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How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles

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Abstract Interpreting and responding to student thinking are central tasks of reform-minded mathematics teaching. This study examined preservice teachers' (PSTs) interpretations of and responses to a student's error(s) involving finding a missing length in similar rectangles through a teaching scenario task. Fifty-seven PSTs' responses were analyzed quantitatively and qualitatively. Analysis results revealed that although the student's errors came from conceptual aspects of similarity, a majority of PSTs identified the errors as stemming from procedural aspects of similarity, subsequently guiding them by invoking procedural knowledge. This study also revealed two different forms of address and teaching actions in PST interventions along with three categories of acts of communication barriers. The broader implications of the study for international communities are discussed in accordance with the findings.

Keywords Preservice teachers · Ratio · Proportion · Proportional reasoning · Additive reasoning

1 Introduction

Ratio and proportion are core topics in elementary as well as secondary mathematics education (NCTM, 1989, 2000; Kilpatrick, Swafford, & Findell, 2001). The NCTM (1989) document describes proportionality to be "of such great importance that it merits whatever time and effort that must be expended to assure its careful development" (p. 82). Very often multiplication and division tasks in the lower grades are presented in unit-rate form, which is a special form of ratio and proportion. In the middle grades, word problems involving equivalent fractions and fraction comparisons can also be thought of as ratio and proportion. The ability to recognize structural similarity and multiplicative comparisons illustrated in such proportional reasoning processes is the cornerstone of algebra and more advanced mathematics (Confrey & Smith, 1995; Lamon, 2007; Streefland, 1984, 1991).

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Nevertheless, research has consistently shown that many students have difficulty developing proportional reasoning. Hart (1984), for example, reported that less than 42 % of students in grade 7 succeeded in solving simple problems of enlargement. The most common source of error is additive reasoning, where students focus on the difference between the given quantities rather than the proportionality illustrated in the context (Hart, 1981, 1984; Karplus, Pulos, & Stage, 1983; Lamon, 1999, 2007).

Recognizing and responding to student errors appropriately is one of the main tasks of teachers in teaching mathematics (NCTM, 2000). Although some view student errors as futile, teachers are increasingly being called to use such errors as catalysts for mathematics learning (Ashlock, 2006; Borasi, 1994; Lannin, Barker, & Townsend, 2007). The NCTM (2000) documents, for example, stress that mathematical errors should not be seen as "dead ends" but rather as "potential avenues for student learning." Ball (1990) also emphasizes that teachers should move beyond a superficial "right or wrong" analysis of tasks. Rather, teachers should use student errors as windows into student understanding, endeavoring to help student understanding of the conceptual basis of their errors.

In this study, I set out to investigate elementary and secondary pre-service teachers' (PSTs') reasoning, their responses to student errors on the topic of ratio and proportion, and the relationship between their knowledge and pedagogic approaches. I used the aforementioned student error, *additive reasoning*, and examined how PSTs would interpret and respond to such an error for a student finding a missing length in similar rectangles as well as how the PSTs' approaches related to their mathematical knowledge. Although a growing body of research has focused on teachers' treatment of student errors (e.g., Schleppenbach, Flevares, Sims, & Perry, 2007; Son & Sinclair, 2010; Stevenson & Stigler, 1992), there are few studies on how teachers treat additive reasoning, in particular in regard to similarity. In addition, PSTs' responses and their teaching strategies have received limited attention in the research literature. If teachers are called to use student errors as springboards for inquiry into mathematical concepts, it is important to explore PSTs' responses and strategies to address student errors in order for teacher education programs to prepare the teachers to make better use of them. The purpose of this study, therefore, is not to add to the collection of studies documenting PST weaknesses but rather to inform the design of teacher education. Such an exploration will enrich the dialogue among reformers, teacher educators, and professional developers about the means to help PSTs teach mathematics in ways that promote understanding. In the next section, I address the related literature and the framework on which this study is built.

2 Theoretical framework

2.1 Student strategies in solving ratio and proportion tasks

Understanding ratio and proportion begins with the ability to understand multiplicative relationships, distinguishing them from relationships that are additive. Formally, a ratio is a comparison of two things with respect to size and can be represented by a fractional expression, i.e., a/b. A proportion is a statement of the equality of two ratios, i.e., $a/b=c/d$. Proportional reasoning describes any kind of reasoning that focuses on the relationship between two ratios, which is a complex of ideas (Vergnuad, 1983). According to *Curriculum Focal Points* (NCTM 2006), students in the USA begin their formal exploration of ratio and proportion in the middle grades, and by the end of grade 7, they are expected to develop a deep understanding of ratio and proportion and the ability to apply them in a wide range of contexts, including situations involving similarity, constant rate of change, slope, and speed (Heinz & Sterba-Boatwright, 2008). Three types of tasks are reported in the literature for developing and assessing proportional reasoning (for examples, see Cramer, Post, & Currier, 1993):

- Missing value problems, where three of the four values are given in the proportion ($a/b=c/d$) and the task is to find the fourth or missing piece of information
- Comparison problems, where the four values are given (a, b, c, and d) and the goal is to determine which of two given ratios represents more or less
- Qualitative prediction and comparison problems, which require students to evaluate the effect on a ratio of a qualitative change in one or both of the quantities involved

Among these three tasks, this study narrows the use of ratio and proportion to missing value tasks, which is the most common task in the US mathematics curricula (Kilpatrick et al., 2001). Missing value tasks, called isomorphism-of-measure problems by Vergnaud (1983), involve "reasoning in a system of two variables between which there exists a linear functional relationship" (Karplus, Pulos, & Stage, 1983, p. 21). Table 1 shows the development level of proportional reasoning in missing value problems drawn from Baxter and Junker (2001) and a list of associated students' incorrect (1–3) and correct strategies (4–5) with examples.

Example A pole that is 4 ft high casts a 6-ft shadow. The task is to find the length of a shadow that a 10-ft high pole casts.

Height	Shadow		Measure 1	Measure 2
4	6	Value 1	*a*	*b*
10	?	Value 2	*c*	(*d*)?

In solving the problem presented in Table 1, a student conceives of two "measure spaces," where each space represents one of the units in the problem. In such situations, each variable remains independent of the other; parallel transformations are carried out within or between variables, thereby maintaining their proportional values. According to Vergnaud (1983), the sophistication of proportional reasoning is determined not only by the student's ability to recognize the correct relationships between quantities but also by the strategy that the student employs when solving problems (Baxter & Junker, 2001; Moss & Case, 1999; Streefland, 1991). Baxter and Junker (2001) proposed a model of five developmental stages for proportional reasoning, starting with a qualitative understanding of quantity (e.g., more and less) and converging on a full understanding of proportionality as an invariant relationship between pairs of changing quantities. Key among the changes is a shift from additive to multiplicative reasoning and from use of context-specific strategies to a generalized understanding of functions (Lamon, 2007; Vergnaud, 1983). For this reason, additive reasoning (or strategy) was chosen as the focus for this study.

