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Mathematical situations of play and exploration

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Abstract The mathematical situations of play and exploration introduced here have been developed as an empirical research instrument for the longitudinal study "erStMaL" (early Steps in Mathematics Learning). They are designed as situations that allow children and a guiding adult to construct situation-related knowledge in common dialogue processes and in mutual conditions. In addition, the situations focus on the activity of the persons involved, as well as the meaning of artefacts as objects of different cultural worlds. Besides relevant theoretical aspects for the development of the mathematical situations of play and exploration, exemplary analyses are introduced to clarify the conceptual aims.

Keywords Elementary mathematical education · Qualitative research methods · Childlike mathematical activities

1 Introduction

Preschool children are faced with mathematical challenges in a variety of different ways. These challenges are identified, negotiated, solved, or reinterpreted and are processed when dealing with others, in the arrangement of the space, within time limits and with the materials available. In which contexts do children encounter these mathematical challenges and in which form?

Preschool life is made up of situations dominated by child's play and also by situations that reflect and support the culture of the respective society. Such activities as sharing meals, starting the day together, and celebrating holidays make up the latter. Both types of situations are highly ritualized and at the same time provide children with ample learning opportunities. Similar to "family learning cultures", "learning cultures" in day care centers presume that "knowledge learning is linked to existential learning and negotiation learning is linked to social learning" (Wulf & Zirfas, 2007, p. 323, translated by R.V.; see also, Delors, 1996).

Mathematical challenges are also present in these situations. This can be an explicit offering of mathematics independent of the particular pedagogical concept the center employs. But it can also occur subtly, i.e. everyday situations initiate mathematical problem-solving processes in the children. For example the kindergarten teacher asks

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whether a child is missing today, whether the number of chairs is correct, or has children set the table where every child is supposed to get a plate, a cup and a spoon.

In the project "erStMaL"[1] the mathematical situations of play and exploration are developed as a type of empirical research instrument (see Vogel, 2013), which gives the children the opportunity to express their mathematical (creative) potential. From a social-constructive perspective on learning and developmental processes this means for the purposes of developing the mathematical research instrument, that participants should have the possibility to negotiate situational meaning of the existing objects (materials) as well as dealing with these objects (see Brandt & Höck, 2011, p. 247).

So the mathematical situations of play and exploration, which were developed in the project "erStMaL" have less normative character and serve to elevate learning standards, and offer the foundation for the development of mathematical thinking from the perspective of mathematics education (see Krummheuer, 2011; Vogel & Huth, 2010).

After presenting the theoretical background of the mathematical situations of play and exploration developed in the project an exemplary analyses is shared that illustrates how the conceptional implementation was achieved.

2 Theoretical framework

The formation of learning and educational space in the field of early learning depends on the perception of human thought development. The "typology of development theories" according to Montada (2008, p. 10) provides a guide (see Fig. 1). The four "prototypical model families" differ in the way in which "the subject and/or environment is allowed to contribute to shaping the development or not, […]" (Montada, 2008, p. 9; translated by R.V.). Educational approaches can be assigned to the different models, which in turn serve as a guide for the actual paedagogic work, the formation of learning environments, and also for day-to-day interaction (Fig. 1; Fthenakis, Schmitt, Daut, Eitel & Wendell, 2009).

The "transactional systemic models" seem a suitable theoretical framework for the basic concept of mathematical thinking.

> The common core assumption of this model is, that the person and its environment form a whole system in which both the developing subject and also its environment have an active and interconnecting influence on one another. Changes to one part also lead to changes in other parts and/or the whole system and these changes have a knock-on effect too. All persons are in a constant state of development, not just children and young people. We all gain new knowledge, new insights, modify our view of ourselves, our view of the world, our ideas, our normative convictions, etc. (Montada, 2008, p. 12; translated by R.V.)

The educational approach of co-construction is assigned to this theoretical framework. This approach makes it possible to accentuate "the importance of social processes for individual processes of learning and construction" (Brandt & Höck, 2011, p. 248, translated by R.V.) The co-construction approach is closely associated with the concept of "educational

[1] The projekt "erStMaL"(early Steps in Mathematics Learning) is a longitudinal study in the field of early mathematical education. The project is established at the IDeA centre (Inidividual Development and Adaptive Education of Children at Risk). It is an interdisciplinary research center in the context of the LOEWE initiative of the federal state government of Hessen. LOEWE is a national initiative in the development of scientific and economic excellence.

		environment	
		active	inactive
subject	active	interactionist transactional systemic models → *approach of the co-construction*	actional and constructivist models → *approach of self-education*
	inactive	exogenist models → *(cooperative) approach of mediation*	endogenist models → *approach of self-development*

Fig. 1 A typology of development theories (Montada, 2008, p. 10) supplemented by educational approaches (*italics*) according to Fthenakis et al. (2009, p. 19) (translated by R.V.)

scaffolding" (Brandt & Höck, 2011; Bruner, 1985a; Verenikina, 2003). This concept brought to bear in the mathematical situations of play and exploration in the sense of Wells (1999):

> In particular, we are all in general agreement about the importance of three features: the essentially dialogic nature of the discourse in which knowledge is co-constructed; the significance of the kind of activity in which knowing is embedded; and the role of artefacts that mediate knowing (p. 127).

