year, 1 football field, and 42 miles.
Which, if any, of these circles is a unit circle?

Fig. 11 The "Which circle?" problem

concluded that each circle could be thought of as an instantiation of the unit circle if that circle's radius is the unit magnitude. Deb provided the following response to *the* "*Which circle*?" *problem* (Excerpt 4).

Excerpt 4: *Deb Responds to the* "*Which circle*?" *Problem*

Deb: The unit circle doesn't even require a specific unit other than the radius. I guess that's why it's called the unit circle is like the radius is always just one unit… If we made three feet our radius we would just think about the circle in terms of radii instead of feet then it would be a unit circle. Every circle has a radius, so if you just want to talk about the circle in terms of that unit, the radius, then every circle is a unit circle…as long as you are considering that the radius is one unit [*holding her hands apart to signify a length*]. Like perhaps it's not one unit, I mean it's not one meter in length, but it's one radius in length.

Deb tied the unit circle to using the radius as a "specific unit" and reasoned that any circle can be thought of as having a radius of one radii. This stands in stark contrast to her earlier action of considering the unit circle in terms of measures in more standard units (e.g., feet and kilometers). Both students seemed to understand the "one" associated with the unit circle as "one radius in length," which enabled the students to understand the unit circle as representative of any given circle and any given circle as an instantiation of the unit circle.

6.5 Later sessions – given circles as instantiations of the unit circle

As the teaching experiment moved forward, the students' unit circle meanings supported them in using the unit circle in fundamentally different ways than the first session. Namely, the students no longer tried to relate given circles to a distinct unit circle. Instead, they anticipated using a given circle's radius as a unit magnitude to determine measures in radii and use trigonometric functions. To illustrate, consider Dexter's solution to *the ski problem* (Fig. 12).

Dexter began, "Well, 1 rad is 2.5 km. So by changing [the unit] we're going to multiply the values of the problem by a factor of 1 over 2.5." Dexter followed his statement by drawing a new circle, claiming, "the size of the circles should be the same" and that only "the units" differ between the two circles; his drawn circle was the given circle with measures in radii. Dexter's subsequent calculations emerged from coordinating unit magnitudes to conclude that measures in radii are 1/2.5 times as large as measures in kilometers for the specified circles. He also reasoned that the sine and cosine functions output values "in terms of radii" (e.g., $\sin(\alpha)$ and $\frac{0.6513}{2.5}$ represent equivalent radii measures), enabling him to use these functions fluently in the context of the given circle.

An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers. A skier started skiing from position (2.4136, 0.6513), measured in kilometers, and skied counter-clockwise for 13.09 kilometers where he paused for a brief rest. Determine the ordered pair on the coordinate axes that identifies the location where the skier rested.

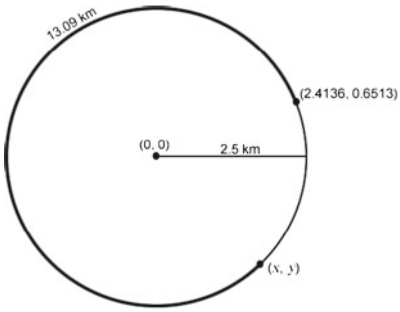


Fig. 12 The ski problem. We note that we designed the problem to intentionally not include coordinate axes for the purpose of determining how the students imposed coordinate axes

Compatible with Dexter's actions, Deb used the given circle's radius as a unit of measure as opposed to relying on dimensional analysis as she did during the initial sessions. She first labeled the coordinates (1, 0) on the given circle to indicate radii measures (Fig. 13) and then calculated the radii arc measure swept by the skier from the initial position to the rest position by dividing the arc length in kilometers by the radius. She stated, "It's telling us that when we divide this arc length by this 2.5 km it's telling us how many radii have made up this arc length." She continued by maintaining a focus on measuring quantities in radii and using trigonometric functions to relate these measures.