2.2 Knowledge needed in finding missing values in similar figures

This study particularly focused on additive reasoning in missing value tasks of similarity because through such tasks, students gain a full understanding of proportionality and similarity both conceptually and procedurally, providing a thread that integrates proportionality and geometry (NCTM, 2000). According to *Connected Mathematics*, the most widely

Table 1 Baxter and Junker's developmental levels of proportional reasoning associated with correct and incorrect strategies for proportion tasks

Development stage	Characteristics	Strategy	Example
		Invalid	
1. Qualitative	Possesses a good idea of knowledge about quantity and more and less concepts. Uses guess or illogical computation.	Intuitive	
2. Early attempts at quantifying	Focuses on the difference between the given quantities rather than a constant ratio. Uses additive strategies to find a missing length.	Additive reasoning	The shadow cast by the 10-ft pole would be 12 ft long because 4+2=6.
3. Recognition of multiplicative relationship	Realizes intuitively that proportion is involved but does not correctly express the proportionality or calculates incorrectly.	Proportion attempt	$\frac{4}{6} = \frac{}{4}, = 3\frac{1}{2}$
		Valid	
4. Accommodating covariance and invariance	Begins to develop a multiplicative change including covariance and invariance. Views a ratio as a single unit to which basic arithmetic operations may be applied. Relies upon informal strategies such as halving or doubling.	Informal	4 plus 1/2 of itself gives 6 and 10 plus 1/2 of itself gives 15
5. Functional and scalar relationship	Has well-developed conceptions of covariance and invariance. Has a repertoire of generalizable strategies. Finds a ratio in one context and applies it to another context.	Within ratio	$\frac{4}{6} = \frac{10}{}, = 15$
	Finds a ratio between contexts and applies it to find a missing value.	Between ratio (functional)	$\frac{4}{10} = \frac{6}{}, = 15$
	Finds a scale factor (or unit-rate) and applies it to find a missing value in other contexts.	Scale factor	$\frac{6}{4} = \frac{3}{2}, 10 \times \frac{3}{2} = 15$

used reform curriculum in grades 6–8 in the USA (Dossey, Halvorsen, & McCrone, 2008), students first develop a conceptual and a practical understanding of similarity in grade 6. In working with similar shapes, students develop an intuitive notion of similarity: Similar shapes have congruent angles but not necessarily congruent sides. As they study ratios and proportions, they extend this understanding of similarity to be more precise, noting, for instance, that similar shapes match exactly when magnified or shrunk (idea of enlargement) as their corresponding sides are related by a scale factor. Students then solve similar figure problems by setting up proportions using fractions that relate corresponding lengths of objects or relationships of lengths within an object. Finding a missing value in similar figures involves: (1) understanding the concept of similarity; (2) recognizing the proportionality embedded in similar figures by comparing lengths and widths between figures (*between* ratio) or by comparing the length to width within a rectangle (*within* ratio) or determining a scale factor, called *the conceptual aspect of similarity*; (3) representing the relationship between two similar figures using a ratio, a proportion, or a scale factor; and (4) carrying out related procedures, called *the procedural aspect of similarity.*

In this study, I used a written classroom scenario where a student provides an additive strategy in finding a missing length in similar rectangles, i.e., a 4×6-cm rectangle is given and in another similar rectangle, only the width of 10 cm is provided. The hypothetical student, Sally, determines the missing length as 12 cm since 4+2=6. It is the combination of the conceptual and the procedural that makes additive reasoning so challenging. Students do not understand the meaning of similarity from the *conceptual* viewpoint, and as a result, their *procedural* knowledge is limited to focusing on the differences. Such a complicated error demands an appropriate, multi-faceted teaching response.

2.3 Teacher knowledge and approaches in ratio and proportion

Shulman (1986) stated that pedagogical content knowledge (PCK) differentiates expert *teachers* in a subject area from the *subject area* experts. PCK is a form of practical knowledge that is used by teachers to guide their actions in highly contextualized classroom settings, built upon teachers' subject matter knowledge in combination with their knowledge of pedagogy. In this study, PCK includes knowledge of common student conceptions, misconceptions, and difficulties; knowledge of the possible sources of these misconceptions and difficulties; and knowledge of strategies that could be used to address the issues (Ball & Bass 2000; Hill, Rowan, & Ball, 2005). Teachers' PCK is closely related to interpreting and responding to student error.

Much research has focused particularly on teachers' PCK in mathematics. Some researchers have examined teachers' PCK in regard to common mathematical concepts (e.g., Berk, Taber, Gorowara, & Poetzl, 2009; Lim, 2009). Lim (2009), for instance, investigated 28 PSTs with four types of invariance in missing value problems: ratio, sum, product, and difference. He found that the PSTs had different levels of understanding depending on problem types: PSTs had less difficulty with the ratio and sum problems versus missing value tasks involving product and difference. In particular, he reported that PSTs generally do not pay close attention to the meaning of ratios when solving a missing value problem, referring to the use of the same approach regardless of context.

Other researchers focused on teachers' knowledge and pedagogical approaches to student misconceptions as well. Hines and McMahon (2005), for example, investigated characteristics in PSTs' interpretations of middle school students' strategies in proportional reasoning. The PSTs were given samples of middle school students' work and asked to classify them according to which showed more developmentally advanced reasoning. They were also asked to explain how they had made their decisions. The researchers reported that the PSTs did not immediately recognize middle school students' use of additive reasoning as a less developmentally advanced reasoning strategy. In addition, the PSTs tended to assign writing an equation for a proportional relationship to the highest developmental level and to devalue the less symbolic concrete/pictorial and numeric/tabular strategies. Schleppenbach et al. (2007) investigated the use of errors in classroom discussion by comparing lessons of Chinese and US teachers. They report that while US teachers tended to avoid and hide student errors, Chinese teachers tended to redirect students to think about the original question in conceptual ways. Indeed, repeating the procedure until students recognize their errors is a well-known, typical strategy that US teachers employ when dealing with their student errors in class (Borasi, 1994; Santagata, 2005; Stevenson & Stigler, 1992).

While these previous studies provide multiple insights about PSTs' understanding and their possible ways of thinking about student error, much research needs to be done in this area. I could not locate any research on PSTs' interpretations and pedagogical approaches related to additive reasoning involving similarity nor any research addressing teacher training related to PSTs' understanding of that topic. The present study in part attempts to address this gap.

2.4 Analytical framework

Table 2 presents the analytical framework utilized in the study. Previous studies have reported that the distinction of "conceptual" versus "procedural" both helps identify student errors and possibly helps instruction that effectively resolves student errors and improves student understanding as well (Hiebert & Lefevre, 1986; Kilpatrick et al., 2001; Rittle-Johnson & Alibali, 1999). *Conceptual knowledge* is defined as the explicit or implicit understanding of the principles that govern a domain and the interrelations between pieces of knowledge in a domain. *Procedural knowledge* is defined as the action sequences for solving problems. In particular, Boero and Garuti (1992) clearly showed that procedure-oriented instruction is not likely to produce improvement in the transition between additive and multiplicative reasoning. Therefore, the conceptual versus procedural distinction drawn from Rittle-Johnson and Alibali (1999) was utilized as the main framework in analyzing PST knowledge and approaches to student error.

In addition to the conceptual vs. procedural distinction, four additional aspects (2–5) drawn from the author's previous work (Son & Crespo, 2009; Son & Sinclair, 2010) were utilized in the analysis of PSTs' teaching approaches. Although the procedural vs. conceptual distinction is well-grounded in the research, the empirical evidence on the effect of conceptual versus procedural instruction is mixed in some areas (e.g., counting or geometry) (Briars & Siegler, 1984; Fuson, 1988; Star, 2005; Wynn, 1990). In addition, such a framework could not provide any further details about the extent to which student errors are utilized. Therefore, in order to give more insight about the nature of PSTs' intervention, the further analyses shown in Table 2 were carried out. *Forms of address* signify whether PSTs deliver verbal or non-verbal (i.e., show–tell vs. give–ask, respectively) information to students in dealing with student errors. *Act of communication barrier* refers to the difficulties students and teachers have in communicating about student errors. More details about the analytical framework follow in the discussion of the findings.

