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Making implicit metalevel rules of the discourse on function explicit topics of reflection in the classroom to foster student learning

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Abstract Despite the existence of extensive literature on functions, fewer studies used sociocultural views to explore the development of student learning about the concept. This study uses a discursive lens to examine whether an instructional approach that specifically attends to particular metalevel rules in the mathematical discourse on functions supports students' learning of the concept in a postsecondary mathematics classroom. The findings suggest that such instruction has the potential to foster learning as indicated by the changes in the ways students talked about functions, and their awareness and modifications of the assumptions shaping their thinking about functions.

Keywords Functions · Teaching experiment · Mathematical discourse · Metadiscursive rules · Student learning · Postsecondary education

1 Introduction

Function is a central concept in mathematics and related branches of science and plays an essential role in K-12 and university-level education. Despite the existence of extensive research on functions, fewer studies used sociocultural frameworks to explore the process of learning about functions in the classroom setting. This study explores student learning of functions in a university-level classroom and uses a sociocultural lens—Sfard's (2008) discursive framework—to examine the changes in student thinking about functions through a teaching experiment. In this study, Sfard's (2008) framework is adopted for two different

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purposes: as an analytical lens to examine student learning and as an approach inspiring the pedagogical choices behind the teaching experiment.

Another incentive for this study comes from the scarcity of research using Sfard's (2008) framework to focus on the teaching practices of university-level mathematics teachers (Viirman, 2013). Among the studies that used Sfard's (2008) framework in the context of postsecondary education, very few explored the interactions between teachers and students (Nardi, Ryve, Stadler, & Viirman, 2014). The practices of the teacher and the interactions between the teacher and students are critical features of this study due to the roles they can play in eliciting student thinking in the classroom. The pedagogical approach used in this study contrasts with those in traditional lecture-based teaching, which is a common form of instruction in university-level mathematics classrooms (Güçler, 2014; Viirman, 2013). A contribution of the study is to demonstrate the potential of Sfard's (2008) framework in helping educators develop innovative teaching approaches in postsecondary mathematics education to enhance student learning.

An additional motivation for this study comes from prior work that examined the classroom discourse on the limit concept in a beginning-level undergraduate calculus classroom (Güçler, 2013, 2014). The findings showed that the metalevel aspects of the instructor's mathematical discourse remained implicit in the classroom, leading to miscommunication among the participants. The results of the prior work led to a hypothesis that making such tacit elements of mathematical discourse explicit topics of discussion and reflection can be a potentially useful pedagogical approach to foster mathematical communication and student learning in the classroom. The current study tests this hypothesis and addresses the following question: How does an instructional approach that specifically attends to particular metalevel rules in the mathematical discourse on function support students' learning of the concept in a university-level mathematics classroom?

2 Thinking about functions

Function is a central concept in K-12 and undergraduate mathematics, and it is a challenging concept for students to learn (Eisenberg, 1991). Researchers exploring student thinking on functions argue that many students at different educational levels struggle with moving flexibly across graphical, algebraic, tabular, and verbal representations of functions (Monk, 1994; Sfard, 1992; Sierpinska, 1992; Tall, 1996). Students have a strong tendency to associate a given function with a single algebraic expression, rule, or formula (Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989) and demonstrate difficulties with piecewise functions (Carlson, 1998; Sfard, 1992). Students also have difficulties interpreting representations and information demonstrating the relationship between two variables (Carlson, 1998; Monk, 1994), which results in challenges in interpreting and representing covariant aspects of functions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Weber & Thompson, 2014).

A factor connected to the aforementioned student difficulties is the ability to view function both as a process and an object (Sfard, 1992). Function is a concept that plays a dual role as a process and object[1], and this duality is necessary to think about the concept flexibly (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991, 1992). However, the dual role of functions presents

[1] Some researchers use *product* instead of *object* to distinguish the processes associated with function from the products of those processes (e.g., Gray & Tall, 1994).

challenges for students, which include associating function with a computational process rather than a permanent entity (Sfard, 1992), inability to deal with operations performed on functions (Dubinsky, 1991; Sfard, 1992), and inability to distinguish between static and dynamic aspects of functions (Carlson, 1998).

Sfard (1991, 1992) uses the term *reification* to talk about the process of converting mathematical processes and actions into mathematical objects and mentions that reification of function is critical and inherently difficult for students. Reification of function was also challenging for mathematicians as evidenced by the historical development of the concept. Sfard (1992) considers historical development of function as “a three-centuries long struggle for reification” (p. 62) as mathematicians turned the dynamic processes of change and variation associated with function into solid mathematical concepts. Although it is not the assumption of this work that the historical development of function parallels its development by individual learners, history can provide useful information when addressing student difficulties with reification, especially in terms of the metalevel rules shaping different realizations of functions. Another potentially useful strategy is to have open discussions on mathematical concepts “and the difference between processes and objects” to challenge students’ views obstructing reification[2] (Sfard, 1992, p. 79). These pedagogical suggestions are fundamental aspects of this work through the utilization of Sfard’s (2008) discursive lens described in the next section.

3 Theoretical framework

Sfard’s (2008) framework views mathematics as a discourse that can be characterized by its word use, visual mediators, routines, and endorsed narratives. *Word use* refers to participants’ use of mathematical words in their discourse. *Visual mediators* are the visible objects used to enhance mathematical communication. *Routines* are the set of metalevel rules describing the discursive patterns in the actions of participants. *Endorsed narratives* are the set of spoken or written utterances about mathematical objects and their relationships the participants consider as true given their word use, visual mediators, and routines. Definitions, theorems, and axioms are among the endorsed narratives of mathematical discourse.

