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Is the derivative a function? If so, how do we teach it?

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Abstract This study investigated features of instructors' classroom discourse on the derivative with the commognitive lens. The analysis focused on how three calculus instructors addressed the derivative as a point-specific value and as a function in the beginning lessons about the derivative. The results show that (a) the instructors frequently used secant lines and the tangent line on the graph of a curve to illustrate the symbolic notation for the derivative at a point without making explicit connections between the graphical illustration and the symbolic notations, (b) they made a transition from the point-specific view of the derivative to the interval view mainly by changing the literal symbol for a point to a variable rather than addressing how the quantity that the derivative shows, changes over an interval, (c) they quantified the derivative as a number using functions with limited graphical features, and (d) they often justified the property of the derivative function with the slope of the tangent line at a point as an indication of the universality of the property. These results show that the aspects of the derivative that the past mathematicians and today's students have difficulties with are not explicitly addressed in these three classrooms. They also suggest that making explicit these aspects of the derivative through word use and visual mediators, and making clear connections between the ways that the quantities and properties of the derivative are visually mediated with symbolic, graphical, and algebraic notations would help students to understand why and how the limit component for the derivative are illustrated on graphs and expressed symbolically, and the derivative is expanded from a point to an interval, and properties of the derivative are investigated over an interval. More explicit discussions on these ideas perhaps make them more accessible to students.

Keywords Derivative · Commognition · Function · Calculus · Instructor discourse · Classroom

1 Introduction

Research on collegiate mathematics education has been growing over the past few years, especially about calculus learning (e.g., Artigue, Batanero, & Kent, 2007; Oehrtman, Carlson, & Thompson, 2008). In calculus, the derivative is difficult for students because it requires understanding other concepts—function, difference quotient, and limit (e.g., Thompson, 1994; Zandieh, 2000). This study investigates three calculus instructors' discourse about the

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derivative of a function as an object, with a focus on the transition between the derivative at a point and the derivative as a function on an interval. Studies suggested that the transition from the point-specific view to the interval view of function is non-trivial for students (Monk, 1994; Sfard, 1992). Building on such research, this study examines how three instructors addressed the derivative as a point-specific object and as a function on an interval. During this exploration, components of the definition of the derivative—function, difference quotient, and limit—are also considered.

Addressing the derivative as a function has been emphasized in various studies (e.g., Oehrtman et al., 2008), but not yet explored in depth especially in classroom discourse. To this end, this study explores introductory derivative lessons, specifically in what ways the derivative gets "a life of its own" as a function (Sfard, 2008, p.181), with the following research questions: In what ways did the instructors:

- Address the derivative as a point-specific value?
- Address the derivative as a function on an interval?

To answer these questions, the *commognitive approach* (Sfard, 2008) was adopted for the analysis of instructors' discourse. This approach has been used to highlight discursive features among different groups of speakers, such as children and adults (e.g., Sfard & Lavi, 2005) or students from two different countries (e.g., Kim, Ferrini-Mundy, & Sfard, 2012). By adopting this approach, this study adds to the body of research on classroom discourse at the post-secondary level (see Artigue et al., 2007).

2 Theoretical background

2.1 Commognition

Commognition (Sfard, 2008) views cognition and communication as two facets of the same phenomenon, and thus views thinking as an individualized version of interpersonal communication. Sfard (2008) defines *discourse* as any verbal or nonverbal communication with others or oneself, and views mathematics as a kind of discourse characterized by four features: word use, visual mediators, endorsed narratives and routines (Table 1).

Table 1 Features of mathematical discourse in commognitive approach (Sfard, 2008)

Feature	Descriptions	Further descriptions
Word use	Use of words signifying mathematical objects	Different speakers can use a word differently. It is an "all-important matter" because "it is responsible to a great extent for how the user sees the world" (p. x).
Visual media-tors	Non-verbal means of communication	Because people attend to visuals in specific ways depending on contexts, mediators need to be viewed as a part of a thinking process rather than auxiliary means representing preexisting thought.
Endorsed narra-tives	Utterances that speakers endorse as true	Students' endorsed narratives are often different from what the professional mathematics community endorses as true.
Routines	Well-defined repetitive pattern in discourse	Patterns can be found in speakers' use of words and visual mediators, or in the process of creating and endorsing narratives.

By explaining someone's mathematical thinking through this lens, Sfard (2008) addresses development of mathematical objects – both historical and individual – through changes in features of the discourse about the objects. Compared to historical development, which is *in general* upward oriented from concrete objects towards more abstract objects, the commognitive approach characterizes learning as individuals' attempting to link a new "concept" to familiar objects by first mimicking more experienced participants' discourse, and then over time participating in the discourse gradually by speaking more "like" experienced participants of the discourse through communicating with them, similar peripheral participants (e.g., peers), or oneself. Successful *learning* would result in a relationship between the new "concept" (more abstract objects) and the familiar objects (related concrete objects), which would be the same as the one developed by mathematicians from history. Despite the difference in direction of abstraction, as shown in numerous studies, there exists a similarity between students' "misconceptions" and "difficulties," and those of mathematicians who first developed the concept in history, which provide valuable insights for individual learning (e.g., Radford, 2009; Sfard & Lavi, 2005).