3 Research methods

3.1 Participants

Participants in this study included 57 PSTs majoring in elementary education, special education, or secondary mathematics education at a large midwestern university in the USA which utilizes a 5-year teacher preparation program. Students interested in obtaining elementary school certification pursue an undergraduate degree in elementary education. As part of the undergraduate major in elementary education, students complete a core sequence

Table 2 Analytical framework for analyzing PSTs' knowledge and responses

Aspect	Categories
1. Mathematical/instructional focus	Conceptual vs. procedural
2. Form of address	Show–tell vs. give–ask
3. Pedagogical action(s)	Re-explains, suggests cognitive conflict, probes student thinking, etc.
4. Degree of student error use	Active, intermediate, or rare
5. Act of communication barrier	Over-generalization, a Plato-and-the-slave-boy approach, or a return to the basics

of educational psychology courses and two mathematics content courses before taking a three credit-hour methods course with parallel field experience in their senior year. They then intern as graduate students for a full year in a professional development school in local school districts partnered with the university.

Thirty-one PSTs were in their senior year of the elementary teacher preparation program, and 26 were mathematics majors seeking secondary certification. Of the 57 PSTs, only five were male (two were elementary PSTs and three were secondary PSTs). The participants ranged in age from 22 to 35 years. The elementary participants had completed two mathematics content courses focused on number sense and geometry. The secondary PSTs had taken a minimum of 39 semester hours of college-level mathematics courses. Participants were recruited from three mathematics methods courses—two elementary and the other secondary. These one semester (14-week) courses were designed to support PSTs' understanding of approaches, strategies, and issues that are relevant to the teaching and learning of mathematics. They were aimed at learning to teach mathematics in a manner focused on student thinking rather than rote memorization. Upon course completion, both elementary and secondary participants were expected to possess sufficient knowledge for teaching mathematics. Through the methods courses, participants had several opportunities to discuss fundamental ideas for teaching and to analyze some examples of children's work. However, elementary participants did not discuss specific student strategies related to proportional reasoning. In particular, both elementary and secondary participants did not explicitly discuss the additive strategy examined in the context of this study. By including two different groups of preservice teachers with different backgrounds and experiences with mathematics, this study intended to collect a broad range of responses to the study's task and to explore the relationship between CK and PCK.

3.2 Tasks

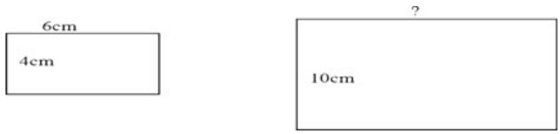
Two tasks were developed based on the literature review (e.g., Hart, 1984) and on textbook analysis (e.g., *Connected Mathematics*). In the first question, called the *similar rectangles problem*, participants were asked to find the missing side of one rectangle, given the condition that two rectangles are similar, and were asked to explain their solution method. This task was developed to assess participants' CK. Next, they were asked to interpret and respond to a student's incorrect solution to the same problem, the main task in this study, in order to assess their PCK. The student's work was presented within a teaching scenario task so as to simulate how mathematical work arises in the context of teaching (see Fig. 1).

3.3 Data collection and analysis

The tasks went through multiple phases of revisions and were piloted with two volunteers who were interviewed in order to check for possible misunderstandings. Once the revisions were made, the final version of the task was then administered as an in-class survey in three mathematics methods courses, two elementary and the other secondary, toward the end of the semester. Only the data of participants who signed the study's consent form are reported.

Table 3 shows an overview of the analytical framework associated with each task for the study. In the first part of the task, responses to the CK task were initially identified based on correctness and then by types of solution strategies used based on Table 2. Participants' explanation of mathematical strategies was further analyzed by looking at the nature of the explanation provided. In particular, the explanation was analyzed to determine whether they pointed to the underlying structure of a situation in which a proportional relationship exists.

You are teaching 6th graders. You asked the students to find the length of the missing side in the similar rectangles shown below. After a few minutes, you asked Sally, one of your students, to explain how to solve the problem. Sally explained that the side would be 12 cm long because 4+2=6.



1. Evaluate Sally's reasoning and explain whether it is mathematically correct or incorrect. If it is not correct, identify the error(s) in Sally's reasoning.
2. How would you respond to Sally? Explain what type of guidance you would give Sally in as much detail as you can.

Fig. 1 Pedagogical task: "What is the length of the missing side?"

Similarly, responses to the PCK task were analyzed examining the range in terms of PSTs' ways of interpreting and responding to student errors. The conceptual versus procedural distinction was utilized first, followed by analysis of whether and how PSTs' identifications of student errors related to their instructional approach. Next, teaching approaches were further analyzed with respect to the four additional aspects in Table 3, exploring the emerging subcategories in each aspect. For example, for the *forms of address* in mathematics teaching, two broad categories—"show–tell" and "give–ask"—were utilized:

- Show–tell approach: delivers verbal or non-verbal information to Sally to hear or see—explaining a procedure, telling the definition of similarity, showing pictures, etc. This type of response usually uses the very words "show" or "tell" but also "explain," "talk to," and "demonstrate."
- Give–ask approach: provides verbal or non-verbal information to Sally to do something with a drawing, computer, or sketchpad or to answer a question. The responses usually use the very words "have," "ask," or "give" but also used "tell," "suggest," "talk to," and "look at."

To ensure the reliability of the coding for these strategies, the researcher met with two other mathematics researchers and shared examples of teachers' responses for each item, with each of the researchers coding PSTs' responses according to the analysis aspects and categories. Initial results of 100 % agreement on the coding of 92 % of the examples were

Table 3 Overview of the initial analytical framework drawn from Son and Sinclair (2010)

Task	Sub-domain	Analysis aspects
CK	1. PST knowledge	Correctness
	2. PST justification	Nature of justification (concept vs. procedure-oriented)
PCK	1. PST interpretation	Conceptual vs. procedural distinction
	2. PST teaching approach	Conceptual vs. procedural distinction
		Form of address
		Pedagogical action
		Use of student errors
		Communication barriers

recorded. Each coding was then documented in excel sheets according to the task sub-domains, and a data table containing all the categorized responses for each participant was developed. Once frequencies (and percentages) were obtained, patterns were explored comparing the tendencies between elementary and secondary PSTs with respect to teacher knowledge, their interpretation of the student's error, and subsequent teaching approaches. Additionally, for individual participants, consistency between PSTs' identification and response to student error in terms of procedural or conceptual approach was examined. In particular, the Wilcoxon–Mann–Whitney test, a non-parametric statistical hypothesis test for assessing whether one of two samples of independent observations tends to have larger values than the other (analog to the independent samples *t* test) (Ott & Longnecker, 2001), was used to examine the differences between elementary and secondary PSTs with a concept-oriented response (code 2), a procedure-oriented response (code 1), and incorrect response (code 0). Using the same coding system, the Wilcoxon signed rank-sum test, the non-parametric version of a paired samples *t* test (Ott & Longnecker, 2001), was also used to trace the tendencies between PSTs' interpretations of and subsequent teaching approaches to student errors if there was no significant different between elementary and secondary PSTs.