Learners’ use of the elements of mathematical discourse can differ from their use by the experts of mathematical communities (e.g., teachers, mathematicians). Sfard (2008) notes that a goal of school learning is to help students modify their discourses so that they can be fluent participants in mathematical communities of practice. From this lens, change in one’s mathematical discourse is an indicator of learning.

Mathematical discourse has a recursive nature as one discursive act turns into the object of another. Such recursion makes it possible to “create utterances about utterances” or “talk-about-talk” (Sfard, 2008, p.103), which indicates the importance of metalevel learning—learning that results in changes in the metalevel rules of participants’ discourses—in mathematics. Sfard (2008) distinguishes between objectlevel and metalevel rules.[3] Objectlevel rules define the “regularities in the behavior of objects of the discourse” (Sfard, 2008, p. 300), whereas metalevel rules characterize the patterns in the activity of participants. Metalevel rules

[2] Teachers need to be aware of different realizations of mathematical concepts and their own positions for such discussions to impact student learning. Otherwise, the discussions would not necessarily be transparent.

[3] Hereon, the terms metalevel rules, metarules, and metadiscursive rules will be used interchangeably.

are “about the actions of the discursants, not about the behavior of mathematical objects” (Sfard, 2008, p. 201). Metarules are broad since they can characterize many different patterns in the activity of participants. In this study, the focus is only on the metarules shaping the objectlevel rules in participants’ discourses. For example, the narratives “A function is a particular type of correspondence between two sets” or “$f(x)+g(x)=(f+g)(x)$” are objectlevel rules, whereas the regular actions participants perform to realize a function (e.g., using particular assumptions such as continuity or dynamic motion when defining and visualizing functions) would be examples of metarules in the discourse on functions. Metarules impact how participants interpret the content of the discourse (Kjeldsen & Blomhøj, 2012).

Sfard (2008) notes that some of the critical junctures in the development of mathematical concepts occur during the process of reification. In her discursive approach, she uses the term *objectification* to refer to the process with which the talk about operations and actions are converted to the talk about mathematical objects. In this process, it is often necessary to change the previously existing metarules of the discourse. On the other hand, metarules are often tacit and “observed in those aspects of communicational activities that are not directly related to the particular content of the exchange” (Sfard 2001, p. 30). The tacitness of the metarules is possibly one of the reasons why experts of mathematical discourse “lose the ability to see as different what children cannot see as the same” (Sfard 2008, p. 59). It is the goal of this study to explore whether explicating tacit metarules of the discourse on function and making them explicit topics of discussion support student learning.

Recently, more researchers have been utilizing Sfard’s (2008) framework to highlight the importance of metalevel learning. These researchers agree that a critical aspect of mathematics education is to create teaching and learning situations in which metalevel rules are made explicit objects of discussion and reflection for learners (Güçler, 2013, 2014; Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014). Emphasizing the inherent difficulty of metalevel learning, researchers argue that it is unlikely to be initiated by learners themselves and requires reflection, imitation, and direct engagement in a discourse featuring the metarules of the discourse (Bar-Tikva, 2009; Kjeldsen & Blomhøj, 2012; Nachlieli & Tabach, 2012; Sfard, 2008). To foster metalevel learning, they highlight the essential roles teachers play in creating opportunities and providing a discursive lead in the classroom so that elements of participants’ discourses become topics of discussion.

Kjeldsen and colleagues argue that history of mathematics can help create teaching situations in which metarules can be turned into explicit objects of reflection, support students’ learning of mathematical concepts, and enhance students’ historical awareness (Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014). In their approach, high school and university students examined history from the perspectives of historical actors and their motivations. The results showed students’ awareness regarding the influence of human actors on the formation of the function concept and some of the metalevel rules shaping its historical development. It was also possible to diagnose students’ metarules that were in conflict with those currently shared by the mathematical communities. Although these studies did not attempt to answer whether this approach resulted in students’ development of proper metadiscursive rules (which is a particular focus of this work), they show how history can be used “in mathematics teaching and learning as a method for turning metadiscursive rules into objects of reflections” (Kjeldsen & Blomhøj, 2012, p. 346). The pedagogical approach used by Kjeldsen and colleagues, which has similarities with the one reported in this study, is one of the many possible ways educators can interpret Sfard’s (2008) discursive theory in relation to classroom practice to support metalevel learning.

In this study, history of mathematics was *not* used to explore any potential parallelisms between historical development and students' individual development—a common assumption of psychological recapitulation, which supports the idea that during their mathematical development, learners go through similar steps as those found in the history of mathematics. Psychological recapitulation views knowledge as removed from its context and reduces history "to a linear sequence of events judged from the vantage point of the modern observer" (Furinghetti & Radford, 2008, p. 649). Schubring (2011) also problematizes the assumption of "a universally homogeneous conceptual development" (p. 93) and states that

> Conceptual developments occur within determinate and specific groups, the so-called scientific communities which have as primary references for their conceptual frames the values and norms of their particular cultural environment, their directly surrounding systems—which one may shortly call 'context'. Therefore, there likewise does not exist an absolute simultaneity or parallelism of conceptual developments in different cultures. (p. 94)

Although there may not be direct parallelisms between historical and individual development, mathematics is a historically established activity, and in order to participate in mathematical communities of practice, students need to adopt mathematical ways of thinking that are compatible with those established by mathematical communities over the course of history. In this work, activities based on the historical development of function were used to promote students' reflections on the metarules shaping mathematicians' discourses to then encourage students to reflect on the metarules shaping their own discourses on function.