According to Sfard's framework, during the development of one's mathematical thinking, the features of the discourse change. Specifically, students, who are relatively new to the discourse apply a course of action, called a routine, that previously worked on familiar objects, to a broader range of mathematical objects. This may lead to a different realization of what course of action to apply and when to apply it (e.g., from subtraction as "an action of taking some number of objects from a larger number of objects" to "a binary operator on pairs of integers"). As a result, what is believed as true, called an endorsed narrative, about the objects may have to change for the new concept (e.g., from "subtraction makes the number smaller" to "subtraction can make the number bigger"). Students' unfamiliarity with the features of the experienced participants' (e.g., teachers') discourse, including their word use and visual mediators, may result in general dissonance in the communication, which Sfard (2008) called *commognitive conflicts*. Through communication with others, students resolve such conflicts by gradually "adjusting their discursive ways" (p. 145).

The history of the derivative contains mathematicians' realizations of the derivative as a point-specific quantity involving the limit and the derivative as a function, as well as their use of two kinds of visual mediators – graphical illustration and symbolic algebra (Grabiner, 2004). One of the ways that an early concept of the derivative emerged in seventeenth-century mathematicians' work including Fermat and Descartes, involved graphical illustrations of the slope of a tangent line to a curve at a point and their attempt to find the algebraic notation to explain how a secant "become[s] a tangent" (p. 220). Later, the derivative as a function was developed as mathematicians sought a rigorous definition through algebra as shown in Cauchy's work (Grabiner, 2004). Among the various mathematicians' work and different approaches that were employed, the historical development of the derivative demonstrates a geometrical approach for developing the concept of the derivative (i.e., finding a tangent line) and its further development with two tools–symbolic algebra and analytic geometry–as one of the mainstreams. In this particular approach, mathematicians' difficulties during this development included expressing the "vanishing" increment (equivalent to h in Fig. 1), and finding a definition of the derivative (with symbolic algebra), which works for any x (Grabiner, 2004, p. 222). These difficulties partially contributed to the exploration of whether and how similar critical transitions such as understanding the "vanishing" increment h through the concept of limit and developing a rigorous definition for the derivative as a function are discussed in today's calculus classrooms and what kind of visual mediators are involved in such discussions.

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Fig. 1 Definition of derivative (Stewart, 2010, pp.107–114)

The analysis undertaken for this study identifies patterns in word use for each component in the definition of the derivative and the relations among the components when mediated with different visuals (symbolic, graphical, algebraic and gestural visual mediators). The analysis also focuses on the objects to which these components were applied while the instructors were transitioning among the views of the derivative as a point-specific value and as a function using various visual mediators.

2.2 Components in definition of derivative

The derivative of a function can be defined at a point or on an interval (Fig. 1). It includes four components – function, difference quotient, limit, and derivative as a function. Each component will be described as a process and object in this section (e.g., Zandieh, 2000).

2.2.1 Function

A function can be seen as a process of mapping one value of the domain to one value of the range, or as an object, that is, the relation between two sets. The commonly used notation *f(x)* shows this dual nature (i.e., *x* shows an input, *f(x)* shows the output, and *f* shows their relation, Oehrtman et al., 2008), which often raises difficulties when students try to understand this notation (e.g. Gray & Tall, 1994; Sajka, 2003; Sierpinska, 1994). Some researchers go further, defining the input–output pair at a point or on an interval (e.g., Dubinsky & McDonald, 2001; Monk, 1994; Sfard, 1992). Despite some differences in classifications and descriptions, these studies all suggest two views for a function—called *point-specific* and *interval views* in this study—and reported that making transitions between them is non-trivial to students (e.g., Monk, 1994).

2.2.2 Difference quotient and limit

The difference quotient can be considered as a process of multiplicatively comparing the changes in the independent variable and the dependent variable or as an object, the ratio itself. The difference quotient is often used with the term, "average rate of change" (ARC) or "slope of a secant line." Then, the limit can be considered as the limiting process on the difference quotient (e.g., ARC) over smaller and smaller intervals, or as an object, that is, the limit that defines the derivative at a point as the end product (e.g., instantaneous rate of change, IRC). Studies have shown that students focus on the algebraic process of finding ARC and do not appreciate what the resulting object means, which may make it difficult to understand IRC as the limit of ARCs (Hauger, 1998). Existing studies have shown that students' misconceptions about the limit (e.g., 0.999999…never reaches 1; Tall & Vinner, 1981) are related to their thinking about the tangent line (e.g., the secant lines never reach the tangent line) (Hahkioniemi, 2005; Katz & Tall, 2011; Orton, 1983). Recently, Zandieh and Knapp (2006) addressed students' thinking about the limit component using *metonymy*, which occurs when one entity is used to refer to another that is related to it (p. 1). Zandieh and Knapp said that although metonymy allows instructors to use familiar situations (e.g., velocity as the derivative), some students' uses of metonymy are not mathematically valid (e.g., tangent line as the derivative).

2.2.3 Derivative

The derivative can also be viewed as (1) a point-specific object; that is, a quantity at a point visually mediated with the slope of tangent line to the graph of the function $y=f(x)$ at the point; (2) as a function at any point usually denoted with a letter (e.g., x) and visually mediated with the notation $y=f'(x)$ and the graph or equation of the derivative of a function. Existing studies have shown that it is hard for students to understand the relation between the point-specific and interval views, and the co-varying nature of the derivative in relation to the original function. For example, Zandieh (2000) described the co-varying nature as "a process of passing through…infinitely many input values and for each determining an output value given by the limit" of the difference quotient (p. 107), and found that only a few of the students she interviewed included some explanation of covariance. Oehrtman et al. (2008) explained "coordinating the IRC with continuous change in the independent variable" as a crucial ability to understand the co-varying nature of a function (p. 358) but students were often unable to make such coordination. Similarly, Nemirovsky and Rubin (1992) explored high school algebra students' understanding about rate in physical settings while they worked with graphs, and found that many students drew a graph for the derivative similar to the original function graph without appreciating the relation between a function and the derivative over the interval.