4 Results

4.1 PSTs' understanding and strategies in similar rectangles

The findings with respect to the CK task are mentioned briefly as they were helpful in providing an initial framework for analyzing the PSTs' responses and pedagogical strategies to student errors. In the *similar rectangles problem*, all of the secondary PSTs answered correctly versus 77 % of the elementary PSTs (24 out of 31). The results of the Wilcoxon–Mann–Whitney test showed a statistically significant difference between two groups with $Z=-2.564$, $p=0.010$. Seven participants who provided incorrect answers used additive reasoning by focusing on the differences between the given quantities. One typical response was as follows:

> I think that the missing side is 12 cm because the difference between 4 cm and 6 cm is 2. Therefore the difference between the 2 sides of the larger rectangle would also be 2. 10 cm+2=12 cm

Three different correct solution methods were used—within ratio, between ratio, and scale factor (see Table 4). Note that strong evidence exists in the literature that "within" ratio approaches are easier than "between" comparisons (Kaput & West, 1994; Weaver & Junker, 2004). Table 5 shows the frequency and percentage of each type of solution strategy used. While the within ratio approach is used most often by the elementary PSTs, the between ratio approach is favored by the secondary PSTs.

As mentioned earlier, finding a missing value in similar rectangles involves not only understanding the concept of similarity but also procedural knowledge of ratio and proportion. PSTs' explanations for their solution strategies were further explored by focusing on whether they had pointed out the underpinning idea of similarity of rectangles (instance of proportionality). While some PSTs provided a concept-based explanation, by pointing out the criteria for similarity, other preservice teachers just restated their solution method by describing how they set up equations and carried out the calculation, namely a procedure-based explanation. Table 6 shows the distribution of concept-based vs. procedure-based explanation among PSTs with correct answers. An example of each explanation approach is provided below:

Table 4 Example of different methods used by PSTs

Between ratio	Within ratio	Scale factor method
$\frac{4}{10}=\frac{6}{x}$. Assuming the 4-cm side is similar to the 10-cm side, I must find a relationship between 6 cm and x. I create a fraction with 10 and 4 cm and set it equal to x and 6 cm. [I cross-] multiply and solve for x.	$\frac{6}{4}=\frac{x}{10}$. If the rectangles are similar, their sides are at a constant ratio. Thus, you can compare the ratio between the width and length and use this proportion to find missing length from the width of same rectangle.	[The ratio] of smaller to bigger similar rectangle is 4:10. $6\times\frac{10}{4},\ 6\times 2\ \frac{1}{2}=12+3=15$

> Concept-based: Similar objects have an equal ratio between the sides of the objects. Since the widths of the rectangles have a ratio of 4/10, the lengths must also have a ratio of 2/5, which implies 6/15. Therefore, the lengths of the larger rectangle is 15.
> Procedure-based: Use ratios and set up a promotion showing the sides that went with each other: 4 and 10 vs. 6 and X. Then I cross multiplied to find missing x.

A significant difference exists between the elementary and secondary PSTs in the nature of justification. A majority of the secondary PSTs referred to the concept of similarity in explaining their strategies, whereas less than half of the elementary PSTs did so. The Wilcoxon–Mann–Whitney test result confirmed the difference between two groups with $Z=-2.545$, $p=0.011$. This trend, in addition to the findings in Table 5, indicates that secondary PSTs in our study not only have a better understanding about proportionality but also provide more concept-oriented justification than the elementary PSTs.

4.2 How PSTs identify Sally’s learning difficulties

After completing the CK task, PSTs were asked to identify the student’s error and to provide guidance. Finding a missing length in similar rectangles involves at least four aspects of knowledge, the first two related to conceptual aspects of similarity and the others related to procedural aspects:

- Understanding the concept of similarity
- Recognizing the relationship between figures by comparing lengths and widths between figures (*between* ratio) or by comparing the length to width within a rectangle (*within* ratio) or determining a scale factor
- Setting up a proportion to represent similarity (or proportionality in similar figures)
- Carrying out the calculation correctly

Table 5 Solution strategy used in finding a missing length by PSTs

	Category	Elementary (n=31)	Secondary (n=26)	Total (n=57)
Incorrect	Additive	7 (22.6 %)	0 (0 %)	7 (12.3 %)
Correct	Within ratio	14 (45.2 %)	7 (26.9 %)	21 (36.8 %)
	Between ratio	7 (22.6 %)	15 (57.7 %)	22 (38.6 %)
	Scale factor method	3 (9.7 %)	4 (15.4 %)	7 (12.3 %)

Table 6 Two forms of explanation about solution strategy by PSTs with correct answers

Reasoning/level	Elementary (n=24)	Secondary (n=26)	Total (n=50)
Concept-based	11 (45.8 %)	21 (80.8 %)	32 (64.0 %)
Procedure-based	13 (54.2 %)	5 (19.2 %)	18 (36.0 %)

In Sally's case, although she recognized a certain relationship between two similar figures, she appeared not to know that the lengths of the corresponding sides in similar rectangles increase (or decrease) by a constant ratio. She therefore focused on the difference, in particular the *within* difference, by comparing the length and the width within a rectangle. Consequently, although she carried out the calculation correctly based on her additive reasoning, she was not able to correctly find the missing length in similar rectangles. Therefore, the fundamental error in Sally's case appears to result from her limited understanding of the concept of similarity.

Three categories were identified from the responses of the participants. In the first type, the error is called *concept-based*, where PSTs identify Sally's learning difficulties using a conceptual approach—in this case focusing on the meaning of similarity in rectangles in which the following hold: Two figures are similar if (1) the lengths of their corresponding sides increase (or decrease) by the same factor, called the scale factor, while their corresponding angles are equal, and (2) the perimeter from one rectangle to another rectangle also increases by the same scale factor. One typical concept-based response is as follows:

> Sally does not understand that similar means proportion or she may not understand what proportion means. A proportion is a ratio of two numbers, whereas Sally looked at the sum (or difference, depending on how you think about it) of sides.

Another type of error identified is called *procedure-based*. The procedure-based approaches merely focus on the procedure (and actions) of finding the missing value in a proportion. The idea of enlargement in similar rectangles where students find the missing side length geometrically is also included in the procedural approaches. In this approach, PSTs may indicate the need for a ratio, a proportion, or a scale factor for calculation to find a missing length without understanding the meaning of similarity, as illustrated in the following:

> Sally did not calculate the ratio of corresponding sides, i.e., 4 cm/10 cm=ratio of sides. What Sally did was determine a 6−4=2 cm difference then add 10 cm+2 cm=12 cm. This [method] is incorrect not setting up a ratio.

One important distinction between the concept-based and procedure-based indication lies in the different use of verbs in describing Sally's difficulties. Preservice teachers categorized as using procedure-based approaches might have the same recognition about Sally's difficulties (notice they also pinpointed not using a ratio or a scale factor). However, their focus was on setting up a numerical expression that presents equality in two rectangles. For instance, while PSTs with concept-based approaches used the verbs "recognize" or "see," those using procedure-based approaches included verbs such as "use," "calculate," or "set up." This difference indicates that PSTs in concept-based approaches tend to think that Sally did not *understand* or had limited understanding of the meaning of similarity so that she did not *see* or *recognize* the proportional relationship between two figures. In contrast, in the procedure-based responses, the respondents assumed that Sally knew something about similarity but did not *use* a constant ratio to correctly represent the proportional relationship between sides and lengths in similar figures.

A third category involved responses that misdiagnosed Sally's error(s) either based on additive reasoning or incorrect focus. In most cases, PSTs indicated Sally's errors as deriving from finding the difference *within* rectangles, as illustrated in the following:

> Sally is explaining the relationship between the sides 4 and 6 rather than first comparing sides 4 and 10 then 6 and x. So she should be looking at how side 4 is related to side 10, then use that same relation with side 6 to get side x.

The concept-based and procedure-based approaches were further subdivided based on whether PSTs focused on a within ratio, a between ratio, or a scale factor. Table 7 shows the subcategories of each approach and its distribution. The data dispersion for the secondary and elementary PSTs was comparable for all categories, with no statistically significant difference between two groups. Accordingly, the results are presented without distinction between the two groups. Table 7 shows that although Sally's errors appeared to have originated from limited understanding about the conceptual aspects of similarity, more than half of the PSTs referred to Sally's error(s) from procedural aspects.