4 Methodology

4.1 The design of the teaching experiment

The study followed a teaching experiment methodology as outlined by Steffe and Thompson (2000) with the purpose of exploring and explaining students' mathematical experiences and development. This methodology involves experimentation with the methods influencing learning to provide an account of students' progress in the context of teaching (Steffe & Thompson, 2000). The researcher plays an active role in students' experiences and development to test and generate hypotheses about their learning. The main focus of the teaching experiment conducted in this work was to make particular metarules of the discourse on calculus concepts explicit topics of discussion in the classroom to explore if this approach resulted in changes in students' discourses. The larger study explored student learning on functions, limits, derivatives, and integrals over the course of 13 weeks, whereas this paper is about the discussions on functions during the first 3 weeks.

The activities used in the classroom were designed to create instances in which students could act according to different metarules.[4] Those instances had the potential to reveal students' existing discourses on function and provide them with opportunities to reflect on their assumptions. The lessons on functions focused on the metarules shaping students' endorsed narratives and visual mediators. Due to space constraints, this paper is only

[4] Sfard (2008) refers to the instances "in which different discursants are acting according to different metarules" (p. 256) as *commognitive conflict* and considers such conflicts as essential in metalevel learning (p. 258).

about students' endorsed narratives in the form of definitions and their discursive acts when defining functions.

Existing literature on functions indicates that students use various definitions of function that may be different from, and in conflict with, the definitions accepted by the experts of mathematical communities (Sfard, 1992; Vinner & Dreyfus, 1989). Students' definitions provide contexts in which it is possible to examine the metarules and level of objectification in their discourses. To elicit the metalevel rules in students' definitions, the focus was on their discursive actions when defining functions. While students' verbal definitions of function revealed their endorsed narratives, their elaborations on how they realized functions in their definitions revealed their metarules (regularities in the *act of defining*).

The students were given an initial survey that asked them to define functions in their own words. Besides the survey, two classroom activities, which will be discussed later, were used to help students reflect on different definitions of functions and the metalevel rules shaping those definitions. In the first activity, students reflected on their own definitions of functions, whereas in the second, they reflected on some definitions of functions generated by mathematicians during the historical development of the concept. The second activity is consistent with the approaches used by Kjeldsen and Blomhøj (2012) and Kjeldsen and Petersen (2014). However, there are also differences between this study and theirs in terms of the utilization of history to foster metalevel learning. This study did not focus on the sociological contexts in which the mathematicians lived in similar detail and did not expect students to examine the historical development of function on their own. Further, this work mainly focuses on using particular assumptions (e.g., assumptions of regularity, continuity, change and variation, motion, discreteness) as metarules in different realizations of functions.

4.2 Participants, data collection, and analysis

This study was conducted at a public university in eastern USA. The participants were one pre-service and seven in-service high school teachers, hereon referred to as *the students*, taking a mathematics content course on calculus for their initial or professional licensure programs. All of the students taking the course volunteered to be included in the study. Except for the pre-service teacher, who had no prior teaching experience, the participants' experiences ranged 4–12 years. This was the only mathematics course the in-service teachers took during the study and, on average, the participants indicated that it had been more than 5 years since they last enrolled in a calculus course. However, all of the in-service teachers taught some form of algebra involving basic characteristics of functions at the time of the study. The researcher was the instructor of the course.

For this study, the data consisted of an initial survey, three video-taped classroom sessions (each lasting 2.5 h), weekly journal reflections, and audio-taped semi-structured interview sessions conducted at the end of the semester. The initial survey was designed to gain some information about the students and their thinking about function before instruction. The interviews provided similar information while also giving opportunities to examine whether the participants were aware of the metarules shaping their discourses on function at the end of instruction. The journals were structured by the students and helped them reflect on the lessons. The interviews were conducted with each student individually and they lasted 45–75 min. The interview sessions and classroom discussions during which students talked about defining functions were fully transcribed. One graduate student assisted the researcher during data collection and analysis throughout the semester. Three other graduate students assisted the team with the analysis after the course.

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), the teacher and the graduate student assisting the data collection met weekly to discuss their observations about the teaching episodes and other data sources. Individual records were analyzed and compared after each teaching episode to facilitate the generation, testing, updating, and retesting of hypotheses throughout the experiment. The records were also analyzed retrospectively with three additional graduate students after the experiment to examine whether, and how, the students' discourses changed.

The analysis did not only explore students' realizations of functions through their definitions but also *how* they realized functions in those definitions (metarules behind the definitions). A critical focus when examining students' metarules shaping their definitions was the degree of objectification in their word use. The students were considered to realize a function as an object if they talked about it as a mathematical entity using a noun. They were considered to realize a function as a process if they talked about the processes and actions they associated with functions or described what a function does rather than what it is. For example, if students talked about function as a set of ordered pairs, rule, correspondence, graph, or concept, they were considered to realize function as an object. If they talked about changing one variable with respect to another, the process of mapping the input values to the outputs, or what a function does to particular input values, they were considered to realize function as a process. If the students talked about functions both as processes and objects depending on the context, they were considered to have both realizations in their discourses.

Another focus when examining students' metarules was the assumptions shaping their definitions. For example, different students can define a function as a rule but their assumptions in their realizations may differ. If their definitions are based on the assumption of regularity, they may realize the rule as a single formula. If their definitions are based on the assumption of arbitrariness, they may realize the rule as an arbitrary correspondence between the domain and range instead of a single formula. Since participants can use similar discursive features in different ways (Güçler, 2013), the examination of metalevel rules can help highlight instances of miscommunication in the classroom.