2.3 Commognitive approach as theoretical framework

This section details how this study connects point-specific and interval views for the derivative to discursive features described in commognition (Sfard, 2008), namely, word use, visual mediators, routines, and endorsed narratives.

2.3.1 Word use and visual mediators

Point-specific view of derivative The derivative of a function at a point is described with words "the instantaneous rate of change" (IRC) and "the slope of the tangent line" of a function at the point, which is often visually mediated with the tangent line to the graph of a function. In the symbolic notation, it is visually mediated with the limit of the difference quotient (DQ) (Fig. 1). This limit component can be considered as a process, a result, and an operator. Specifically, if we consider $f'(1)$, the DQ expression $\frac{f(1+h)-f(1)}{h}$ can be illustrated with a secant line; the limit symbol $\left(\lim_{h\to 0}\right)$ is added to the DQ notation presented with several secant lines; and finally the result of this process written as $f'(1)$ is presented with the tangent line on the graph. This limit can be also considered as an operator on the symbolic expression, $\frac{f(1+h)-f(1)}{h}$; on the algebraic expression for DQ of a specific function; or on the secant lines in a graphical illustration. Textbooks may show the limit process with numerical values for the slopes of secant lines approaching a specific number and the limit as the result with the number (the slope of the tangent line). The word "slope" plays an important role connecting the graphical and symbolic mediators for both limit as a process and as the result, especially when these numerical values are not provided. Several studies show the importance of such connections for the slope by stating that its algebraic expression and its visual representation such as graphs or diagrams cannot be simply taken as isomorphic (Lobato, Rhodehamel, & Hohensee, 2012; Zaslavsky, Sela, & Leron, 2002). However, the word "slope," by and large, is used in a tacit undefined manner by instructors; in other words, the term is not always explicitly used with the terms "secant lines" and "tangent line," which will be shown later in my data.

Interval view of derivative Once the derivative at a point is defined, the concept of derivative is extended as a function defined on an interval. The transition from the point-specific view to the interval view of the derivative is often made with the derivative at a point expressed as a letter (e.g., $f'(a)$ for a generic point a) or at multiple points (e.g., $f'(2), f'(2.5)\ldots$). In both cases, words used with the letters in the symbolic mediators play an important role in this transition. For example, for the notation in Fig. 1, one textbook uses the words "let[ting] the number a vary" and "replac[ing] a by a variable x," to "obtain…the derivative of a function" (Stewart, 2010, p. 114). Also, the plural form of words such as "slopes" refers to the derivative at multiple points and visual mediators such as multiple tangent lines may be used before the complete graph of the derivative is drawn. Sometimes, the derivative at a point (e.g., $f'(a)$ for a generic point a) and the derivative at multiple points are both used in transition to the function $f'(x)$ (Font, Godino, & D'amore, 2007, p. 5).

Once the derivative of a function is extended to an interval, the derivative gets "a life of its own" as a function on an interval. In an animation, it can be visually mediated with tangent lines constantly changing, and a corresponding point moving on another x-y plane. However, since drawings in the book and on the board are fixed, this process can be mediated with illustrations of or gestures towards multiple and discrete tangent lines. The derivative as a function can also be addressed while discussing its properties such as its dependence on x and where it exists. These properties can be compared with properties of the original function with respect to their notations ($f(x), f'(x)$) or behaviors (e.g., "differentiability" and "continuity"). Discussing and proving the differentiation rules are also examples of addressing the derivative as a function. In general, the rules are written with notations involving the letter x paired with graphical and algebraic notations justifying the rules.

2.3.2 Routines

Routines are operationalized as patterns in the instructors' discourse about the derivative. For example, in explaining differentiability on graphs, an instructor might regularly use the phrase "tangent line" while drawing the graph of or making the gesture of secant lines approaching a tangent line. Another instructor may show the tangent line only without secant lines on examples while using the phrase "the derivative." A third might use word "the limit" instead of "derivative" with a gesture following the original function rather than imitating the secant lines or tangent line. To be coded as a routine, a similar course of action had to be observed during at least three different lessons to reflect its repetitive nature in the data collected within the limited time frame.

2.3.3 Endorsed narratives

Endorsed narratives are operationalized as the statements considered as true, including definitions, theorems, and justifications identified in the classroom discourse. Endorsed narratives of interest are typically those that are taken for granted by the instructor, but perhaps not understood by the students, or vice versa. For example, the instructor may endorse the constant multiple rule about the derivative of a function by comparing the slope of one tangent line to the slope of the corresponding tangent line for the graph of the constant multiple of the function, and consider that this illustration would be sufficient to address any point on the domain. However, students may not see this illustration as proof of a rule over the interval.

3 Research design

The data for this study come from a bigger study that includes classroom observations, student surveys, and interviews with instructors and students. This study reports the analysis of the instructors' discourse during the first three lessons about the derivative when the derivative was defined and applied to functions.

3.1 Site

This study was conducted in three *Calculus I* classes at a large Midwestern university. Each class had around 30 students mainly majoring in natural science and engineering. One class used *University Calculus* (Hass, Weir, & Thomas, 2008) and the other two used *Thomas' Calculus* (Weir, Hass, & Giordano, 2006). These books were written by the same author team and included the same sections for the lessons covered in this study.

3.2 Participants

The three instructors, Tyler, Alan, and Ian, were chosen based on different backgrounds and teaching styles.[1] Ian had a Ph.D. in mathematics and had taught *Calculus I* several times. Tyler and Alan were doctoral students with masters' degrees in mathematics, who were teaching *Calculus I* for the first time, but who had taught a reform-based calculus course several times. All three lessons were lecture-based; Ian and Tyler used a blackboard, and Alan used PowerPoint presentations.