According to Nührenbörger (2009; translated by R.V.), "…learning mathematics in dialogue with others aims at learners' understanding of their own and external—even irritating—points of view of structures" (p. 112).

The situations of play and exploration initiated by the research project "erStMaL" often have this *discursive character* of a conversation (see Vogel & Huth, 2010). So the action and spoken impulses given by the guiding adult (a trained person from the research group) are often initially perceived by the children as somewhat irritating. So the children pursue their own ideas first for solving the mathematical task. Subsequently, at the beginning of the situation, it can be observed that individual children or groups of children use several parallel paths for tackling the mathematical problem. Even so, all the activities in the group are observed by the parcipants.

The role of the activities becomes important in the mathematical situations of play and exploration because it is these situations where the participating children will become mathematically active. Becoming active should be understood in the sense of "doing mathematics" (see van Oers, 2004, p. 322). What does this mean? Mathematical problems are solved by using suitable tools. The tools themselves are assigned meaning in the interaction between the participants or the objects themselves (see Fetzer, 2010) in order to use them in accordance with a predefined or newly developed set of rules for tackling the task or the problem-solving process. For this purpose, the position of the available tools is changed in line with a set of rules where changes are made to the individual objects or the individual objects are placed in a particular relationship to one another (see also, van Oers, 2004) From this perspective, mathematical activities take place within the sense of van Oers' (2002) formulated assumptions on mathematical activities:

> […] mathematical activity is basically a special form of semiotic activity, i.e. an activity of reflecting on signs, meanings and the interrelationships between signs

> and meaning. Semiotic activity occurs in both play activity and learning activity, but in different forms with respect to their regulation, and with different levels of strictness and consciousness (p. 32).

The recognition of patterns in the reorganisation of the objects or the reconfiguration often provides stimuli for solving the mathematical problem or the mathematical task. Changing the rules or adapting the problem is also conceivable and even desirable.

The objects or materials have the character of artefacts, since they are something specifically *sought* for the situation. These situations, which have been created in an educational context and often contain aspects (e.g. the culture of living together) that are the result of a cultural process, such as the miniature crockery set used in the mathematical situation of play and exploration "crockery". On the one hand, this material gives the children scope to play out and internalise appropriate scripts, in this case that of eating together; but it also offers a way of making real allocations in a mathematical sense. So a plate and a spoon can be allocated to each cup. This enables the creation of an "overlapping area" between the mathematical world and the world of children's experience, in which the adult represents the world of mathematics (see Prediger, 2001; Vogel, 2013).

Thus, the selected material-spatial arrangement has the function of "culture tools" (Bodrova & Leong, 2001, p. 9) within the sense of Vygotsky (1978). The selection of objects and their arrangement are designed on the one hand to appeal to the children in their childish world of ideas and on the other hand to support dealing with the mathematical problem. In this way the material-spatial arrangement mediates between two parallel sign systems (see Bartonlini Bussi & Mariotti, 2008). The accompanying person as an expert of both systems offers an introduction to dealing with and interpreting the material-spatial arrangement. In contrast to Bartolini Bussi & Mariotti (2008), the described material-spatial arrangements in the situations of play and exploration are not used for the staging of systematic learning processes.

Altogether, this manner of conception is based on the "activity-oriented approach" for early mathematical education described by van Oers (2004; see also Vogel, 2013). This approach assumes that

> … mathematics [is] learnt in interaction in the context of significant activities. For young children, there is often a direct association to their daily activities, when they, for instance, eat, get dressed, try or play something new. Play in particular is a significant concept to have conversations with children and thus steer their attention towards certain procedures or aspects of the situation (see e.g. Beardsley & Harnett, 1998). (van Oers, 2004, p. 317)

In this way, an exploration environment is created in which both situative and collective thought systems can be developed. This means the interpretation of signs and the relationship between signs and meaning by the persons involved (see van Oers, 2002). This enables the expression and further development of individual mathematical thought processes in the interaction (see Sfard, 2002, 2008)

The play components of the mathematical situations of play and exploration are shown in the particular type of actions which are possible in these situations and which are considered in advance, initiated and accompanied. The observable activities in the situations can be called actions because in many cases, they are targeted and relate to an object, the materials (see Oerter, 2011, p. 3). The particular characteristic of the actions, which is also what marks the play nature of the mathematical situations of play and exploration, lies in the purposelessness. Purposelessness in play means that in the chain of play actions, i.e. "objective–action–result–consequence," the result is missing (Oerter, p. 6). Oerter refers to Bruner (1985b) here, who in this connection talks about "[…] the loose connection between action and result" (Oerter, 2011, p. 6).