In this study, student learning was evidenced by the changes in students' discourses regarding how they talked about and defined functions, and their awareness and modifications of the assumptions on which their thinking about functions was based. Such changes were examined for each student by exploring their responses in the initial survey, classroom discussions, and individual interview sessions at the end of the semester. The journal entries, which gave students opportunities to reflect on any aspect of the course without an imposed structure, were used to triangulate[5] and complement the results from the other data sources.

5 Results

In this section, students' discourses on definitions of function are analyzed through their responses in the survey and classroom activities. This is followed by the analysis of students'

[5] Students' journal entries were checked and compared with the interview responses to look for possible discrepancies in order to avoid providing limited descriptions of students' discourses based only on the interviews.

Table 1 Students' definitions of functions in the survey

Student	Definition
[1] Carrie:	A relation between independent and dependent where each independent is paired with exactly one dependent.
[2] Fred:	A relation between two variables; for each input there is only one output.
[3] Lea:	The dependency of one variable on another.
[4] Martin:	A mathematical concept where a domain must adhere to a specific rule to provide a given range.
[5] Milo:	The behavior of a graph.
[6] Ron:	For every input, there is only one output.
[7] Sally:	One to one, for every x there exactly one y value, passes vertical line test.
[8] Steve:	An equation-like concept where there must be no repeating y-values for two different x-values, and there must be no gaps (or discontinuity).

Steve was the only pre-service teacher in the classroom

discursive changes using their interview responses in conjunction with their journal entries. The purpose is to examine whether and how the students' realizations of functions evolved from the beginning until the end of the course.

5.1 Students reflecting on their own definitions of functions

During the initial survey, which was administered at the beginning of the first lesson on functions, the students were asked the following question: "Please define function in your own words." The written responses students[6] provided for this question are shown in Table 1.

Note that what could be inferred about student thinking from this survey is limited since the survey did not reveal students' actions, but there are some observations that can be made from these initial responses. Table 1 indicates that four students objectified function by referring to it as a mathematical entity and describing what a function *is*, whereas others used phrases they associated with the concept without explicit referral to what a function is. The former students described a function as a relation or concept ([1–2], [4], [8]), whereas the latter talked about it as signifying a dependency [3], characterizing the behavior of a graph [5], or used phrases about some properties of the concept [6–7]. Two students provided mathematically incorrect narratives about function (e.g., it has to be continuous [8] or one-to-one [7]). Steve also seemed to have confused the fact that a function needs to be well-defined with that it has to be one-to-one [8]. Three students' realizations of function were based on the (implicit or explicit) use of a graph ([5], [7–8]), whereas others seemed to use the assumption of a function-machine through their elaboration on input and output ([2],

[6] All the student names used in the study are pseudonyms. Throughout the paper, students' incorrect or incomplete utterances are preserved to keep the originality of their word use and narratives.

[6]). At this point, it was hard to tell whether the students who defined function as a relation [1–2] realized it as a process (e.g., the process of mapping the inputs to outputs) or an object (e.g., a correspondence between two sets). This initial glance at students' discourses on function indicates that not all of them objectified function as a mathematical entity at the beginning of the course.

Two classroom activities were used to focus on defining functions. The first activity, which consisted of two parts, was a modified version of that used by Schoenfeld and Arcavi (1988). In the first part, students were asked to use *one* word that they believed captured the meaning of the term function by completing the sentence "A function is a…" In the second part, the students provided a definition of function using as many words as they wanted. This activity was presented during the first lesson on functions. The goal was to focus on students' initial discourses on function to elicit the similarities and differences between their responses as well as the assumptions on which those responses were based.

After working on the first part of the activity, the students generated the following words: relationship, rule, process, model, mapping, graph, and pattern. To get more insights about the metarules shaping these realizations, the teacher asked students to elaborate on their thinking about function in relation to the words they generated:

> [9] Fred: [referring to relationship] Um, between two variables. I was thinking two variables because function is a particular kind of relation…For every input, there is only one unique output that defines a function.
> [10] Martin: [referring to rule] I think the function itself is the rule of …in order to get this output or this range from a given domain you have to do this so it's like that is the rule and it's clearly defined in that specific function.
> [11] Fred: [referring to process] I used that word. It is kind of based on a rule. The inputs must go through and define the process to get to the output.
> [12] Steve: [referring to model] I said that because I continue to go back to the real life phenomena, like modeling real life phenomena. So it is like graphing and mapping; it is using mathematics to model something in the real world.
> [13] Ron: [referring to mapping] It is just the process of mapping the domain to the range for the output.
> [14] Fred: [referring to graph] It's for the representation of the relationship.
> [15] Martin: [referring to pattern] It is kind of…how it is changing is always the same.
> [16] Sally: Yeah, no matter what function you are working with, there is a certain pattern.
> [17] Teacher: You seem to suggest some sort of regularity here. Do you think a function needs to be regular?
> [18] Sally and Martin: Yes.
> [19] Teacher: Okay, these ideas are very rich.