3.3 Method

All derivative lessons taught by the three instructors were videotaped and supplemented with field notes. The author transcribed classroom videos and coded them, identifying the items such as speakers, words, writing, drawing, and gesture, which are consistent with commognitive approach (Sfard, 2008). "Speakers" were the instructors or students. "Words" included the terms for the four components and the views for the derivative. "Writing" included symbolic and algebraic mediators. "Drawing" mainly included graphs of a function or the derivative. "Gesture" included the instructors' gestures imitating a function's behavior or pointing to what was on the board. Then the transcripts were divided into several episodes based on the topics of the discussion. A routine table was made for each episode by (a) identifying the instructors' actions and placing them in the second column of the table in chronological order, and (b) identifying examples (e.g., different graphs or problems) on which these courses of action were used and placing them in the first column of the table, and (c) placing what the instructor and students said and did (e.g., drawing, writing, and gesturing) in each cell of the table (See Table 2 for an example of a routine table). Then the instructors' or

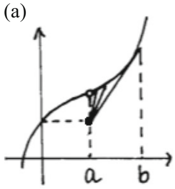
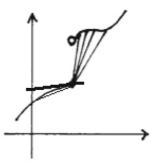
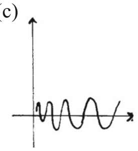
[1] These are pseudonyms of the instructors.

students' word uses and visual mediators addressing the different views of the derivative were coded in the table. Finally, how each view of the derivative was addressed and which components were involved was summarized for each table. Once the summaries for each routine table were made, the similar features of the instructors' discourse were combined together (e.g., the definition of the derivative at a point and differentiability were presented together because the instructors' word use and visual mediators were very similar in their discussion about these two topics).

4 Results

This section addresses instructors' word use, visual mediators, routines, and endorsed narratives – about the derivative as a point-specific value and as a function. Symbolic notations include expressions, $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, or parts of them. Algebraic notations include expressions and equations to which the symbolic notations were applied. The instructors' discussion of the derivative at a point covers the function, DQ and

Table 2 Ian's visual mediators, routines, and word use in discussing the existence of the limit, $\lim_{b\to a}\frac{f(b)-f(a)}{b-a}$

Visual Mediator	Routine	Words and gestures
(a)  [*f(a)* exists. $f(a)\neq\lim_{x\to a}f(x)$]	1. Drawing secant line.	I will take…*b* and let it get closer and closer to *a* [drawing *secants*]. Do you see what happens to all these *secant lines*? What happens to those *slopes?* [Hand *vertical*]
	2. Describing limit of the secant lines	Student: becomes vertical? Yes, the line becomes pretty much vertical.
	3. Closure	*The slope* blows up. There is *no finite limit* to this expression.
(b)  [*f(a)* exists. $\lim_{x\to a}f(x)$ does not exist.]	1. Drawing secant line.	[From the right] Those *secant lines* will look like this [drawing *secants*].
	2. Describing limit of the secant lines	They become vertical.
	3. Drawing secant line.	The secant lines on the other side will converge. [drawing tangent and *secants*]
	4. Describing limit of the secant lines	We will have the limit from other [left] side.
	5. Closure	*Both side limits* need to exist and to be the same. But I only have a *limit* from this side.
(c) 	1. Drawing secant line.	What will happen to the *secant lines*? [*Hand oscillating*]
	2. Describing limit of the secant lines	*It* will not converge to any thing [*Hand oscillating*].
	3. Closure	So, *no tangent line.*

limit components, specifically, (a) the relation between a function and the derivative at a point, and (b) applications of the limit of the difference quotient as a process, result, and operator. Their discussion about the derivative as a function covers (c) transitioning from the point-specific view to the interval view of the derivative, (d) illustrating the derivative as a function given as a graph, and (e) justifying the differentiation rules on graphs. Many routines and endorsed narratives were identified for these topics, but the results report only those features pertinent to the research questions.

4.1 Point-specific view of the derivative

Historically, finding the tangent line on graphs, which visually mediates the derivative, and symbolizing its slope with notation were pivotal advancements in the development of the derivative. Similar steps were seen in these three classrooms. The rigorous definition of the derivative using the concept of limit went through several iterations, including limit as a process, as a result, and as an operator. The instructors discussed the derivative at a point in relation to the original function, and then with respect to the limit using both graphical illustrations and symbolic notations.

4.1.1 Local relation between function and derivative a point

The local properties of a function revealed by the derivative were highlighted by the instructors' words used with graphical mediators (e.g., tangent line, slope). Where derivative was defined was highlighted by words used with symbolic mediators (e.g., at an instant t_0). Examples illustrate instances where this occurred in each of the classrooms.

Ian only explained the local relationship between a function and its derivative with the phrase, "slope of the tangent line." In contrast, both Alan and Tyler provided much more detail in their explanations. Alan used four visual mediators (Fig. 2) taken from an earlier chapter in the book. He used words "instantaneous rate of change" for these visuals, and "how fast is something changing at an instant" for the relation between the function and the derivative, without specifying what he meant by "something." He emphasized how to find the IRC with the phrase "getting h [to] zero," and he used the words "instant," and "where" along with the words "a specific time," and "exact point," for literal notations "t_0" and "x_0."