> For the child it is not so important to actually achieve the original objective associated with the play action, instead it is the activity itself which takes precedence, it is varied and revisited over and over again. To a certain extent, the child forgets the original purpose in the course of the activity. In this way, according to Bruner, skills are practised and combined which would probably never have been tried under functional pressure (i.e. that of actually achieving a result). (Oerter, 2011, p. 6; translated by R.V.)

The purposelessness originates in the common construction of a new reality by the participants, one which differs from everyday reality and enables the achievement of results whose consequences are without meaning for the play situation (see Oerter, 2011). This means that situations can occur in which things that are thought may also appear impossible. For the situations of play in the project "erStMal" it is exactly this purposelessness, which is important because it offers participants the scope to find meanings and to construct contexts for which there is no right or wrong category of assessment and whose nature is unconventional. The situations of play in the project "erStMal" have precisely the function of creating such an "agreed framework" (Oerter, 2011, p. 11), which to a certain extent also represents a sanctuary space. Particularly important for the presented research context is the correlation formulated by Oerter (2011; translated by R.V.) in his action-theoretical approach to the psychology of play:

> It is not imagination and fantasy, which are the reasons for the play activity, but rather the opposite: the game creates the framework conditions for development of imaginative and fanciful activity (p. 14.)

3 The design of the mathematical situations of play and exploration

The mathematical situations of play and exploration developed by the project "erStMaL" are aimed at stimulating children to mathematical activities. The starting point of the mathematical conception, the conceptions in mathematics education and the arranging of every exploration situation are the domains of mathematics described in the various educational standards[2] (KMK (Kultusministerkonferenz), 2004; NCTM, 2000): numbers and operation, geometry and spatial thinking, measurement, patterns and algebraic thinking and data and probability. These were applied to the areas of early education within the context of NCTM (Clements, Sarama & DiBiase, 2004).

Children tandems or groups of four children and an adult person are involved. The guiding adult represents the framework conditions of the situations of play and exploration. Mathematical interpretations are carried out by the participants in the set frame. The situations are not intentionally designed as a learning situation, but as a situation of play and exploration where the child in the situation considers his or her words. This consideration allows different mathematical interpretations to be possible.

[2] The indicated mathematical domains refer to educational standards of the Standing Conference of the Ministers of Education and Cultural Affairs of the Federal States of Germany (KMK) for the primary school, grade 4. In the "Principles and standards for school mathematics" of the National Council of Teachers of Mathematics (NCTM) the mathematical domains are formulated for overlapping grades.

The situations of play and exploration can be characterized structural by three components: (1) the mathematical task or problem, (2) the material-spatial arrangement, and (3) the multimodal stimuli (spoken language, gestures, and acting) of the guiding adult (see Vogel, 2013).

The *mathematical task* of a mathematical situation of play and exploration is based on a mathematical domain and focuses on basic mathematical activities such as systematic ordering or sorting, determination of quantity, the recreation and continuation of patterns or manipulating geometric forms of plane and space. In order to process the mathematical task, an appropriate *material-spatial arrangement* must be chosen or developed that, on the one side, emphasizes the mathematical importance and stimulates mathematical activities but still offers the participants ample leeway in forming their own means of processing. The *multimodal stimuli* of the guiding adult provide the biggest challenge. The aim here is to generate an appropriate repertoire from which the guiding adult can deduce the correct mathematical and situational response the context requires. The choice is framed around a comprehensive description of the mathematical content and possible applicability to other mathematical domains. This allows the guiding adult to diagnose the children's mathematical activities and to provide appropriate suggestions.

These initiated encounters with mathematics are described in the context of the "erStMal" project in the form of "design patterns of mathematical situations" with reference to the "Didactic Design Pattern" format (Wippermann & Vogel, 2004; Wippermann, 2008; Vogel, 2013). This structured way of describing the situation allows the material-spatial arrangement, as well as the accompanying multimodal stimuli (spoken language, gestures, acting) to be determined and instructed in advance to offer the guiding adult, orientation, assistance, and reaction scope. At the same time, they also guarantee that the created mathematical activities do not vary too much in their fundamental intention.