Note that some of these words (e.g., process, mapping, model, and pattern) were not present in students' survey responses, and the discussion helped students elaborate on their realizations of function. Fred preserved his initial definition of function (Table 1, [2]) as a relation between two variables during the classroom discussion [9]. However, he also connected that definition to the realizations of function as a process [11] and a graph [14]. In this excerpt, both an objectified view (function as a particular kind of relation [9]) and a process view of function (the process inputs go through to get to the output [11]) were present in Fred's discourse. It can also be seen that Ron's initial definition where he used input-output (Table 1, [6]) was based on his realization of function as a "process of mapping" the former to the latter [13]. Martin and Sally's comments indicate that they realized function as a single rule [10] or pattern [15–16] that signifies a regularity.[7] Although Martin's response indicated an objectified view of function as a mathematical concept in the survey (Table 1, [4]), he talked about the process of what one needs to do to get to an output [10] and the process of dynamic change [15] when referring to function in the classroom. Sally added another realization of function as a pattern [16] to her initial phrases about function (Table 1, [7]). Steve initially objectified function as an equation-like concept without gaps (Table 1, [8]) and then talked about it through the processes of modeling, graphing, and mapping during the classroom discussion [12]. This discussion suggests that the students—including those who objectified function—realized it as a process when asked to further elaborate on the concept.

In the second part of the activity, the students provided a definition for function using as many words as they wanted. Those definitions were as follows:

[20] Lea: The dependency of one variable to another (that is one-to-one and onto).
[21] Fred: Passes the vertical line test.
[22] Carrie: A relation or a rule that takes any given input and produces one unique output.

It was Sally who suggested adding one-to-one and onto to Lea's definition. All the students agreed that these three definitions captured how they would define a function. Their responses indicated that some students were considering one-to-one and onto as inherent properties of functions. When asked to talk about what they meant by those terms, the students were silent. Sally said "it is hard to put those in words; you just show them but can't explain them," which suggests that the students were not yet able to provide explanations for some of the mathematical terms they associated with functions.

Given the lack of immediate connections between their definitions in the first and second parts of the activity, the teacher asked the following question:

[23] Teacher: It looks like we generated at least ten ways to define functions. Are these all the same?
[24] Steve: I now think the relationship captures all. Graph, mapping, and pattern are all components of the relationship.
[25] Sally: Relationship can be taken in questionable ways like the relationship with people. Seeing relationship and all the words below it, people will understand they are talking about a function as a relationship.

Here, we see some changes in Steve and Sally's thinking about function compared to their initial definitions in the survey (Table 1, [7–8]) and the one-word definition Steve provided during the discussion [12]. They were now considering the word "relationship" as the main characteristic of function and others as components [24] to define the concept more precisely to differentiate it from its everyday use [25]. By doing so, they were talking about the metadiscursive rule of defining in mathematics and referring to relationship as a subsuming discourse that is connected to the other realizations of functions. Indeed, *saming* is a metalevel rule that helps generate and expand mathematical discourse through the similarities of various realizations of a concept (Sfard, 2008). On the other hand, to flexibly view function both as a process and object, it is also important to distinguish the differences among its realizations depending on the context.

[7] Later during the lesson, it also became clear that Martin and Sally assumed continuity for all functions.

Students' overall responses to the first activity suggested that they were able to see the similarities among the definitions they generated. However, there was no indication in their discourses that they were also aware of the differences among those definitions. There was also no indication that the students differentiated between an abstract mathematical object (function) and the visual mediators used to realize the concept (Sfard, 1992). These observations prompted the teacher to ask the following questions with the goal of helping the students reflect on the similarities and differences among their realizations of functions: "If a function *is* a graph, then is a graph a function? If a function is a graph, then can we also define a function as a table or algebraic expression? If a function *is* a process, is every process a function?" These questions initiated discussions revealing more about students' metalevel discourses on functions. For example, the students did not consider every graph or process as a function. They were comfortable considering function as a graph or algebraic expression but not as a table. When asked to elaborate why, they mentioned that "a table is only a representation" after which they mentioned that graphs and algebraic expressions are also representations. The teacher then asked if a function is the same thing as its representation.[8] The students' responses to the question revealed that they "always assumed continuity," which made it "natural [for them] to think about graphs and equations as functions." They also mentioned using tables to draw graphs of functions, but they did not consider a table as a function because it was "just a set of values."

There are a couple of observations that can be made from these discussions. First, students explicitly talked about the assumption of continuity as a metalevel rule shaping their endorsed narratives about functions. Second, the students talked about a table almost as a tool to get to a graph (through the assumption of continuity), and the discrete nature of the table itself did not match with their existing definitions of function. Third, this was the first time students used the word "set" when talking about function. This discussion was a good opportunity for the students to think about the similarities and differences among different realizations of functions and the metalevel rules shaping those realizations.

Note that, until that point, the teacher was shaping the dialogue through the questions she initiated but did not modify or correct students' narratives on function. Instead, she elicited a representative inventory of the different ways in which students realized function. Using the ideas students generated in the classroom (e.g., process, assumption of continuity, set of values), she then took a more direct approach in explicating the different metadiscursive rules on which students' definitions were based. For example, she talked about the connections and differences between a definition of an abstract mathematical concept (function) and the visual mediators used to realize the concept. While doing so, she used students' consideration of function as a graph and rejection of any graph as a function to encourage them to think about how to define function in a way that removes ambiguity. She also talked about the differences between realizing function as a process and a mathematical object using the definitions students generated. She concluded the discussion by mentioning that the assumption of continuity may support particular realizations of functions but can be in conflict with others.

[8] In the classroom, the word *representation* was used instead of the term *visual mediator* due to the students' familiarity with the former.

5.2 Students reflecting on mathematicians' definitions of functions

Students worked on the second activity about defining functions during the next lesson on functions. They were presented with sheets that included four different definitions of the function concept throughout its historical development and two modern definitions taken from current calculus textbooks. The historical definitions used in the activity were formulated by Euler in 1748 and 1755, Dirichlet in the 1800s, and Bourbaki in 1939 and were taken from Kleiner (1989). The students were asked to examine these definitions with a focus on their similarities and differences. The goal was to help students reflect on the metarules shaping mathematicians' endorsed narratives about functions to then reflect on their own assumptions. Before discussing students' responses during the activity, it is important to examine the historical definitions in terms of the metarules on which they are based.