Tyler mainly used graphs and gestures to highlight the local features of the relationship between the function and its derivative. The two phrases he used were "the direction the graph is heading," which was mediated with an illustration of and gesture toward the tangent line at a point, and "how fast function was changing," mediated with a gesture pointing to the word "slope" written on the board. Although the word "slope" connects the two phrases "direction" and "how fast," he did not make this connection explicit by quantifying "slope" or "how fast" as a number, or mentioning its sign (i.e., positive/negative) in relation to the function behavior (e.g., "increasing" or "decreasing").

4.1.2 Limit as process, result, and operator

Limit as process and result The instructors mainly addressed the limit as a process using both words and graphs, but there was a brief instance of one instructor using symbolic notations with these words. They addressed the limit as a result using words with algebraic mediators in computations. For example, all three instructors used similar graphs as visual mediators illustrating a function, secant lines, and tangent line (see Fig. 2b) to explain the limit in the definition of derivative at a point (Fig. 1). Their word use highlighted a process: the secant lines “approach” the tangent line at a point ($x=a$), as another point “gets closer and closer” to the point, or the length of the interval between them (h) “gets smaller and smaller.” Each instructor used these terms for the limit process multiple times while defining the derivative at a point or illustrating at which points the derivative exists on graphs. Their illustrations and words almost exclusively addressed a single point on the graph rather than multiple points. Although one instructor used similar words “gets smaller, smaller, and smaller, close to zero” for the numerator and denominator of the symbolic notation $\lim_{b\to a}\frac{f(b)-f(a)}{b-a}$ once, he did not connect the graphical with the symbolic regarding the quantities getting “smaller and smaller” versus the point getting “closer and closer.” The other instructors did not use such terms (e.g., “getting closer and closer”) with symbolic notations. Rather, they used the word “limit” as a result in the algebraic computation of the derivative: “when h goes to zero, the limit is 2.”

In summary, when teaching the limit as process, instructors primarily used graphical mediators and words such as “approaching” and “getting smaller and smaller.” In teaching the limit as a number or result, instructors applied symbolic mediators to equations of functions, and used words such as “the limit is.”

Limit as operator All three instructors used all three visual mediators – algebraic, symbolic, and graphical – when discussing limit as an operator. Whereas their words used with symbolic and algebraic mediators explicitly show the objects on which the limit operated, their words used with graphical mediators did not always show the objects. Specifically, while the instructors applied the limit on the written expression for the difference quotient (e.g., $\frac{f(a+h)-f(a)}{h}$), on algebraic notations (i.e., algebraic expressions when the symbolic notation was applied to equations of functions), their word use and gestures highlighted the objects on which the limit was applied. They applied the limit on symbolic notations and graphs simultaneously while defining the derivative, and then applied it on the algebraic notations later while finding an equation or showing the existence of the derivative. Their word use and gestures while using algebraic notations clearly pointed to the objects the limit was applied to (i.e., DQ as “a continuous function of h,” where 0 can be “plug[ged] in”). However, while they

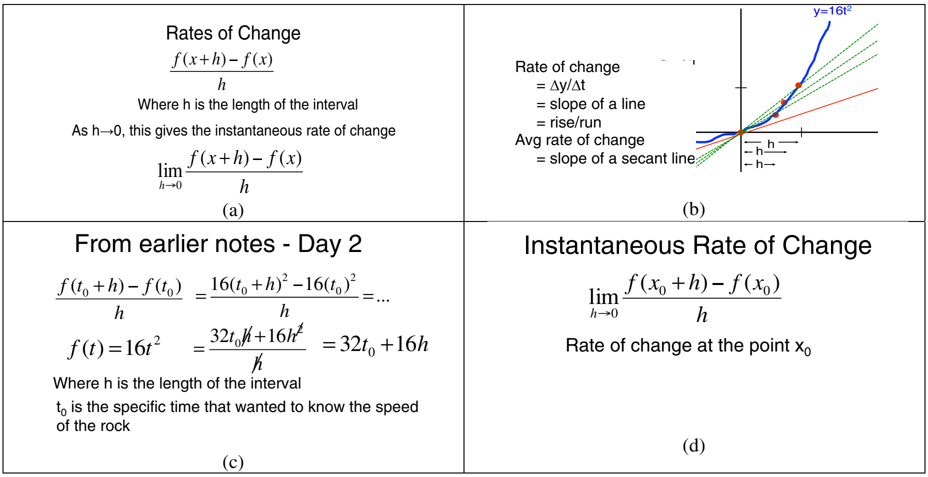


Fig. 2 Alan’s words and visual mediators for IRC included in PowerPoint

were illustrating the limit with a graphical mediator, their word use and gestures did not clearly identify the slopes of secant lines as the objects on which the limit operated, as illustrated in next few examples.

First, the instructors' word use did not explicitly indicate the object on which the limit operated when illustrating the limit process for defining the derivative or determining where the derivative exists. They all used "slope(s)" with "secant line(s)," but did not use "slope" with "tangent line" in most cases. Alan's explanation of Fig. 1b was:

> The average rate of change is the green one that's *the slope of the secant line* [showing one secant line]…Getting closer and closer for *h* to get into zero, we got really closer [showing *more secant lines* for each *h* value], h got smaller, even smaller until we get *h* going to zero, so we get *the tangent line* [showing the tangent line]. (Ian, 09-14-2009, Italics added)

In cases where the derivative did not exist, they explored the existence of "the derivative" using the existence of the tangent line, rather than mentioning the undefined "slope." All three instructors had routines for discussing the limit or the existence of the derivative at a point. For example, Table 2 illustrates Ian's routine in discussing the existence of "slope of tangent line" written as $\lim_{b \to a} \frac{f(b)-f(a)}{b-a}$.