The central elements of "design patterns of mathematical situations" and their functions will be discussed shortly in the following section (see also Vogel, 2013). However, here the individual elements (italic) are summarized into three function groups:

- Organizational factors

  A *short summary of the situation* which includes the description of the application field (e.g. for which ages the situation is intended and whether the situation is meant for children in pairs or groups), the description of the *material-spatial arrangement* and mentioning the *literature* that was used for developing the mathematical situations of play and exploration.

- Realization-related hints

  The mathematical task is described in the introductory situation and also, if needed, in the possible narrative contexts. By means of the description categories *"possible stimuli in the introductory situation"* and *"potential points of reference,"* the guiding adult is provided with a repertoire of multimodal stimuli that can be used for the concrete implementation of the situation. The guiding adult can use these impulses; however, the adult has the possibility of simultaneously modeling the situation.

- Background knowledge

  The execution of the situation requires a large amount of mathematical competence to identify the children's mathematical ideas and support them with the appropriate multimodal stimuli. Comprehensive descriptions of the *mathematical content* of the situation and *explanations* are helpful here in determining which *other mathematical domains* are possible in this situation.

The individual descriptive elements are described according to specific central questions and are described separately from one another. The several description elements only take effect in their interaction in the production of the mathematical situation of play and exploration. Individually described aspects can be chosen for the concrete situation, then assessed, and linked. The guiding adult can use the description of the mathematical content of the situation and is prepared, for example, to better identify the children's mathematical activities, pick up on them multi-modally, and further continue them. A sophisticated linguistic, gestural, and action repertoire is available to the guiding adult from which he or she can make a situational-appropriate choice. This way, a component-oriented description simultaneously creates freedom and orientation for forming mathematical situations of play and exploration. The work with "design patterns of mathematical situations" was tested in the teacher education. Using such descriptions student teacher could carry out comparable teaching and learning environments.

4 Analyses of selected mathematical situations of play and exploration

On the basis of selected situations, it will be shown in what way the mathematical situations of play and exploration in the project "erStMaL" fulfil the theoretical and conceptional requirements described. Potentially, the following three research questions generate an empirical field in each case. In the context of this paper answering these questions has the aim of clarifying the conception of the mathematical situations. For this reason, the following three questions are answered exclusively by the generic analysis of a selected exemplar situation

(1) *Will the conception of mathematical situations of play and exploration create mathematical learning spaces, which will enable the children to become mathematically active in an "overlapping area" between the worlds of mathematics and childhood?*
(2) *What range of mathematical interpretations can be reconstructed in the mathematical siutations of play and exploration?*
(3) *In what way can the playful nature of the situations be expressed?*

4.1 The description of the selected mathematical situations of play and exploration

The selected situations of play and exploration demonstrate various levels of freedom with regard to the structural components "mathematical task" and "material-spatial arrangement" (see Vogel, 2013). The level of freedom is shown, e.g. in the scope of action which is allowed by the material-spatial arrangement. The "golden treasure" situation offers in view of the material-spatial arrangement little scope because the exact rules for the game are established here. The children have freedom only in their choice of the wheel and the colour. The playful arrangement, however, allows explanatory statements from the world of mathematics, from the world of chance and from the everyday world. The coloured wooden sticks in the situation "wooden sticks" can be redefined in their function as a unit of pattern and can be used lightly, e.g. to lay out two-dimensional buildings. The cubes and pyramids in the situation "solid figures" can mutate into building blocks to build structures.

"Wooden Sticks"

The mathematical play and exploration situation "wooden sticks" can be allocated to the mathematical field of "patterns". The mathematical task is to form rows of patterns using the given material. The children are provided with a large number of coloured wooden sticks, which differ only in colour. So colour is the element, which can form a pattern. Geometric patterns are ribbon patterns, which are created through various combinations of depictions of congruence applied to the pattern unit (basic figure) (Hülswitt, 2006; Vogel, 2005).

"Solid figure"

The mathematical play and exploration situation "solid figure" can be allocated to the mathematical field of "geometry and spatial thinking". The mathematical task in this situation involves getting to know different geometrical figures and their properties. The material-spatial arrangement provides the children with prisms, pyramids, cones and spheres in an appealing design (materials and colours). In order to enable the children to focus their attention more easily on the properties of these geometric figures, a cloth bag in which the children feel the geometric figures is used during the first and second data collection point. At the same time the fact that the children cannot see the objects and have the task of describing them means that they are prompted to find terms for the properties of the figures they feel or find parallels to similar everyday objects.

