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Teachers modify geometry problems: from proof to investigation

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Abstract We explored transformations that teachers made to modify geometry proof problems into investigation problems and analyzed how these transformations differ in teachers who use a dynamic geometry environment (DGE) in their classes and those who do not. We devised a framework for the analysis of problem transformations and types of teacher-generated problems. We introduce distinctions between static and dynamic transformations of geometry problems. By observing differences in the transformations the teachers made and the types of problems they produced, we suggest that teachers who use DGE in their classes develop a better understanding of geometry investigation tasks and have no difficulty in transforming proof problems into investigation discovery problems through teaching. Furthermore, we suggest that working with DGE leads to more changes in the givens of the problems and to more dynamic transformations of a problem. From the differences we found in relation to the various problems used in this study, we conclude that problem transformations are problem dependent. Finally, we argue that problem transformation is teachable but requires special training.

Keywords Investigation problems · Problem transformations · Dynamic and static transformations · Teachers' experience and skills

1 Introduction

In the last two decades, the mathematics education community has strongly emphasized the importance of inquiry-based learning environments for promoting active learning on the part of students (Brown & Walter 1993, 2005; Da Ponte, 2007; Jones & Shaw, 1988; Leikin, 2004; Silver, 1994; Wells, 1999, 2001; Yerushalmy, Chazan & Gordon, 1990). By contrast, the majority of Israeli textbooks contain almost no investigation problems. Nevertheless, teachers can transform conventional problems provided in the textbooks into investigation tasks. Task transformation is not a simple assignment, however, and demands of the teacher specific

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problem-posing skills and mathematical understanding of the concepts and processes associated with the given problem. Based on these observations, we focused our attention on the question: "How do teachers transform mathematical proof problems into investigation problems when they are required to do so?"

2 Theoretical background

2.1 Inquiry-based learning

Researchers point out that inquiry-based learning improves the quality of mathematics learning by providing learners with multiple opportunities of raising and testing conjectures using multiple examples, receiving quick feedback, using multiple representations, and being involved in the modeling process (Da Ponte, 2007; Yerushalmy et al. 1990).

The dictionary definition of inquiry is seeking knowledge, information, or truth through questioning (e.g., http://dictionary.reference.com/browse/inquiry). According to Wells (2001) inquiry is a way of teaching and learning which integrates wonderment and puzzlement and arouses interest and motivation in learners.

> Motivated and challenged by real questions and problems, their attention is on making answers and solutions. Under these conditions, learning is an outcome that occurs because the making requires the student to extend his or her understanding in action—whether the artifact constructed is a material object, a demonstration, explanation, or theoretical formulation. (Wells, 2001, p. 17)

Mathematics education researchers studying inquiry-based learning use terms such as "investigation", "exploration", and "experimentation", while the boundaries between these terms are rather vague. Borba and Villarreal (2005) use the term "experimentation" denoting procedures that include leading to the discovery of mathematical results previously unknown to the experimenter [learner]. Da Ponte (2007) considers explorations and investigations as special kinds of problem solving associated with problems that are not completely formulated, in which the student has to define which mathematical question to pursue. Following Wells (2001), in this study we consider investigatory activities as the core element of inquiry-based learning.

Taking into account that special attention is given to inquiry-based learning and teaching in geometry (Healy & Hoyles, 2001; Yerushalmy et al. 1990), we decided to focus our research on geometry investigations. Based on Yerushalmy et al. (1990), we consider investigations in geometry as activities that include experimenting (to arrive at a conjecture), conjecturing, testing (the conjecture), and proving (or refuting) it.

Inquiry-based learning gains more power with the use of technological tools (Borba and Villarreal, 2005; Hoyles & Lagrange, 2010; Schwartz, Yerushalmy & Wilson, 1993). Investigations in geometry are naturally associated with the use of dynamic geometry environments (DGEs) (Mariotti, 2002; Schwartz et al. 1993; Yerushalmy et al. 1990). Numerous studies have explored the role of DGEs in the instructional process, specifically in concept acquisition, geometric constructions, proofs, and measurements (e.g., Hölzl, 1996; Jones, 2000; Mariotti, 2002), and in understanding the dynamic behavior of geometric objects (Talmon & Yerushalmy, 2004). Dragging is usually considered to be a problem-solving tool that can include construction, searching for commonalities, conjecturing, and proving and refuting conjectures (Healy & Hoyles; 2001; Hölzl, 1996). At the same time, teachers "have been neglected players in research on the relationship between digital technologies and mathematics education" (Healy & Lagrange, 2010, p. 287). Following this observation, our study pays attention to teachers' ability to adapt regular mathematical problems to investigations in geometry; it further attempts to understand what teachers learn from their experience of implementation of DGE in their classes. For this purpose, the research population in this study combined teachers who do and do not implement geometry investigations with DGE in their teaching practice.