In (a), Ian drew secant lines converging to a tangent line, and used the word "slope." In the other examples, he did not use "slope", instead said "no tangent line," or "different limits." His routine was to draw a graph, draw or indicate a number of secant lines, describe the limit, and reach a conclusion.

Second, the instructors' gestures also did not explicitly indicate the object where the limit was applied on the graph of a function. They said "limit" or "derivative" while gesturing toward the graph of a function, but not toward the tangent line. This combination becomes problematic when both the limit of the function and the derivative do not exist at the point. Alan's routine in discussing non-differentiability of $y = \sqrt{x}$ at (0, 0), and Ian and Tyler's routine in discussing non-differentiability of an oscillating curve (Table 2c) included such gestures. For example, in Table 2c, Ian's use of a hand gesture following the graph of the original function and a single pronoun "it," were connected to his closure "no tangent line." This gesture could be interpreted as the limit of the original function at (0, 0), rather than the slope.

In summary, in addressing the limit as operator on the symbolic and algebraic mediators, the instructors clearly indicated the object on which the limit operated with words for DQ such as "a continuous function of *h,*" where 0 can be "plug[ged] in." However, when using graphical mediators, their words and gestures did not always indicate the object where the limit was operating. Rather, their word use emphasized the "secant line" as the object, and the "tangent line" as the result, instead of their slopes. Also, their gestures of following the graph of the original function did not make explicit non-differentiability at the point.

4.2 Interval view of derivative

Historically, one of the ways to look at the development of the concept of the derivative is the transition from point to interval views; the derivative as a point-specific object developed first and the derivative as a function, whose definition works for every *x* on its domain, developed later. With the idea of limit already in place, and providing a rigorous definition of the derivative at a point, the instructors addressed the derivative as a function with symbolic notations using a literal symbol (x), intentionally different from the one used for a point (a). They also addressed the derivative as a function with a graphical mediator while graphing the derivative over the interval and justifying the differentiation rule on graphs.

4.2.1 Definition of derivative as a function

Evidence suggests that while addressing the derivative as a function with symbolic notations, the instructors used letters in the notation and words such as "points" and "variable" to explain where the derivative is defined. Specifically, their word use highlighted how they related the point-specific view to the interval view, and the relation that the derivative as a function describes. All three instructors started with a letter a or x_0 and word use "at point" for derivative at a point. Then, they changed it to another letter meant to be more general (x), and/or changed their word use from "point" to "any point." Table 3 summarizes such transitions.

In Alan's classes, two different notations x_0 and x were paired up with different words "point," and "any points." In Ian's classes, the same notation a was attached to the word "fixed," and "any point." These notations and words were consistently used while Alan and Ian discussed derivative as a function. Alan compared the two notations in Table 3 as "the slope of tangent line *at a single point*, so it will give us a single value," and "slope of tangent lines *at all points*…so this is gonna give us some formula." Then, Alan transitioned back to the point-specific view by saying, "if we *plug in* the point x_0, it will give us that value," and his word use highlighted the literal symbols (x_0 vs. x) and the algebraic procedure ("plug in"). In comparison, Ian's word use highlights the "dependent relationship" that the notation $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$ entails:

> This expression *depends on a.* If we *change a*, we will have *a different limit.* We will have different slopes of tangent line on different points on the curve. In other words, this expression is *a function…of a,* change a [writing "$=f'(a)$" on the board], it will change the value. (Ian, 09-14-2009, Italics added)

Here, his word use "depends on a" and "a function of a," and writing "$=f'(a)$" all address the derivative as a function of the variable a. Ian also used gestures for this "dependent relation" on the graph of a function while saying, "as a changes, as a moves around" [moving a hand along the x axis]. There was no letter change in the definition. He later changed the letter "a" to "x" while applying $\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)\right)$ to the equation of a function ($f(x)=c$) by saying, "we will have the *same argument* as x (pointing to the notation $f(x)$ and $f'(x)$).

Tyler also made a transition with the literal notations, but not with word use. He used the letter "a" with the words, "that particular point," but did not use any word indicating point-specific or interval views with "x." He did not provide a further justification for this literal transition besides his mention of "$f'(x)$" as "what always you wanna" compute. Tyler later addressed the derivative as a function while he was transitioning from algebraic computation of the derivative (i.e., computation of $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ for the equation of a function) to graphical illustration of the derivative. His word use and gestures highlighted the symbolic notation in the transition:

> This is the way to define *the derivative at one point* [circling $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$], but we can do *simultaneously at any point* of the graph, and then we get *a new function.*

> Starting with some function *f*(*x*), [Pointing to the equation $y = \sqrt{x+2}$] we are getting *a new function f′*(*x*). *We can graph this function as well.* (Tyler, 09-30-2009, Italics added)

Here, he transitioned from the point-specific to interval view using the notations $f'(1)$ versus $f'(x)$ and words "one point" versus "any points," and then stated that the end product $f'(x)$ is "a new function," which can be graphed.

While applying the definition of derivative to a function, each instructor connected the interval view to the point view, explaining the interval view as a general case of the point-specific view. They used the symbolic notation for a specific function, simplified the equation, and took the limit to find the value(s). One common action was using a specific value of x to calculate the derivative at a point. For example, Alan's word use highlighted the relation between the two notations $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ versus $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ in Fig. 3.

Here, Alan used the phrase "function and the formula" for the notation $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, and "plug in" for its relation to $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$. Ian and Tyler both made a similar connection while finding the slope of the tangent line at specific points using the equation of the derivative function.