"Golden Treasure"

The mathematical play and exploration situation "golden treasure" can be allocated to the mathematical field of "data and coincidence". This play situation uses a board on which two playing pieces must be moved over a course from a starting point towards their respective homes. The winner is whoever gets their playing piece home first (see Vogel, 2012). All players start from the same square on the board. The direction taken from there is dictated by the wheel of fortune. The circular wheel of fortune has two sectors of different colours. These are the colours of the two playing pieces. The players take turns to spin the wheel. If the wheel shows the colour of the player who spun the wheel, this player may move his or her piece in the direction of home. If this is not the colour shown, the other player takes a turn. There are various wheels in which the coloured sections are of different sizes. The division ratios are as follows: 1 : 1, 1 : 3, 5 : 7. There are two versions of the division ratio 1 : 1 (see Fig. 2).

Fig. 2 The mathematical situation of play and exploration "Golden Treasure" (data collection point T1)

4.2 Empirical frames of the mathematical situations of play and exploration

During the longitudinal study “erStMaL” children are accompanied over a period of approximately four years. The data collection started in 2009 with around 144 (see Acar Bayraktar, Hümmer, Huth, Münz & Reimann, 2011). The children are about 4 years old at this time. At each of the six data collection points, two mathematical situations of play and exploration are carried out with every child. The situations are arranged in tandems or in groups up to four children. The design of the study intends that one of the two situations for each participating child originate from the same mathematical domain at all six data collection points. The second situation is coming from two other alternating mathematical domains. The reality of the survey shows that this cannot always be adhered to. Some children do not take part in individual data collection points because they are moved away or due to illness. In addition, it becomes tempting also to keep the mathematical situations of play and exploration constant within a mathematical domain for the children. For this, the situations are developed further within the mathematical and content frame, e.g. by enlarging of the mathematical task or by the variation of the material-spatial arrangement. All the mathematical situations of play and exploration are developed by the researcher team of the project “erStMaL” and are implemented through trained persons. The realisations of individual play situations are videotaped. The video material then provides the database for various analyses carried out within the research team of the research project “erStMaL”.

4.3 Analysis methods

The selected exemplar mathematical situations of play and exploration are analysed with the following methods.

(a) *Segmentation analysis*

The foundations for this type of video analysis are the processes, which can be identified on the macro level of the interaction. By way of a stepwise limitation of segments, the course of the interaction process is structured (see Dinkelaker & Herrle, 2009). If a particular “overlapping area” is identified, the three following distinction features are offered for establishing the segments: changes in the constellation of space and bodies, changes in the pattern of speech exchange and changes in the subject matter (see Dinkelaker & Herrle, 2009). The circular segments thus established are “briefly described and symbolised with a typical still. This sequence of stills enables the structure of the interaction to be compared at pattern level.” (Vogel, 2011, p. 862; translated by R.V.).

(b) *Videocoding*

The analysis method of videocoding according to a coding guideline is used to work out the mathematical statements of the participants, expressed multimodally (spoken language, by gestures and activities with the material) and then quantified. The used form of analysis is based upon qualitative content analysis (Mayring, 2007, 2000). The coding guideline was developed in a reduction of inductive and deductive categorisation. The main categories thus identified are: determination of quantity—operation (QO), mathematical structures (MS), pattern (pattern units, band ornaments, tessellation) (PA), topological fundamentals and activities (geometric topology) (TP), components of spatial thinking (ST), geometric shapes and 3-D figures—transformations between plane and space (GE), measurement (ME), data (DA), chance (probability) (CH), combinatorics (CB) and miscellaneous (MC) (Miscellaneous: exploring the material (artefacts); playing with the material without attention to the stimuli; observing the situation without action). The main categories are more diversified in further sub-categories (see Vogel & Jung, 2013). Coding always refers to a time unit of 30 s. For this period, participants' spoken and gestural statements and activity with the material are allocated by fitting sub-categories (at most two sub-categories per time unit).

The video documents are structured into work phases before the videocoding. Sequences from the videos, which do not deal with the mathematical task of the situation, e.g. conversations with the children about the weather or upcoming birthdays, are not considered. The coding results are represented by relating to the situation, which is quantified in the main categories or sub-categories or in the course of time. In the course of time means that the codings are represented separately, by the participants in 30 s intervals (Vogel & Jung, 2013).

(c) *Language-analysis based method*

This kind of analysis is orientated as a dialogue (conversational) analysis after Henne and Rehbock (2001) and the interaction analysis after Krummheuer (2012). The mathematical concepts of the involved persons expressed in the dialogue and hence in the interaction process, are at the centre of the consideration. The interaction process is transcribed and is subdivided for the analysis. The analysis of individual turns is aimed at reconstructing the exploration of mathematical ideas and the mathematical concepts of the individual participants. The results are represented in a synoptic interpretation (see Vogel & Huth, 2010, p. 188).