Comparison between the nature of PSTs' explanation (Table 6) and their interpretation of Sally's errors (Table 7) with respect to the conceptual vs. procedural distinction shows a decrease in the frequency of concept-oriented responses from justifying the PST's own solution strategy and to identifying the student's learning difficulties and vice versa. This finding suggests that a relatively small number of participants identified student errors stemming from conceptual aspects regardless of their own solution strategies.

4.3 How PSTs respond to Sally's work

4.3.1 Conceptual vs. procedural intervention

Since Sally's difficulties appeared to stem from a limited understanding of the underpinning ideas of similarity, from my point of view, the focus of instruction should be on the conceptual aspects (Boero & Garuti, 1992; Kilpatrick et al., 2001; NCTM, 2000; Rittle-

Table 7 PSTs' identification of Sally's errors

Type of identification	Subcategory	# of response	Total
Concept-based	1. Meaning of similarity	3	19 (33.4 %)
	2. Recognizing the relationship	8	
	by a within ratio	(5)	
	by a between ratio	(1)	
	by a scale factor	(2)	
	3. Meaning of proportion	8	
Procedure-based	4. Idea of enlargement	2	
	5. Use difference in calculation	14	32 (56.1 %)
	6. Not use a within ratio in calculation	9	
	7. Not use a between ratio in calculation	3	
	8. Not use a scale factor in calculation	4	
Misdiagnosed	9. Additive reasoning	6	6 (10.5 %)
Total		57	57 (100 %)

Johnson & Alibali, 1999), which can be developed sequentially in the order presented in the following:

Step 1: Develop the meaning of similarity in mathematics
Step 2: Help Sally recognize the proportionality between similar figures
Step 3: Guide Sally to determine the ratio or scale factors in a given problem
Step 4: Conduct and carry out the procedures correctly

Thus, the first analysis focused on whether PSTs provided *concept-based* instruction to Sally, namely responses that focused on the meaning of similarities and/or the meaning of proportionality (steps 1, 2, and/or 3), or *procedure-based* instruction related to finding the missing value by applying a ratio or a scale factor (steps 3 and/or 4). As step 3 involves problem solving (i.e., how to solve the given problem), it can be categorized as either concept or procedure-based, depending on the way intervention is provided. PSTs' use of probing questions or giving various tasks was categorized as concept-based instruction, whereas just telling and showing was categorized as procedure-based instruction. In particular, the geometrical approaches (i.e., the idea of enlargement) can be interpreted either concept-based or procedure-based depending on how teachers address such activities or ideas (Moss & Case, 1999). For example, if a teacher presents the 4×6 rectangle embedded in an incomplete rectangle with the "4 side" lying on the "10 side" and then asks students to find the missing side length simply geometrically, the case is categorized as procedure-oriented instruction coded as "Idea of enlargement." Conversely, if a teacher makes or asks students to make a graph with the lengths and widths represented on the axes and then uses these examples to have students to explore the concept of similarity and proportionality and apply it to the given problem, it is categorized as concept-oriented instruction and coded as "Concept of similarity." In the following, an example of each type of instruction is shown:

> Concept-based: I would show her a picture of a rectangle that added two to the width and length and ask her if the rectangle looked the same to show that [Sally's] answer doesn't make sense. Then I would have Sally look at many rectangles that look similar. I would then check using the ratio between the sides. Discuss similarity after this. I would explain that for rectangles to be similar, the length of sides must have the same ratio or fraction. After that, I would ask her to draw 2 new rectangles and that had length n and $n+a$ and one that had $\frac{6}{4}=\frac{x}{y}$. I would help her see that the bigger rectangle [in the second problem] is 2.5 times the size of the smaller one (using 4 cm: 10 cm) so the 6 cm side grows 2.5 times too. The hope being she would see that ratios maintain similarity.
> Procedure-based: Show her [Sally] how to set up ratios and plug in 12 for x (e.g., $6/4=x/10$). See if these are the same. Explain to her that similar rectangles must have similar ratios and therefore her reasoning does not work in this scenario.

In addition to these two categories, a third one involved responses that misdirect Sally based on either the teacher's additive reasoning or incorrect focus. PSTs in this category ask or tell Sally to compare the length in one rectangle with the corresponding length in another rectangle to find the *difference* between rectangles. The following is an example:

> I would tell Sally that although she ended up with the correct answers, she did not use the correct strategy. I would tell her that she needs to figure out how much larger the big rectangle is than the small rectangle. To do that, she should figure out the small rectangle. It is 6 cm side of the small rectangle, $6+6=12$ cm.

As in the analysis of identification of Sally's errors, each approach was further subdivided, as addressed in Table 8. Table 8 shows that about 56 % PSTs provided

Table 8 PSTs' intervention to Sally's error(s)

Instructional focus	Subcategory	# of responses	Total
Concept-based	1. Meaning of similarity	6	18 (31.6 %)
	2. Recognizing the relationship	9	
	by a within ratio	(3)	
	by a between ratio	(2)	
	by a scale factor	(4)	
	3. Meaning of proportion (or ratio)	3	
Procedure-based	4. Idea of enlargement	2	
	5. Finding a missing value in calculation	20	32 (56.1 %)
	by a within ratio	(8)	
	by a between ratio	(16)	
	by of a scale factor	(6)	
Misdirected	6. Guidance based on additive reasoning	6	7(12.3 %)
	7. Incorrect focus	1	
Total		57	57 (100 %)

intervention focused on procedural aspects of similarity. Seven were classified as providing incorrect guidance based on their additive strategy, as they did in identifying Sally's errors.

A comparison between the frequencies between Tables 7 and 8 shows that there is not much difference between the frequencies of concept- and procedure-based approaches for identification and for intervention. The results of the Wilcoxon signed rank-sum test resulted from matching the two values for individual PSTs in Tables 7 and 8 also revealed no statistically significant difference between PSTs' identification of student errors and their intervention with $Z=-0.500$, $p=0.617$. This finding suggests that participants in this study provided their intervention corresponding to the way they recognized student errors. However, more than half preservice teachers identified Sally's errors in terms of procedural aspects of similarity and tried to cope with it by resorting to procedural knowledge even though Sally's errors suggested lack of conceptual knowledge. This apparent disconnect between student errors, teacher interpretations, and teacher intervention may pose several challenges to the teachers and students working together to correct student errors.

4.3.2 Digging deeper: beyond the conceptual vs. procedural distinction

Recognizing several controversial issues surrounding the conceptual versus procedural distinction, further analyses were conducted to gain more insight into the nature of teacher responses (see Table 9).

Form of address Increasingly teachers are being called to engage students in mathematical discussions whereby students share, discuss, and test with each other their mathematical ideas (NCTM, 2000). Two distinct forms of address are apparent in PSTs' intervention toward Sally—(1) the "Show–tell" approach and (2) the "Give–ask" approach. Consistent with the findings from previous studies (e.g., Son & Crespo, 2009), Table 9 shows that the responses were overwhelmingly "show/tell" (39 out of 57, 68 %). This tendency indicates that PSTs in this study prefer to deliver information rather than listening to their students.