In summary, in defining the derivative as a function with symbolic mediators, the instructors expanded the derivative at a point to the derivative on an interval by changing the symbol for a point (a or x_0) to a symbol for a variable (x) or by changing words for the same letter a from "a point" to "variable." The instructors also addressed the derivative as a function as a general case of the derivative at a point, applying the symbolic notation for the derivative of a function to compute the derivative at a specific point. They used algebraic mediators in the computation, and the connection between the interval to point-specific views was made with words "plug-in" or equations that showed this substitution.

4.2.2 Derivative as a function on the graph

The instructors used graphs on the coordinate plane as visual mediators for derivative as a function. While drawing graphs, they used words and numbers quantifying the derivative as positive or negative on an interval or as a numerical value at a point. Specifically, Alan and Tyler graphed the derivative of a function given as a graph highlighting their use of the point-specific and interval views of the derivative and transitions among these views. Ian's lessons did not include graphing the derivative of a function given as a graph. While transitioning between the point-specific and interval views, both Alan and Tyler quantified the "derivative" and "slope" as positive or negative, or as a specific numerical value, but such instances occurred rarely and only with a function that had limited graphical features (i.e. where the derivative was zero, or the original function was linear).

Alan's process for graphing the derivative of a non-linear function included words addressing the interval view, but only once included quantification of the derivative as a number and only where the function has a horizontal tangent. Specifically, while graphing the derivative of a quadratic curve, he mentioned the value of the derivative at the vertex as zero as describing how the value of the derivative changes:

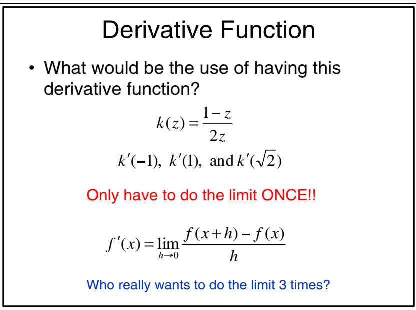
> When we graph this, we've got something that's gonna be *negative*, first. It's kinda *steeper* here. As it *gets lower*, our tangent lines are *flattening out* for each of these. So, we are gonna get eventually to *the vertex down here* the parabola…so, it *gets closer to zero*…The other side, they become more and more positive. So, kind of looks like this [completing an increasing line]. (Alan, 09-28-2009, Italics Added)

Table 3 Transition from point-specific to interval views

	Symbolic notation and key words	
Instructor	First	Second
Alan	Instantaneous Rate of Change $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ Rate of change at the point x_0 "This (circling x_0) is like a point"	Derivative Function The derivative function f(x) with respect to the variable x is the function f' whose value at x is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ provided the limit exists
		"It could be any of x's…any of the points…where this function is defined."
Ian	$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$ derivative of f(x) at x=a "I considered *a* to be *fixed*. Like 5."	$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$ derivative of f(x) at x=a "I can compute the slope of the tangent line *at any point*."
Tyler	Def. The derivative of f(x) at x=a (f'(a)) is the slope of the tangent line. "Slope of the tangent line to the graph…*at that particular point*."	Def. $f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}$ "(Writing $f'(x_0)$) x_0 is what I used to call *a*…I don't like my notation very much …(Erasing $f'(x_0)$ and writing $f'(x)$). Let's try to compute $f'(x)$ because what always you wanna do."

Here, the derivative was first quantified as "negative," then "get[ting] closer to zero" then "zero," and then "more and more positive." Note that there was no quantification of the derivative as a number besides zero. His graphing procedure of the derivative of a piece-wise linear function similarly addressed the interval views. His word uses included the plural form of words such as "the slope of the tangent line…for all these points" and the hand gestures imitating these lines over the interval.

In summary, in discussing the derivative as a function on graph, the instructors quantified the derivative as a number mainly by using functions with limited graphical features such as linear functions or functions with horizontal tangents. They quantified the derivative as "positive" or "negative" on an interval rather than as numbers showing how the derivative changes as x changes.



We want to know the derivative of *f* at these three points. We could do x_0, the top equation *three times…at each of these points*. Or you can do the bottom equation, do it one time get the function, *the formula* for the derivative function and then *plug in these values*. So, we are gonna do that with this one [circling

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ on the board].

Fig. 3 Alan's PowerPoint and discussion for notation differences (09-25-2009, Italics added)

4.2.3 Properties of the derivative as a function

The instructors also talked about the derivative of a function while discussing its properties as a function on graphs. In the graphical justification of differentiation rules, their word use and visual mediators highlighted how they addressed the interval views of the derivative in relation to the point-specific view. Each instructor used a singular or plural form of the words “slope” or “tangent line,” or a single or multiple form of the illustration and gesture of “the tangent line” differently from each other. For example, in the justification of the derivative of a constant function, all three instructors said that the tangent line is the same as the original function, and thus that the slope is zero. However, Alan used several hand gestures on the line whereas Ian and Tyler used the tangent line at a single point to justify the rule. In general, Tyler and Ian’s routine of justifying differentiation rules on graphs included a consistent use of a single tangent line without mentioning or illustrating any other points. For example, Tyler’s visual mediators (Fig. 4) and word use (e.g., “you are looking at the derivative at one point.”) only addressed the point-specific view, although he addressed the interval view by saying the phrase “the derivative of a function” for the notation, $f'(x)=cg'(x)$.

In summary, in justifying the differentiation rules on graphs, the instructors used words and symbolic notations for the properties of a function on an interval, but used a graphical mediator for the derivative at a point without making a clear connection between the point-specific and interval views of the derivative.