The three analysis methods introduced are used for the longitudinal consideration of the data from the study "erStMaL" to respond to different questions (Vogel & Jung, 2013; Vogel & Huth, 2010). In this article they serve as the characterization of the mathematical situations of play and exploration.

5 Results

The results of the analyses of the selected mathematical situations of play and exploration are introduced along with the three presented research questions.

5.1 Mathematical activities in the "overlapping area" of a mathematical and a childlike world

The selected situation took place at the data collection point T3 of the study. The children have already worked with the material and the mathematical task "laying pattern" twice. The four children are approximately 5.5 years old at the time of recording.

In work phase I (see Fig. 3) of the play situation, the box of the wooden sticks is first tipped out in the middle of the carpet. The guiding adult sets out the mathematical task, to

Fig. 3 Second segment of the "work phase I" from the mathematical situation of play and exploration "wooden sticks" (data collection point T3)

lay a pattern with the sticks, on an action level (lays out a snake with a determined sequence, where the sticks are joined at the narrow edges) and also on a spoken level. The impulse is taken by the girl on the right edge of the picture (see Fig. 3). On the other hand the three boys (left side of the picture) interpret the request to lay a pattern in their own way and lay out different shapes such as triangles and numbers. In addition, the act of laying the sticks over one another seems important for all the children. This motif recurs again and again at various times over the course of the whole situation and is incorporated into their laying of the pattern. This is also shown by the figure on the right of the picture which ressembles the shape of a letter and which was laid by the girl in this work phase. The intended activity for this situation, that of laying patterns, in which the colour determines the pattern is either completely ignored by the children in this phase or done only very hesitantly. Immediately before the survey situation in kindergarten the children develop their own patterns, which can also be related to the activities.

Work phase II (see Fig. 4) is dominated by work on the intended work task. The work phase is introduced by the guiding adult's renewed intervention. A pattern sequence is chosen as an example (bottom right of the picture) in which the long sides of the sticks are laid against one another. This type of laying the sticks enables the colour sequences to be seen more easily. All the children in the group now work intensively on generating patterns, and consider what colour stick must be laid next. In addition it is observed that the various patterns are discussed intensively between the children.

It is also clear that for some children the large quantity of sticks is still fascinating. Already at the data collection point T1, the large quantity of sticks are discussed between two of the three involved boys. This is expressed in the described attempt to hold as many sticks as possible, to sift through the large pile of sticks or to pile up a certain number of sticks in front of them (see Fig. 4, left). Also in this situation, one boy tries to hold as many sticks in his hand as possible (see Fig. 4, right).

In work phase III (see Fig. 5) the three boys break away from the common task of laying a pattern and find their own laying activities. A monkey house is laid out. The sticks are now no longer being used to create patterns but instead to depict a monkey house in the zoo. The fictitious three-dimensional monkey house is laid in the form of a plan (two dimensional depiction).

The analysis shows, how the children use the scope for interpretation and action and allow themselves to be inspired by the material or by the actions of other children in their

Fig. 4 Third segment "work phase II" mathematical situation of play and exploration "wooden sticks" (data collection point T3)

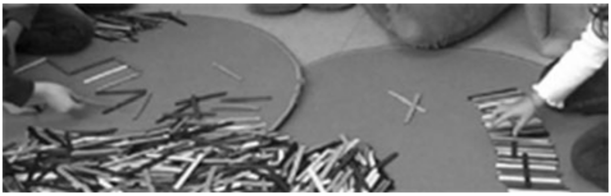

Fig. 5 Fourth segment "work phase III" of the mathematical situation of play and exploration "wooden sticks" (data collection points T3)

interpretation and implementation of the work task. Thus, the available sticks initially used to create geometric patterns and signs, derived from the current space of the children's experience are assigned to the childlike world (see work phase I). In this process, activities are seen which actually belonging to different mathematical domains. These geometric activities are revisited and intensified in work phase III, i.e. the childlike world is picked up again with an example from the zoo. The sticks are used to make plans of fictitious buildings. The boy who introduced this idea into the situation has already used the sticks to reproduce three-dimensional objects two-dimensionally at the data collection point T1. So from a longitudinal view, it is possible to identify types of children in this group who deal with topics from other mathematical domains in a consistent manner across the data collection points.