Table 9 Categories for describing four aspects of pedagogical strategies to student error(s)

Aspect	Categories	%
Form of address	1. Show and tell	68
	2. Give and ask	32
Pedagogical action	1. Tell how to solve the given problems by a constant ratio	61
	2. Engage in Socratic questioning to think and respond	58
	3. Explain a concept of similarity	56
	4. Draw (or have students draw) picture or drawings to scale	43
	5. Redirect (or tell) Sally to focus on the proportional relationship by comparing two figures	28
	6. Present examples of similar figures to search for patterns	12
	7. Present a counterexample to Sally's additive reasoning	3
	8. Use (or have students use) geometry software (e.g., sketchpad)	7
	9. Ask Sally to reflect or check her method (self-checking)	7
	10. Bring up the problem in class and discuss	3
	11. Have students measure the rectangles to see Sally her errors	3
	12. Pose follow-up problems after correcting Sally	7
Use of student error	1. Active use	19
	2. Intermediate use	23
	3. Rare use	58
Communicative barrier	1. Over-generalization approach	64
	2. Plato-and-the-slave-boy approach	24
	3. Return to the basics approach	12

Pedagogical action With the presented goal of considering student errors for insights into student understanding and making efforts to help students understand the conceptual thinking behind their errors, Table 9 shows the distribution of 12 pedagogical actions observed from the PSTs' responses to Sally. Since one teaching action can take place simultaneously with another action, the percentage of each action was calculated out of 100 %. The most prominent teaching action is telling how to solve the given problems by applying a constant ratio. This action was typically initiated after pointing out the formal definition of similarity or redirecting Sally to focus on a constant ratio between the two figures. Eighteen out of 19 PSTs described the concept of similarity. Although they provide a conceptual rationale for procedures, these teachers relied mainly on telling; consequently, there is little indication that the instruction would help Sally to apply the concept in other problem situations.

The second most prominent teaching action is engaging Sally in Socratic questions that provide opportunities for Sally to think and respond. Typical questions used include: "How much of 6 is 4?" "How can this be applied to the second rectangle?" "What is to 10 like 4 is to 6?" This questioning action was initiated in combination with drawings or geometry software to scale or examples of similar and distorted figures that encouraged Sally to ponder and reflect on the questions. Two PSTs asked Sally to self-check her method before providing intervention. Similarly, two participants provided follow-up problems. Thus, although some PSTs attempted to invite Sally to think about the conceptual basis of her errors with multiple representations and additional questions as recommended by NCTM (2000) as adequate intervention, the finding indicate that a large portion of the PSTs tend to prefer telling.

Use of student error As teachers are being called to use student errors as "catalysts" for student learning rather than hiding or avoiding them, three categories were devised to describe the extent to which Sally's errors were utilized in instructions: (1) *active*, (2) *intermediate*, and (3) *rare* use of student error with 11, 13, and 33 responses, respectively. With the active use of student error, PSTs use Sally's additive strategy as a major tool of their instruction and provide Sally opportunities to discuss and test why her method doesn't work. For example, one PST had Sally draw two rectangles—one based on her additive method (10×12) and the original rectangle (4×6), asking Sally to compare two figures to see if they looked similar. The PST then asked questions to have Sally recognize the meaning of similarity. With the intermediate use of student error, Sally's error is used but not as major part of the discussion. Student error is addressed briefly as a stepping-stone to correct Sally. In examples such as "Ask Sally to check whether her method make sense" or "Measure the given two rectangles to see that her method does not work, the major component of the instruction is remediation of student error by providing Sally opportunities to think problems through or give her chances to reassess her answer." PSTs in the rare use category may make a declarative remark on student error, "Your method doesn't work" or not mention the method at all, merely providing a correct procedure. Thirty-three responses were categorized as rare use, including all procedure-based approach responses. Thus, despite the current attention paid to the active use of student errors as a potential avenue for student learning, the responses in this study did not bear out this emphasis.

Act of communication barrier Accordingly, three patterns with potential for causing difficulty in communication when discussing student errors were identified: (1) the over-generalization of student errors, (2) a Plato-and-the-slave-boy approach (Plato's dialog *Meno*), and (3) a return to the basics. Although the three response categories have been identified and described in the author's previous work (Son & Sinclair, 2010), each category was operationalized for ratio and proportion in similarity.

The most prominent category involved the *over-generalization* of student error, which shows a tendency to run ahead of Sally's need. In this category, participants provide too general an intervention and consequently do not address the given student error(s) appropriately. The responses in this category involve "showing"/"telling"/"talking about" the general properties of similarity or tasks, as in the example, "I would tell her that similar means proportional." As Sally's errors appear to come from the limited understanding of the meaning of similarity, she seems likely to need more concept-oriented instruction. Another type involves responding to Sally by invoking the properties of similarity at a more general level than that of the actual error, as in the following cases: "I would tell her to use a computer" or "I would show Sally a more extreme example (triangles)." In both types of instruction, Sally may well be able to find the correct missing length in the given problem once she understands or follows the instructions provided in these responses; however, doing so does not necessarily help Sally understand the meaning of similarity or prepare her to face related tasks.

The second most prominent type of response involved a *Plato-and-the-slave-boy approach* in which the PSTs appealed to the notion that Sally actually did know how to find a missing length in similar figures correctly but had simply forgotten. Teachers using this response remind Sally of the properties of similarity or ask her to remember those properties, e.g., "I would have her recall the definition for similarity" or "I would tell Sally that she needs to look back on what it means for two rectangles to be similar." The examples are distinguished from *over-generalizations* due to their unique didactic flavor. Recalling memory may be a more useful strategy in situations where students are faced with what Hewitt (2001) calls "arbitrary" knowledge, such as in fact-retrieving or procedural hurdles.

In this case, asking the student to "remember" may actually (inadvertently) communicate to the student that learning mathematics calls for memory rather than understanding. Although not stated by the problem, except perhaps by saying that Sally was in grade 6, in these responses, the PSTs assume that Sally had already learned about the meaning of similarity and how to find a missing length in similar figures and thus did not need introductory instruction.

The third and final category of responses concerns the tendency to fall short of student needs. They included responses such as "I would show examples of similar figures" or "I would bring up triangles to see if that would help her understand the concept of using a ratio." This category of responses involves a return to underlying principles when, in fact, Sally may not need to go so far back in order to address her error and doing so may either make her forget her original problem or introduce still more problems.

5 Discussion and implications

Despite the importance of understanding teachers' knowledge and treatment of student thinking, relatively few studies have focused on teachers' proportional reasoning (especially for teachers in the USA), in contrast to an extensive body of literature on students' understanding of the same topic. The current study addressed the need for more research focusing on preservice teachers' CK and PCK related to proportional reasoning, in particular regarding additive reasoning. Due to limits created by the small sample size, results should be considered insights rather than generalizations. In addition, preservice teachers were being asked to imagine a student they knew little about. Thus, the responses they provided may not fully coincide with the actions they would take in an actual classroom teaching situation. Nevertheless, the study contributes to the continuing effort to optimize mathematics learning for all students by focusing on the mathematical knowledge teachers need to teach effectively.

Results indicated that elementary and secondary PSTs had developed different levels of understanding and ability to explain their solution strategies to the specified ratio and proportion problem. However, the differences became less distinct when it came to both identifying Sally's incorrect strategy and providing good intervention. While Sally's difficulties appeared to originate from insufficient understanding of the concept of similarity, a majority of the PSTs identified her errors as stemming from procedural misunderstandings. The findings thus revealed certain aspects of PST responses that indicate more general beliefs and tendencies that play an important role—alongside content knowledge and the ability to recognize student errors—in developing response strategies for students.

This study points to a subtle relationship between preservice teachers' CK and their ability to identify and respond to Sally's incorrect answers. PSTs who used additive comparison in their own problem solving generally provided inappropriate interventions, suggesting that a lack of mathematical knowledge limits a teacher's instruction (Carpenter et al., 1988; Spillane, 2000; Son & Crespo, 2009). In addition, we also observed that the distributions of PSTs' identification of Sally's error and of their intervention were very similar, suggesting that teachers tend to intervene to compensate what they think is lacking in children's thinking.