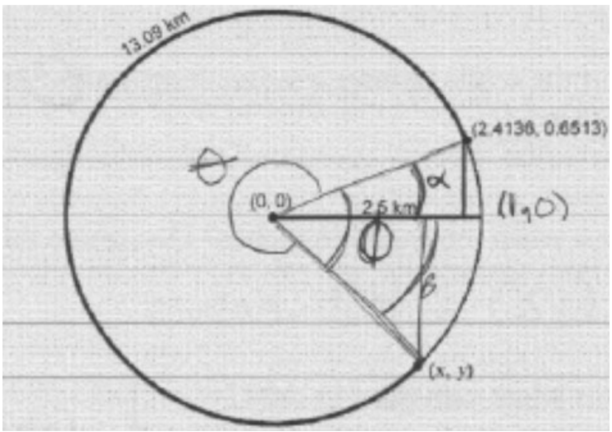


Fig. 13 Deb's work on the ski problem

7 Discussion

Weber (2005) argued that students should come to understand the unit circle as a tool of reasoning. Our study provides insights into how particular meanings influence students' abilities to understand the unit circle in such a way. Namely, Deb's and Dexter's actions highlight the role that measurement and measurement conversions can play in students' unit circle meanings.

7.1 Measuring in radii

Measuring in radii did not form a foundational basis for the students' meanings for the unit circle during the first session. Instead, their meanings relied on calculational strategies to connect given circles to a circle distinct from these circles. Because the students' meanings foregrounded calculational strategies, they understood the unit circle in ways that were tied to particular features of each situation. In Deb's case, the unit for numbers associated with the unit circle changed based on the unit given in the problem. In Dexter's case, he was primarily concerned with identifying a number that he could use to scale given values to those on the unit circle. For both students, their meanings did not entail understanding the unit circle as representative of all circles at once, nor did their meanings entail considering given circles as an instantiation of the unit circle.

As the study progressed, it was not until each student reasoned that all circles have a *radius of one radius length* (versus a *radius of one*) that he or she understood given circles and the unit circle as essentially one in the same. By understanding that any circle's radius can be used as a unit magnitude and that this unit resulted in a radius of measure 1, the students' meanings for the unit circle gained the capacity to assimilate any specified circle (e.g., Excerpt 4). Reflecting a shift in the students' unit circle meanings, and corroborating Moore's (2013) work with precalculus students, their reasoning transitioned from foregrounding the execution of calculational strategies to anticipating calculations and solutions through coordinating measures in radii, measures in other units (e.g., inches), and associated magnitudes. In the former case, their unit circle meanings involved comparing values or numbers on the unit circle with values on a given circle in a case-by-case basis to identify the proper calculation. The students generalized such calculational strategies, but the unit circle remained distinct from given circles in this generalization. In the latter case, the students thought of the unit circle in terms of a quantitative relationship (i.e., measuring relative to the radius) that encompasses all specified circles. Considering Deb's and Dexter's progress, a unit circle meaning that encompasses conceiving any specified circle's radius as a unit of measure appears to have supported their ability to "mentally or physically construct geometric objects to help them deal with trigonometric situations" (Weber, 2005, p. 103).

7.2 The unit circle and unit conversions

Measurement is an important and complex foundation to mathematics. For this reason, it is not surprising that the students' notions of measurement significantly influenced their unit circle meanings. As the students' actions illustrate, a unit circle meaning based in measuring in radii involves understanding: (a) a circle's radius as a viable unit of measure; (b) that lengths simultaneously take on several measures (including radii), each of which convey the same magnitude; (c) a relationship between measures in common linear units and measures in radii; and (d) that when measured in radii, corresponding lengths on all circles have the same numerical value. Most notable was the importance of the students relating the unit circle to comparisons between unit magnitudes and using these comparisons to convert between equivalent measures in different units.

Compatible with Reed's (2006) observation that dimensional analysis can circumvent important meanings, we found that Dexter's and Deb's unit conversion schemes did not entail the aforementioned unit circle meaning at the onset of the study. Instead, their conversion schemes involved pairing equivalent measures and basing their calculations on tracking and cancelling units associated with measures (i.e., dimensional analysis). Such reasoning did not necessitate that they compare (or give explicit attention to) unit magnitudes, nor did it involve coordinating how the measure of a quantity changes with respect to changes in unit magnitudes. Furthermore, it did not appear that their conversion schemes included the radius as a viable unit magnitude, which inhibited our attempts to promote their reasoning about the radius as a unit magnitude.