Euler's definition in 1748 viewed a function of a variable quantity as "an analytical expression composed in any manner from that variable quantity and numbers or constant quantities" (Kleiner, 1989, p. 284). Some of the metarules behind this endorsed narrative were assumptions of regularity and continuity. Euler's definition in 1755 was more comprehensive in that it characterized a function as "all the modes through which one quantity can be determined by others" (Kleiner, 1989, p. 288). While doing so, he talked about the dependence of quantities so that "if the latter are changed, the former undergo changes" (Kleiner, 1989, p. 288). He did not use the term "analytical expression" in his second definition and realized function as a process based on the assumptions of change and variation. Although Euler's second definition was an attempt to objectify function as a mathematical concept, he still talked about the processes and actions performed on the quantities instead (Sfard, 1992).

Dirichlet talked about a correspondence between two variables x and y by also laying out the conditions for the correspondence to be a function. He restricted x to an open interval (without explicitly mentioning the term *domain*) and highlighted that "it is irrelevant in what way this correspondence is established" (Kleiner, 1989, p. 291). In Dirichlet's definition, the realization of function as an analytical expression was eliminated. This change in the endorsed narrative on function was based on the changes in the previously existing metarules in the discourse on functions such as replacing the assumptions of regularity and continuity with arbitrariness. Bourbaki defined function as determined by a functional relation between the elements x and y of two sets and referred to y as the value of the function at x (Kleiner, 1989, p. 299). In Bourbaki's definition, the metarules of the discourse on functions related to dynamic motion (variation, direction, approaching) were replaced with "static terms using only real numbers" (Lakoff & Núñez, 2000, p. 308). Dirichlet's and Bourbaki's definitions were products of the period called arithmetization of geometry during which mathematicians, in order to enhance mathematical rigor, changed the metalevel rules of the discourse on various concepts of calculus (Güçler, 2011; Sfard, 2008). Those changes in the metarules of the discourse also impacted the words, visual mediators, and endorsed narratives used to realize function (e.g., using "set" or "correspondence" instead of "variable" or "change"; using algebraic symbols instead of geometric visual mediators).

When working on the activity, the students initially contrasted Euler's and Bourbaki's definitions and concluded that the latter explicitly described the nature of the relation between the input and output, whereas Euler's definitions were about a relation rather than a function. To encourage students to examine Euler's definitions more closely, the teacher initiated the following discussion:

[26] Teacher: Does everyone have a clear idea what Euler's first definition indicates in terms of function? We had different ways of defining and thinking about functions that emerged from our discussions last time. Which of those ideas do you associate this definition most closely with?
[27] Steve: I would say equation. It says analytical expression, which is like an equation…that is how a function is shown.
[28] Teacher: And 7 years later he changes his definition. Why may that be the case?
[29] Sally: Like a transformation. If something changes, you need to now change the rule and definitions.
[30] Milo: [interrupts] I see a process view [shows Euler's second definition].
[31] Teacher: That is interesting. Do you see a process view in his first definition?
[32] Milo: No, but I see it in the second. It says "a variable quantity that changes" whereas the first one is like an equation.
[33] Sally: Did somebody do something that proved the first definition wrong? Or like it didn't say enough?

The students' responses in this excerpt are quite rich in terms of the observations they made regarding the metalevel rules of the discourse on function. The students connected Euler's first definition to their realizations of function as an equation [27]. They were able to identify the differences between Euler's definitions by noticing the process view based on dynamic change in Euler's second definition ([30], [32]). Sally's comments about the need to change the "rule and definitions" of the discourse based on "transformations" indicate her awareness of how the definitions change based on the changes in metarules ([29], [33]). Sally's comments also indicate another metarule in her discourse: Mathematical definitions have a truth value [33].

After examining Euler's definitions, the students talked about the definitions formulated by Dirichlet and Bourbaki:

[34] Fred: Through intervals, Dirichlet is restricting the inputs to part of a set since it [the function] might behave differently elsewhere.
[35] Ron: Dirichlet's function is defined on an open interval; not looking at the whole function. Euler was looking at a single expression when talking about the whole function.
[36] Martin: Dirichlet's use of correspondence seems like the emergence of the idea of mapping.
[37] Steve: Bourbaki's definition is meant for universalizing function so that everyone can understand it in depth.

By focusing on the added emphasis on intervals in Dirichlet's definition, Fred and Ron noticed that Dirichlet abandoned the realization of function as a single rule [34–35]. Steve's consideration of the Bourbaki definition as a way to universalize function [37] was based on his reflection on the previous classroom discussions during which characteristics of mathematical definitions were discussed. Finally, although the students' earlier discussions on function rarely included a set-theoretical approach, their word use when talking about Dirichlet's and Bourbaki's definitions was about "sets" [34], "intervals" [34–35], and "correspondence" [36] rather than process, change, and movement. These observations suggest that, although the students did not explicitly talk about all the metarules shaping mathematicians' definitions (e.g., adopting or eliminating the assumptions of dynamic motion and continuity), they gained important insights about some of them.

At the end of the activity, the teacher highlighted the similarities and differences among mathematicians' definitions by summarizing the metarules students noticed as well as explicating those that were not mentioned by the students. She also talked about the situations that gave rise to the transformations in mathematicians' realizations of functions (e.g., the vibrating string problem, changing notions of rigor). At the end of the discussions, Sally questioned whether learning about functions initially through the static approach in the Bourbaki definition was an effective approach. Her question led to a dialogue among the students, who decided that it should not be taught before students have experiences with the dynamic view.[9] The teacher assigned the examination of modern definitions of functions as an exercise and encouraged students to explore how function is defined in current textbooks to gain more information about the metalevel rules behind those definitions.