However, the relationship between the content knowledge and the identification/ intervention of PSTs who had sufficient mathematical knowledge about similarity was not straightforward. This study indicates that good mathematical knowledge does not guarantee good teaching practice. A large portion of PSTs who showed a strong knowledge of ratio and proportion focused on procedural aspects of similarity when identifying student error. This study also revealed that the lack of ability to identify the problem fed into the lack of ability to correctly respond to students. Indeed, some of the PSTs considered Sally's error to be a procedural one and tried to tell her the right procedures even though they understood similarity conceptually. Thus, we cannot simply conclude that "a positive relationship exists between teacher knowledge and their pedagogical strategies" in this case. Conceptual knowledge, for our PSTs, did not necessarily equal conceptually oriented teaching. In addition, although elementary and secondary PSTs showed differences in their understanding about proportionality and types of justification, they did not show differences in their identification of students' errors. This result seems to suggest that there is a complicated relationship between teachers' content knowledge and their teaching practices. These findings are supported in several previous studies that reported no direct relationship between teachers' content knowledge and pedagogical strategies (e.g., Begle, 1979; Stacey, Helme, Steinle, Baturo, Irwin, & Bana, 2001; Sanchez & Linares, 2003; Son & Sinclair, 2010).

The complex, disconnected relationship between PSTs' understanding, interpretation, and teaching strategies may be explained in four ways: (1) PSTs' insufficiently developed knowledge, (2) the known/familiar mathematics (i.e., procedural) serving as a source of security for PSTs, (3) difficulty in creating concept-oriented instruction, and (4) resistance of PSTs to teaching mathematics for understanding. First, not only the PSTs' insufficiently developed mathematical knowledge but also their lack of knowledge about correctly assessing student's problems and addressing them contributed to this complex relationship. PSTs either overlooked or misapplied teaching strategies that directly deal with student errors and misconceptions. In addition, a specific explanation for the overuse of procedure-based approach when working with proportions appears likely on the part of some: Hines and McMahon (2005) reported that PSTs tend to view writing an equation for a proportional relationship as the highest developmental level and to devalue less symbolic proportional reasoning strategies. Similarly, due to this view, PSTs in this study might have been reluctant to provide concept-oriented identification and instruction, instead focusing on more valued procedures containing abstract equations. Moreover, some PSTs' difficulty with a concept-based approach likely makes such instruction less preferable still (Ma, 1999; Brown & Borko, 1992; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Son & Sinclair, 2010). Eisenhart et al. (1993) examined the relationship between teacher knowledge and their teaching practice and reported that even teachers with a strong knowledge of mathematics often have much more difficulty articulating a conceptually oriented approach. Furthermore, PSTs' resistance to teaching mathematics for understanding may lead to a complicated relationship between teachers' content knowledge and their teaching practices. Many PSTs come from traditional mathematics classrooms where procedural knowledge is revered over conceptual understanding and investigation (Ma, 1999). Despite teacher education program experiences and professional development experiences that focus on the ideas of teaching mathematics for understanding, some teachers, especially secondary teachers, prefer to teach mathematics procedurally. Brendenfur (2008), for example, found that elementary teachers were often much more willing to teach mathematics conceptually than secondary teachers because of different views of teaching mathematics between elementary and secondary teachers. Similar to the findings and sources from previous studies, secondary PSTs in this study may have preferred traditional ways of teaching, although their content knowledge was slightly stronger. Due to the compound effects of these four factors, preservice teachers may have struggled to come up with powerful means of representing mathematics to Sally.

This study has implications both for teacher educators working to design mathematics courses for preservice teachers as well as for researchers interested in better understanding teacher knowledge and teaching strategies. First, this study highlights the need for better content knowledge acquisition in elementary PST education programs. The preservice elementary teachers involved in this study were seniors and had completed all their mathematics and mathematics education requirements. However, some produced the same types of errors that their students might, consequently providing inadequate interpretation of student error and interventions to students. Teacher educators must ensure that PSTs, elementary and secondary, are fully prepared to address student misconceptions. Research has documented numerous misconceptions and error patterns that students possess regarding the mathematics they learn. To increase levels of CK on the part of PSTs, teacher educators must examine their programs to ensure that the misconceptions identified in this and other studies are addressed. It is important to examine the mathematical perspective of these misconceptions, e.g., conceptual aspects and procedural aspects of proportionality.

In addition, teacher educators should provide learning opportunities for PSTs to accurately recognize student errors and plan more appropriate instruction. Because many elementary school students adopt the additive strategy (e.g., Streefland, 1991), many teacher educators have recommended that preservice and inservice teachers need to know about this misconceived strategy. Teachers should also be able to help change students' informal understanding of proportion into more formal ones (e.g., Ben-Chaim, Keret, & Ilany, 2012; Fernandez, Linares, & Valls, 2011). Yet, this study revealed PSTs' frequent reliance on telling and showing when responding to student errors. We also saw that these teachers rarely used or discussed student errors as part of their instruction. A large portion of the PSTs did not provide any opportunity for Sally to reflect on her method, which might have provided a beginning step to help her make sense of her ways of thinking about mathematics. Son and Crespo (2009) reported that PSTs do not often listen to their students, preferring instead of deliver information by repeating the procedure until students recognize their errors. Considering PSTs' limited understanding of ratio and proportion and their limited exposure to identifying student errors, focusing on rules and procedures might be more appealing to them in identifying the source of errors and responding to errors. This calls for mathematics teacher educators to provide more opportunities for PSTs to analyze students' written work. A model might be found in Bright, Chambers, and Vacc (1999), who reported how to use children's work effectively in enhancing PSTs' knowledge for teaching. In addition, Ben-Chaim et al. (2012) suggest a model for teaching ratio and proportion using authentic investigative activities. Several prior studies reported that examining student work samples can help teachers attend to student thinking and connect it to teaching (e.g., Blythe, Allen, & Powell, 1999; Chamberlain, 2005; NCTM, 2000). Adding to these findings, this study stresses that teacher educators need to use incorrect student examples as a tool to promote PSTs' understanding. For example, in preservice elementary mathematics methods courses, an instructor can demonstrate students' use of the additive strategy, ask preservice teachers to think about how to support these students, and then discuss pre-service teachers' responses, focusing on the importance of the cognitive-conflict approach. Although this approach might have already been adopted in some countries, such learning opportunities are not common in the teacher education programs in many countries, in particular the USA (Mewborn, 2000). More in-depth learning experiences focusing on children's thinking, in particular, their misconception and errors during mathematics methods courses, could lead to better understanding and provide PSTs with a better framework for identifying important instructional strategies as they begin their first year of teaching.

Furthermore, another finding, the three categories of acts of communication barriers, also calls for special attention by teacher educators and in teacher educator programs. Despite current reform efforts aimed at helping students learn mathematics conceptually, the PSTs in this study asked Sally to recall procedures rather than encouraging her to think through the problem or to reason about mathematics. Conversely, the potential existed for the PSTs to perceive mathematics as a statement of end products—definitions and procedures—and memorization. More consideration is needed in teacher education programs and research studies regarding how to help preservice teachers develop their knowledge and teaching approaches in ways that allow them to identify errors correctly and provide appropriate interventions.