Although dimensional analysis was restrictive from our perspective, it is important to note that the students did not initially hold this same belief. The students initially emphasized comfort with dimensional analysis and used it to obtain solutions that they perceived as correct. As the students experienced situations that required (sometimes at our request) they not use dimensional analysis, both students expressed that dimensional analysis made it difficult to consider other ways to approach problems. They also could not recall comparing unit magnitudes to draw conclusions about measurement conversions during their previous mathematical experiences. Their recollections may or may not be true, but their reliance on and propensity to use dimensional analysis suggests that their previously encountered instruction (intentionally or unintentionally) supported such reasoning. Deb's and Dexter's actions also indicate that their previous experiences did not have a fundamental or sustained focus on coordinating changes in a unit magnitude and the inversely proportional relationship between unit-measure pairs (e.g., if the unit magnitude is doubled, then the measure is halved).

In addition to circumventing important measurement ideas, using dimensional analysis forms an inherent trap that inhibits conceptualizing a circle's radius as a unit magnitude. To illustrate, consider an arc length of 4.2 in. on a circle with a radius of 1.4 in. Dividing 4.2 by 1.4 determines the measure of the arc length in radii. However, dimensional analysis (e.g., $4.2/1.4 = 3$) yields a unit-less number. Such thinking can lead to students concluding that radian measures are unit-less. Although this issue can be reconciled by treating the 1.4 as a number of inches per radius, we use this example to echo Reed's (2006) observation that dimensional analysis foregrounds a system of rules and calculations as opposed to the quantitative relationships that might underlie calculations. For this reason, we consider unit conversion schemes that primarily entail dimensional analysis as unlikely to support unit circle meanings based in coordinating unit magnitudes and measures.

8 Concluding remarks

Our purpose is not to argue that introductions to trigonometric functions involving the unit circle are superior to introductions involving right triangles. In fact, Kendal and Stacey (1997) identified that the opposite can be true. Instead, we argue that the unit circle should not be taken as a given no matter when working with students (or teachers). If students are to understand the unit circle in a way that encompasses all circles, it is necessary that they have experiences that warrant the construction of meanings that entail such generality.

In the present work, we identify that students' measurement schemes–and particularly their capacity to coordinate changes in unit magnitudes with changes in a quantity's measure as described in Sections 4 and 6–are critical to students coming to understand the unit circle as representative of all circles, and particular circles as an instantiation of the unit circle. Because students' measurement schemes have a wide range of variability both in complexity and sophistication (see Steffe & Olive, 2010), a pertinent next step is conducting future studies with a broader range of participants (in both age and ability). Such work has the potential to form more detailed descriptions of relationships among students' measurement schemes and their unit circle meanings, including how differences in students' measurement schemes contribute to differences in their unit circle meanings. Furthermore, and based on Deb's and Dexter's actions, research in this area has the potential to identify how instruction on the unit circle might simultaneously draw on students' measurement schemes.

We close with a reviewer observation: the present study has connections with research on *advanced mathematical thinking* (Tall, 1991), and specifically Harel and Sowder's (2005) distinctions of advanced mathematical thinking. Harel and Sowder claimed that *advanced mathematical-thinking* can and should occur at any level, and that such thinking should be approached as an evolving process that is developmentally coherent and leads to *advanced-mathematical thinking*. Although the present study involved undergraduate students, our focus was on advanced thinking relative to topics (e.g., measurement and the unit circle) introduced in K-12 mathematics. Our results and results reported elsewhere (Moore, 2013) highlight that students' meanings for the unit circle and angle measure can entail ideas related to equivalence classes, which is a central topic of advanced-mathematics courses. An implication of our work is that the students' unit circle meanings potentially provide a foundational understanding for the notion of equivalence classes. This is only a conjecture on our part that future research should explore, but we use the example to emphasize that if educators expect students to make sense of and work with abstract notions and definitions, then students must have experiences and meanings that warrant these expectations.

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