5.3 Changes in students' discourses on functions

The previous sections elaborated on students' initial and evolving thinking about functions in the classroom. The students also reflected on functions in their weekly journal entries and individual interview sessions conducted at the end of the semester, which provide further evidence for their learning. This section is primarily based on the interviews. The journal entries were also examined and used to triangulate and complement the interview responses.

In the interviews, the students talked about all of the topics in the course. For functions, they were asked to provide a definition, discuss learning challenges associated with the concept, and how they would address those challenges as teachers. The follow-up questions varied depending on the students' responses. When applicable, the students were asked to elaborate on their journal entries in which they referred to the dual nature of the function concept. Although not all the students realized function as a mathematical object at the end of the course, how students talked about function in the interviews indicate changes in their discourses compared to their realizations at the beginning of the course. The detailed transcripts of the interviews during which the students talked about their realizations of functions can be found in the Electronic Supplementary Materials. Table 2 summarizes students' definitions of functions in the context of the interviews.

The definition Carrie provided during the interview (Table 2, [38]) was similar to the one she provided in the survey (Table 1, [1]), where she talked about a particular relation where an independent and dependent (she did not use the term *variable*) were paired with each other. However, the definitions were different in that she now used terms like "set" and "map" when realizing a function. Her interview responses revealed that she was not using the word "rule" to refer to a single formula but to the condition for a given relation to be a function. In a journal entry, she mentioned that she "always thought of function as a process but not as a product[10]," but during the interview, she said "a function could be viewed as a process or product." When asked to elaborate, she said she could view function as a process "like a kind of machine. You put in a number and it spits out another number…like a number machine." She viewed function as a product in various ways: "I guess there are a lot of options. Are we talking about the functional value as the product? The graph of your algebraic representation of the

[9] Since the students in the classroom were also teachers, they were not only reflecting on their own discourses but also on how to teach the concept to their students. This was the main motivation behind Sally's question.

[10] Consistent with the literature on the duality of mathematical concepts, both the terms *object* and *product* were used in the classroom depending on the context. Carrie used the latter during the interview.

Table 2 Students' definitions of functions during the interviews

Student	Definition
[38] Carrie:	A rule that takes elements from one set and relates them or maps them to elements of another.
[39] Fred:	It's a process and object. I still want to use the input/output model because that is a strong model in my head.
[40] Lea:	A function is the dependency of one variable on another. As something changes, it is causing something else to change with respect to that.
[41] Martin:	For every input there is one and only one output.
[42] Milo:	One-to-one and onto come to mind.
[43] Ron:	For a mathematician, I would talk more about a mapping versus a student…I would stick with for every input you have to have a unique output.
[44] Sally:	A relation between an input and an output where for every input there's exactly one output.
[45] Steve:	A set of values where no input has two outputs.

function?" She then contrasted graphing, which she viewed as a process, with the graph she viewed as a product. These comments indicated that she was able to realize different aspects of function depending on the context; she could focus on mapping as a process and also refer to the outcomes as discrete numbers belonging to sets. Her definition reflected a preference rather than lack of awareness about the different metarules in the discourse on function.

A similar conscious preference to talk about function in one way rather than another was also present in Fred's discourse during the interview. When asked to define a function, he still wanted to use the input/output model (Table 2, [39]). He then mentioned that the reflections in the classroom helped him also view function as a process and object. When talking about the process view, he mentioned the input/output model and said "as you increase the value of x, the functional value will change. That's also a process model." His realization of function as a process was based on the assumption of dynamic change. When asked what he meant by the object view, he mentioned "just looking at a set of ordered pairs or like, hey, I see a line here. It's a static graph. It's there and now I am going to analyze it." Fred's views of the input/output model and dynamic change as processes were consistent with the definitions he provided in the survey (Table 1, [2]) and the classroom ([9], [11]). However, by the end of the course, he could also use the assumption of discreteness when realizing function as a set of ordered pairs and a graph as a static mathematical object.

Lea's definition during the interview (Table 2, [40]) was consistent with those she provided in the survey (Table 1, [3]) and the classroom [20]. She referred to function as a dependency of two variables as one variable changes with respect to another. Her realization was clearly based on the assumption of dynamic change, and Lea referred to this view of function as "a process of one variable depending on another." Yet, she was also aware of the discrete view of functions during the interview, which was also evidenced by her journal entries where she said a function could be both a process and product. When asked to elaborate on her journal response during the interview, she said that her students might think of functions as "just the outputs" or "a set of pairs" and she wanted them to "understand the process behind it as well" rather than viewing function as a "static set of values." Similar to Carrie and Fred, Lea's definition indicated a conscious choice regarding how to define a function. Such choice seemed to be based on particular criteria concerning the participants' personal or pedagogical preferences.

Martin's definition in the interview (Table 2, [41]) was somewhat similar to the one he provided in the survey (Table 1, [4]), but it was different from the one he provided in the classroom, which signified function as a single rule ([10], [15], [18]). His responses when talking about the student difficulties indicated that he was aware that not all functions had to be continuous and defined through a single rule. In a journal entry, he said "we have a tendency to assume continuity despite not being shown that a function is continuous" and mentioned that he will address this issue with his students. In the interview, he did not realize function as an object by talking about what a function *is*, and his definition was still based on a process view through the "inputs producing the outputs." However, his word use depicted the process accurately compared to his initial realization of function as a single rule.