5 Discussion and conclusion

Building on accounts of the historical development of the derivative and existing studies about student thinking about the derivative, this study looked at how the three instructors defined the derivative at a point using the concept of the limit, and transitioned to the derivative of a function on an interval. Applying the commognitive framework (Sfard, 2008) revealed key features of their word uses and gestures to address the four components – function, limit, DQ, derivative – with different visual mediators – symbolic, algebraic, graphical. Several results stand out:

- Their discussion of the point-specific view of the derivative highlighted their use of symbolic notation for the limit along with a graphical illustration, the tangent line, without explicit connections between how the symbolic and graphical mediators are related to each other. The only connection mentioned between these two visual mediators was the word

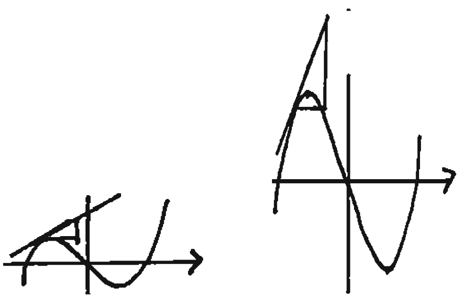


Fig. 4 Tyler’s illustration of constant-multiple rule

"slope," which the instructors used inconsistently when illustrating and talking about limits, secants, and tangents.

- Their discussions of the derivative as a function highlighted limited use of visual mediators. They mainly used symbolic notations to transition from point-specific to interval views by changing letters (e.g., "*a*" for "a point," versus *x* for "variable") with little explanation of how the value of the derivative changes. They did not address the co-varying nature of the derivative with algebraic notations.
- Graphical mediators for the derivative as a function were limited to illustrations of zero, positive, and negative values of the derivative rather than numerical values that change over an interval.
- They justified properties of the derivative function with single examples – that is, showing the value of the derivative at a single point as an indication of the universality of the property.

These features of the instructors' use of words and visuals were not just found in a snapshot of a class. They were consistently reflected in routines and endorsed narratives in three instructors' discourse.

These results showing the instructors' uses of various visual mediators without explicit connections between them, their limited discussion on how the derivative as a function varies, and their dependence on symbolic and algebraic notations, seems related to some well-known student *difficulties* with the derivative. First, their use of the tangent line as a visual mediator for the derivative without making a clear connection to or mention of the quantity that is visualized is related to studies showing that illustrating the tangent line does not automatically connect to the slope as a quantity (e.g., Zaslavsky et al., 2002) and specifically supports students' description of the derivative as "tangent line" (e.g., Park, 2013; Zandieh & Knapp, 2006). Second, the transition from the point-specific to interval view of the derivative based on literal notations without addressing the co-varying nature of the derivative is related to studies showing that such transitions are nontrivial to students (e.g., Monk, 1994), and students tend to use algebraic notation without knowing *what it means* for the derivative to be a function (e.g., Oehrtman et al., 2008).

The connections made here between the results of this study and students' difficulties found in existing studies do not imply a causal relation between instructors' teaching and students' learning. However, the commognitive view of learning as a process in which students develop their thinking by communicating with others, especially with more experienced participants in the discourse (Sfard, 2008), emphasizes the role of instructors in learning because of their unique participation in the discourse. Also, from the commognitive view that takes similarities between the historical and individual development of discourse about mathematical topics into account for learning (Sfard, 2008), the difficulties that past mathematicians had with writing a rigorous definition of the derivative that includes the limit component and works for any *x* implies that these aspects of the derivative cannot be considered as trivial to today's students. However, the data seem to imply that the instructors assume that both the relationship between symbolic and graphical notations of the derivative and the transition between the point-specific and interval views of the derivative are clear to their students, for they did not make these aspects explicit in their routines identified across various examples while they were endorsing narratives in their class. Despite various explanations for these phenomena, including instructors' goals for teaching and limited resources for visualization provided in classrooms, this study provides examples in which instructors' expertise blinds them to the difficulties students have in understanding what seems obvious. This indicates a disconnect between the endorsed narrative of the teacher and the students' abilities to comprehend what the teacher is saying.

Sfard expressed it this way: "we [teachers] lose the ability to see as different what children [our students] cannot see as the same" (2008, p. 59).

Use of the commognitive framework highlighted specific disconnects between, on the one hand, mathematical concepts and, on the other hand, the words, symbols, graphs and gestures used to communicate them. Instructors say "secant" when they mean "slope of the secant" and "tangent" when they mean "slope of the tangent." They draw graphs to illustrate the derivative at a point, but no longer use graphs when defining the derivative of a function. They limit the repertoire of functions used to illustrate the graph of the derivative of a function to those that are somewhat trivial. These disconnects, and others noted in the results above, all point to ways in which instructors do not make explicit the mathematics they are aiming to teach, leaving some fairly difficult steps implicit for the students. Making these connections clear through careful use of words and visual mediators along with close examination of one's assumptions (endorsed narratives) and routines could reduce the implicitness of some aspects of the derivative, and perhaps make them more accessible to students.

Although this study contributes to the field by illuminating the importance of explicit discussions on the derivative through word use and visual mediators, I recognize at least three limitations. First, the ambiguity between the point-specific and interval view of the derivative may come from the colloquial use of the derivative in English (e.g., "Is the derivative positive?"). The discourse in other languages in which "the derivative at a point" and "the derivative function" do not share a common word (e.g., French, Japanese, and Korean), may produce different characteristics from the results of this study. Second, the instructors' classroom discourse can be explored from other angles such as their assumptions about what their students already know or beliefs about what is important to cover in class. Such data would complement this study by explaining instructors' word use and visual mediators during class. Third, I mainly analyzed the first three lessons from each class. Although this analysis has highlighted some important discursive characteristics, analyzing their discourse over a longer time period could inform our understanding about how these characteristics change over time. These topics provide important future directions for continued research.

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