5.2 Reconstruction of diverse mathematical interpretations in the mathematical situations of play and exploration

The main categories percentage distribution over the coded work phases of the situation "solid figures" (see Fig. 6) shows that the mathematical domain of "geometric shapes" (GE),

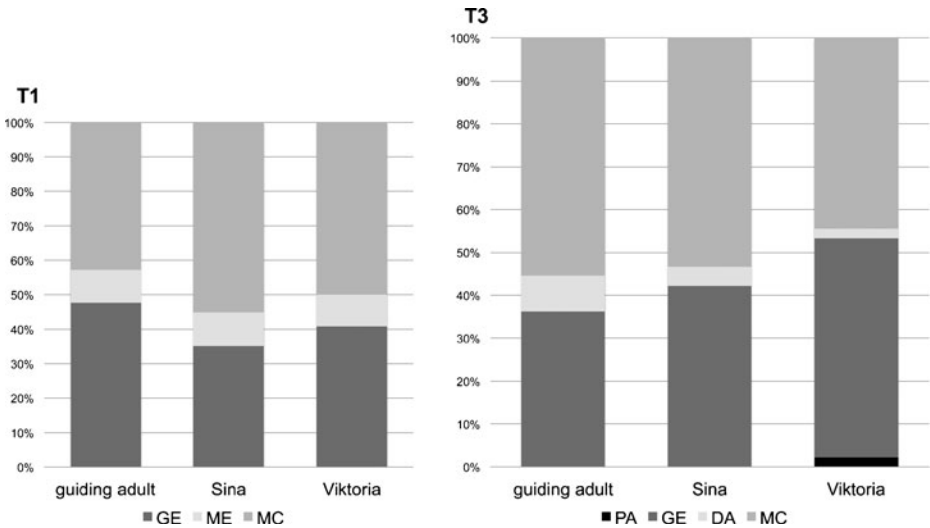

Fig. 6 Percentage distribution of the main categories over all the coded work phases of the situation. *Left* is the result for the situation "solid figures" at the data collection point T1 and on the *right*, the result for the data collection point T3. The children Sina and Viktoria and a guiding adult are involved in the situation

tackled in the concept, dominates in both data collection points (T1 and T3). The domain of "measurement" (ME) is added at the data collection point T1. Here the children compare the size (height) of the various shapes. At the data collection point T3, the domain of "data" (DA) and "pattern" (PA) are accrued. Temporarily, the children sort the forms according to specific characteristics. Viktoria uses the geometrical figures even for the construction of a pattern sequence.

In addition, it appears that at both data collection points T1 and T3, the main category "miscellaneous" engaged a large proportion. This includes the exploration of the material, playing with the material without recognizable mathematical activities and observing the event without active participation.

On the one hand, the differentiation of the main category "geometric shapes" (GE) reflects the develpers' focus for each data collection point and at the same time, the priorities set by the children are revealed (see Fig. 7).

At the data collection point T1, geometric figures were explored with the fingers in a cloth bag, which are named (GE1) and the differences between the figures established (GE11). The comparison of the different geometric figures dominates. In addition, the geometric figures are compared with three-dimensional everyday objects or are used to recreate this (GE7 + GE10). At data collection point T3 the comparison is decreased and the representation of everyday objects with the geometric figures is increased.

The frequencies of the codings permit an evaluation of the mathematical situations of play and exploration. In addition, these results give an indication of the children's variety of mathematical interpretation. Furthermore, the coding shows how the children's mathematical employment of the geometric figures is developed over time. The different geometric attributes of the figures decrease and the preparing of references to everyday life in the occupation with the figures are emphasized more and more.

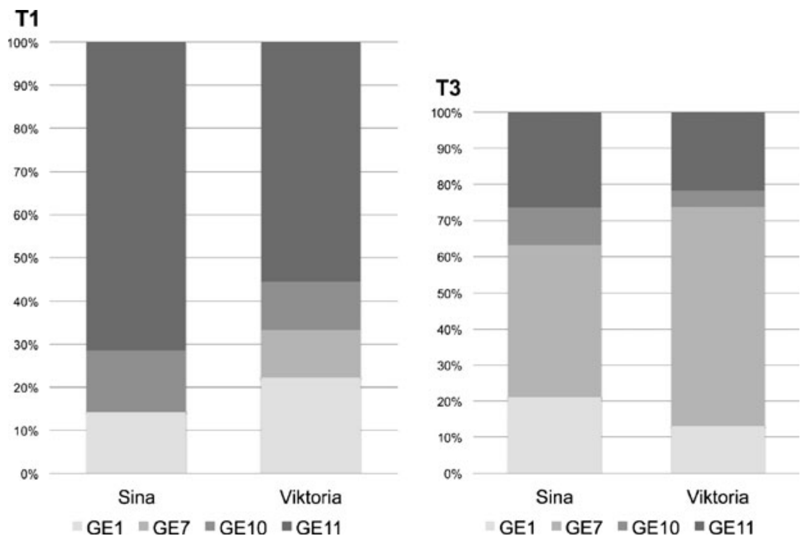

Fig. 7 Percentage distribution of the relevant sub-categories, within the main category "geometric shapes" over all the coded work phases of each situation. On the *left* is the result for the situation "solid figures" at the data collection point T1 and on the *right*, is the result of the data collection point T3. The children Sina and Viktoria and a guiding adult are involved in the situation

5.3 Play-like character of the mathematical situation of play and exploration

The third selected example clearly shows the play-like nature of the mathematical situations of play and exploration. It becomes clear how many of the children create new realities, which they use to interpret the situation.