Milo's definition indicated a limited view of function (Table 2, [42]) after which he said "I know that is not the definition of a function, but that's all I can bring up." Although he was not able to elaborate on a definition of function, he said "something that I would have to focus on is my discourse [as a teacher] and the definition and how we use them, what is a function and what is not." Milo's lack of explanation on functions contrasted with the elaborate reflections he provided in his journal entries where he talked about introducing functions to his students initially through Euler's definitions and introducing Bourbaki's definition only when the previous definitions did not fit his students' existing knowledge. Yet, at the end of the course, Milo did not objectify function and was still struggling with various aspects of functions.

Ron's interview responses indicated that he would use different definitions for mathematicians and students due to his pedagogical preference (Table 2, [43]). The latter definition was consistent with that in the survey (Table 1, [6]), whereas the former was consistent with the one Ron provided in the classroom [13]. However, neither definition was about what a function is; they only consisted of particular words or phrases. Later in the interview, it became clear that Ron considered mapping as a process when he said, "I am mapping the input to the output," which was very similar to his initial realization of function [13]. During the interview, he talked about the process and object view of functions, but his explanations indicated that, for him, the object signified the "answer" to a given problem and the process signified the "process of solving the problem." He wanted his students to "understand how to solve the problem" (the process) rather than "just get the answer" (the object). There was no indication that he considered these views as different ways to realize functions in his journal entries and interview responses.

Sally's objectified definition (Table 2, [44]) was very different than those in the survey (Table 1, [7]) and the classroom ([16], [18]). In the survey, Sally did not provide complete mathematical sentences; she used phrases associated with the function concept. In the classroom, she realized function as a single rule with a regular pattern. During the interview, she mentioned that she was no longer viewing function as a single rule and wanted to challenge her students' similar realizations of the concept. In the interview, when asked to elaborate on her journal entries, she realized function both as a process and object by saying that a function can "have certain processes like generating a graph or rule, or table, but the function itself is the definition of a function; it's a concept."

Steve's definition in the interview (Table 2, [45]) contrasted sharply with the definition he provided in the survey (Table 1, [8]) and the classroom discussions [12]. He no longer realized function as an equation without gaps; instead, he defined it through the assumption of discreteness by referring to it as "a set of values." When asked to elaborate on his answer, he said "a function is as a thing in itself; not an actual process," indicating that he was able to realize function as an object.

The results of the interview sessions and the journal entries suggest that Carrie, Fred, Lea, Sally, and Steve objectified function and, among those, four[11] also realized function as a process and object/product. Although Martin did not objectify function at the end of the course, his elimination of the assumption of continuity in his discourse on functions helped him change his mathematically incorrect narratives about function to a mathematically correct process view of the concept. These six students referred to the classroom discussions and activities as shaping their realizations of functions (see Electronic Supplementary Materials). Although the course may not be the only factor playing a role in students' learning, the comments in the journals and interviews indicate that some of the discursive changes the students demonstrated were triggered by the discussions in the classroom.

6 Conclusions and discussion

The findings of this study indicate that instruction that explicitly attends to the metalevel rules of the discourse on functions has the potential to support student learning. In this work, learning was evidenced by the changes in the way students talked about functions and their awareness and modification of particular metalevel rules in their discourses. This study demonstrates how the assumptions of Sfard's (2008) theoretical framework can be implemented in actual classroom practice to foster student learning. Such pedagogical approaches may provide the discursive transparency needed to enhance mathematical communication in the classrooms. Teachers play critical roles in such transparency in terms of eliciting participants' discourses and making the discourse a topic of reflection.

The students' learning of particular metarules of the discourse on functions does not mean that they did not demonstrate difficulties about the concept. Their work during the classroom sessions, their responses in the journals, and the final interviews suggested that some students were still struggling with various aspects of functions at the end of instruction. On the other hand, they could identify which aspects of functions they struggled with, the nature of those difficulties, and in what ways they needed to change their thinking to become more fluent participants in the discourse on functions. These findings are in accordance with the previous studies (Nachlieli & Tabach, 2012; Sfard, 1992, 2008) indicating that reification and objectification are inherently difficult processes while also demonstrating that it is possible to teach these aspects of functions to students. The results also confirm the tacit nature of metarules and their role in mathematical communication since the students in the study often mentioned that they did not learn about the assumptions shaping their discourses on function in their prior education.[12]

[11] Steve's interview responses indicated that, at the end of the course, he mainly realized function as an object rather than a process (See Electronic Supplementary Materials).

[12] The theoretical framework of the research was not discussed with the students during or after the study.

The pedagogical approach used in the study helped participants reflect not only on their own thinking but also on how to teach the concept to their students. In this respect, this study may have implications for teacher education in terms of providing some ideas and activities that can be used in teacher development for prospective and in-service teachers. It can be argued that a reason why the students in this study were elaborate in their reflections was because they were also teachers who regularly reflect on their knowledge to make it accessible to their students. Another possible explanation may be due to the fact that they were adults who had familiarity with functions prior to this study. It is critical for future studies to examine whether the metalevel learning demonstrated in this study would also be applicable to younger students as they are learning about the concept.

Currently, traditional lecture-based teaching is still dominant in university-level mathematics education (Güçler, 2014; Viirman, 2013). When discussing the origins of traditional teaching methods and their resilience, Sfard (2014) encourages educators to think about the possible impacts of discursive approaches on the teaching and learning of mathematics in postsecondary education "if one strives for helpful and sustainable pedagogical innovation" (p. 202). The work reported here can be considered as an effort in that direction.

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