Regarding methodological limitations, benefits would have resulted from more human interaction. The feedback provided by students to the written task was sometimes very brief. Clearer findings could have resulted had the participants been interviewed face to face. Another drawback to the written task was the lack of human interaction involved in responding to an imaginary student whom PSTs knew little about. Thus, their responses may not fully coincide with the types of responses they would give in a live situation. It is unclear to what extent participants would have actually handled the scenarios presented in an actual classroom setting in similar ways. As other studies (e.g., Ma, 1999; Hines & McMahon, 2005) have shown, however, the inclination to act in certain ways can indeed provide important insight into the reasoning and practices of teachers, contributing to the collective understanding of such complex phenomena.

Future research that continues to investigate preservice teachers' interpretation of and responses to students' errors is needed. This study investigated these complex topics through only one simulated teaching task and with a relatively small number of participants. The results are consistent with others' findings but also raise questions for further investigation. Future studies might examine the ways teachers interpret and respond to student errors with topics other than ratio and proportion. More specifically, relevant to the implications for teacher education programs, future study may examine preservice teachers' responses to student errors with different research tools and in different contexts, using an interview or observations, for example, as teachers in real situations might respond and react differently. Studies that examine inservice teachers' responses to student errors would also contribute significantly to understanding preservice teachers' ways of thinking. Furthermore, researchers and teacher educators should collectively address the challenge of the disparity between PSTs' mathematical understanding and their ability to identify and intervene in student errors. Such exploration of PSTs' interpretations of and responses to student ideas, and in particular, student errors, will enrich an increasingly multifaceted dialogue among reformers, teacher educators, and professional developers about how we may help PSTs learn to teach mathematics in ways that promote student understanding.

References

Ashlock, R. B. (2006). *Error patterns in computation: A semi-programmed approach* (9th ed.). Columbus, OH: Charles E. Merrill.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132–144.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Baxter, G. & Junker, B. W. (2001). Designing developmental assessments: A case study in proportional reasoning. Paper presented at the Annual Meeting of the National Council of Measurement in Education, April 2001, Seattle, WA.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Ben-Chaim, D., Keret, Y., & Ilany, B. (2012). *Ratio and proportion: Research and teaching in mathematics teachers' education (pre-and in-service mathematics teachers of elementary and middle school classes)*. Rotterdam, the Netherlands: Sense.

Berk, D., Taber, S., Gorowara, C., & Poetzl, C. (2009). Developing prospective elementary teachers' flexibility in the domain of proportional reasoning. *Mathematical Thinking and Learning, 11*(3), 113–135.

Blythe, T., Allen, D., & Powell, B. S. (1999). *Looking together at student work: A companion guide to assessing student learning*. New York: Teachers College Press.

Borasi, R. (1994). Capitalizing on errors as 'springboards for inquiry': A teaching experiment. *Journal for Research in Mathematics Education, 25*(2), 166–208.

Brendenfur, J. L. (2008). Connecting elementary teachers' mathematical knowledge to their instructional practices. *The Researcher, 21*(2), 1–18.

Briars, D. J., & Siegler, R. S. (1984). A feature analysis of preschooler's counting knowledge. *Developmental Psychology, 20*, 607–618.

Bright, G. W., Chambers, D. L., & Vacc, N. N. (Eds.). (1999). Incorporating cognitively guided instruction into elementary preservice teacher education programs: Results from a National Science Foundation research project. Madison, WI: University of Wisconsin, Wisconsin Center for Education Research.

Brown, C., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*, 385–401.

Chamberlain, S. P. (2005). Recognizing and responding to cultural differences in the education of culturally and linguistically diverse learners. *Intervention in School and Clinic, 40*(4), 195–211.

Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Currier, S., Cramer, K., & Post, T. (1993). *Learning and teaching ratio and proportion: Research implications* (pp. 159–178). New York: Macmillan.

Dossey, J., Halvorsen, K., & McCrone, S. (2008). *Mathematics education in the United States 2008, a capsule summary fact book*. Reston, VA: National Council of Teachers of Mathematics.

Eisenhart, M., Borko, H., Underhill, R. G., Brown, C. A., Jones, D., & Agard, P. C. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*, 8–40.

Fernandez, C. Linares, S., & Valls, J. (2011). *Development of prospective mathematics teachers*' *professional noticing in a specific domain*: *Proportional reasoning*. In the Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 329–336). PME.

Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer.

Boero, P., & Garuti, R. (1992). *A sequence of proportionality problems: An exploratory study in the proceedings of the International Group for the Psychology of Mathematics Education, XVI* (Vol. 1, pp. 225–232). Durham, NH: University of New Hampshire.

Hart, K. (1981). Ratio and proportion. In K. M. Hart (Ed.), *Children's understanding of mathematics: [ages] 11–16* (pp. 88–101). London: John Murray.

Hart, K. (1984). *Ratio: Children's strategies and errors*. Windsor, England: NFER-Nelson.

Heinz, K., & Sterba-Boatwright, B. (2008). The when and why of using proportions. *Mathematics Teachers, 101*, 528–533.

Hewitt, K. (2001). Blocks as a tool for learning: A historical and contemporary perspective. *Young Children, 56*(1), 6–12.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Hines, E., & McMahon, M. T. (2005). Interpreting middle school students' proportional reasoning strategies: Observations from preservice teachers. *School Science and Mathematics, 105*(2), 88–105.

Kaput, J., & West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235–287). Albany, NY: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. (1983). Early adolescents' proportional reasoning on 'rate' problems. *Educational Studies in Mathematics, 14*(3), 219–233.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Lamon, S. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Hillsdale, NJ: Erlbaum.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2007). How students view the general nature of their errors. *Educational Studies in Mathematics, 66*, 43–59.

Lim, K. H. (2009). Burning the candle at just one end: Using nonproportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School, 14*, 492–500.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Mewborn, D. S. (2000). Learning to teach elementary mathematics: Ecological elements of a field experience. *Journal of Mathematics Teacher Education, 3*(1), 27–46.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122–147.

NCTM. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: National Council of Teachers of Mathematics.

Ott, L. R., & Longnecker, M. T. (2001). *An introduction to statistical methods and data analysis* (5th ed.). Pacific Grove, CA: Duxbury.

Rittle-Johnson, B., & Alibali, W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175–189.

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education, 6*, 5–25.

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and US mathematics lesson. *Teaching and Teacher Education, 21*(5), 491–508.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and U.S. mathematics classrooms. *The Elementary School Journal, 108*(2), 131–147.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Son, J., & Crespo, S. (2009). Prospective teachers' reasoning about students' non-traditional strategies when dividing fractions. *Journal of Mathematics Teacher Education, 12*(4), 236–261.

Son, J., & Sinclaire, N. (2010). How preservice teachers interpret and respond to student geometric errors. *School Science and Mathematics, 110*(1), 31–46.

Spillane, J. P. (2000). A fifth-grade teacher's reconstruction of mathematics and literacy teaching: Exploring interactions among identity, learning, and subject matter. *The Elementary School Journal, 100*(4), 307–330.

Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Pre-service teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education, 4*, 205–225.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36*, 404–411.

Stevenson, H. W., & Stigler, J. W. (1992). *The learning gap*. New York: Simon & Schuster.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process. Part I. *Educational Studies in Mathematics, 15*(3), 327–348.

Streefland, L. (1991). Fractions, an integrated perspective. In L. Streefland (Ed.), *Realistic mathematics education in primary school, 93–118*. Utrecht, the Netherlands: Freudenthal Institute.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York: Academic Press.

Weaver, R., & Junker, B. W. (2004). Model specification for cognitive assessment of proportional reasoning. Department of Statistics technical report #777. Pittsburgh, PA: Carnegie Mellon University

Wynn, K. (1990). Children's understanding of counting. *Cognition, 36*, 155–193.