The selected situation takes place during the data collection point T1 of the study. The two children, who in the selected sequence are involved in transporting the golden treasure around a board, are around 4 years old at the data collection point.

	B	None/Lara do you have any idea why you always … able to go and
<		not Josip/
<	Lara	Josip, Josip always makes so/*rotate slowly the arrow on yellow* so Josip must make quickly *grasp the red bear by Josip* and so *puts the red bear one playing field forward* Josip quickly make *rotates on the turntable, thereby the arrow slipped from the turntable* quickly make *go on turning the arrow, although this be located besides the turntable, incomprehensibly, lays the arrow again on the turntable* I always fast *rotates on the turntable, so that the arrow rotates rapidly* look/

The transcript[3] comes from the conversational phase in which the children are playing with the wheel of fortune with the divisional ratio 1 : 3. Lara is playing with the bright (yellow) colour, which takes up three quarters of the overall surface of the wheel. The guiding adult asks Lara whether she has any idea why she can always advance on the board and Josip can't (see line 7–8). Lara's explanation model is the speed with which the arrow is spun. She jumps out of the game situation, and its rules, and converts the play situation into an experimental situation in which the explanation model as to why Lara's colour comes up so often on the wheel, is explained. The game context enables what is known as a new reality to be created for a short period, in which explanations on the progression of the game can be made clear.

6 Summary

With the three qualitative analyses of the selected exemplar videotaped mathematical situations, the specific children's dealings with the situative challenges can be worked out and clarified. Along with the formulated research questions, the represented results are summarized. In this way final characterizations of the mathematical situations of play and exploration are explained.

(1) The mathematical situation "wooden stick" enables by the chosen material of the colored sticks, both a mathematical and an everyday life interpretation. These interpretations show a smooth transition so that "overlapping areas" arise between the mathematical world and the world of experience of the children. At first, the children test the possibilities of the sticks by shaping figures and signs, whose meaning for the observer remains unclear or the children are fascinated by the large number of sticks.

[3] Transcript of the mathematical play and exploration situation "golden treasure", data collection point T1 (legend: actions are in italics, < … happened at the same time, / …lifting the voice)

The children pick up the initiated laying of patterns after a phase of exploration and they pursue these activities for a certain time. Mathematical abilities such as the two-dimensional representation of three-dimensional objects become clear at the same time. The mathematical task does not provide these activities, however, the selected material empowers the children to make this mathematical interpretation, so they can express their fantasy world.

(2) The situation "solid figure" continues the childlike world only such that the children encounter the chosen pro-types of mathematical figures in the form of building blocks. The mathematical task aims at the identification and description of the typical figure characteristics. The mathematical variety potential of interpretation is demonstrated by the fact that the children compare the size of the figures and then they lay patterns.

(3) The situation "golden treasure" moves in the childlike world of the game. It is possible in the game to change the purpose of the activity and to show the explanation of chances of winning concretely. Usually, during the game, the wheel only is span, to work out which token is allowed to move on. In the analyzed situation the spinning of the wheel is used for a short moment to explain the chances of winning experimentally.

In addition, the chosen analysis methods support the expressiveness of the executed analyses. The segmentation analysis focuses on the structuring of the interaction course on the basis of defined distinguishing features. With that, the "overlapping areas" can be identified in a specific way. Changes in the constellation of space and bodies, in the patterns of speech and in the subject, make visible in which areas children are changing the world reference, i.e. where they are moving in an "overlapping area" between a childlike and a mathematical world. The qualitative videocoding permits a quantification of the mathematical domains, which are used by the participants in the mathematical situations of play and exploration. A quantification of the variety of the mathematical interpretation becomes possible. Language-analysis based methods allow to follow specific questions at the micro level. It could be shown in detail, in which way the game character of the situations becomes visible.

Altogether, it may be considered that the developed mathematical situations of play and exploration fulfill the requirements. At the same time, the analyses display the potential of the mathematical situations for using them in the mathematical education in kindergarten, pre-school and primary school.

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