2.2 Teachers' role in devolving tasks to the class

Fennema and Romberg (1999), and many other researchers in mathematics education, have stressed that one of the central roles of mathematics teachers is the initiation of meaningful mathematical activities in their classrooms. In some cases, these positions may be associated with the centrality of the teachers' role in devolving good problems to their students as theorized by Brousseau (1997). Consequently, to perform meaningful investigation in geometry class, teachers should choose appropriate problems which facilitate experimentation, discovery, conjecturing, and the testing and proving of conjectures. Wells (2001) pointed out:

> As we have discovered, the choice of experiences that provide the topics for investigation is critical. Not only must they be such as to arouse student interest, engaging feelings and values as well as cognition; but they must also be sufficiently open-ended to allow alternative possibilities for consideration. They also need to be able to provide challenges appropriate to individual students' current abilities. (ibid, p.17)

Usually investigations in geometry are supported by DGE that frequently lead to technological difficulties with the environment, classroom equipment issues, and more (Healy & Lagrange, 2010). Additionally, there is a difference in the availability of instructional materials in "regular" versus investigation classes. In a "regular" geometry classroom, where proving is the main mathematical activity (e.g., Hanna & De Villiers, 2012), teachers choose proof problems from textbooks and other instructional materials. Inquiry-based learning, however, requires devolving investigation problems to the classroom (e.g., Da Ponte, 2007; Yerushalmy et al. 1990), yet often teachers cannot find investigation problems in regular instructional materials.

In general, geometry investigation with DGE requires teachers to rethink teaching: they are supposed to deal with unfamiliar or even new mathematical practices, and "take a more prominent role in designing learning activities for their students" (Healy & Lagrange, 2010, p. 288). One of the ways for designing investigation problems for geometry classes is transforming proof problems from regular textbooks into inquiry problems. Our study analyzes these transformations performed by the teachers as an instance of problem posing associated with generation of a new problem based on the given one.

2.3 Problem transformation as a problem posing activity

Problem posing is a broad concept, usually related to the creation of a new problem by a "poser." Several studies consider problem transformation (also called re-formulation) as an instance of problem-posing activity, (Stoyanova, 1998, with reference to Duncker, 1945; Mamona-Downs, 1993). The present study falls into this category (Leikin, 2004). The assignment presented to teachers in this research asked them to "transform the given proof problem into an investigation problem."

Mathematics educators include both problem posing and investigation problems in a broad range of types of mathematical tasks called ‘open problems’ (Pehkonen, 1995). Pehkonen argues that problem openness depends on the openness of givens and goals (starting situation and goals situation, according to Pehkonen’s terminology) defined by a task. Correspondingly, problem posing related to problem transformation is explored by researchers focusing on systematic transformations of a given problem involving variations in goals and givens. The “what if not?” scheme is the most well-known problem-posing strategy (Brown & Walter, 1993; 2005; Jones & Shaw, 1988; Yerushalmy et al. 1990; Friedlander & Dreyfus, 1993). “What if not?” strategy, which is based on changes in givens, leads to creating space for conjecturing and producing new insights about problem outcomes. Silver, Mamona-Downs, Leung, and Kenny (1996) and Hoehn (1993) have drawn attention to the “symmetry” strategy that leads to the creation of a problem in which the givens and the goals have been swapped. Silver et al. (1996) also describe the “goal manipulation” strategy, in which the givens remain and only the goal is changed. “Chaining” and “special cases” strategies that appear in solutions to a given problem are problem-posing strategies that integrate systematic goal manipulation (Hoehn, 1993). Generalization is a systematic variation of a given problem that sometimes leads to theoretical conclusions based on observation of several specific cases (Hoehn, 1993). Finally, Hoehn (1993) points out that some of the facts/tasks in geometry may be discovered accidentally and not by a systematic approach.

Researchers who focused their attention on problem posing in school mathematics addressed various aspects associated with problem posing. Among these were the characterizations of cognitive processes involved in problem posing (e.g., Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005), problem-posing strategies (e.g., Silver et al., 1996; Stoyanova, 1998), the development of problem-posing skills, and problem posing as a didactic tool (e.g., Brown & Walter, 1993, 2005). The present study explores ways in which teachers transform proof problems into investigation problems, from the perspective of the openness of teacher-generated investigation problems and the changes they performed to the givens and goals of the initial problems.

3 The study

3.1 Research objectives

The research had two interrelated objectives. First, it was aimed at exploring transformations made by mathematics teachers to problems taken from the geometry textbook in order to convert the problems from proof to investigation mode. Second, it examined the connections between these transformations and the teachers’ previous teaching experiences with DGE.

3.2 Population

The problem modification assignment was an unconventional one for the teachers. It required deep and connected knowledge of mathematics and pedagogy. Therefore, all the teachers who were invited to participate in the study were also teachers’ instructors in middle school. Eight of ten in-service teachers who took part in the study had MA degrees in mathematics or in mathematics education. All the teachers had at least 10 years of teaching experience (varying from 10 to 32 years). All the teachers had participated in special training courses for the integration of dynamic software in geometry teaching. Five of the ten teachers implemented computer-based activities in their classrooms (group A) and the other five did not do so (group B). Thus, it was possible to compare performance of the assignment by the teachers who did versus those who did not teach geometry in a computer environment.

3.3 The problems in the study

Three proof problems from a standard geometry textbook were chosen according to the following criteria: all the tasks belonged to one geometry topic (quadrilaterals) taught by all the participating teachers. The problems differed in their level of complexity: problem 1 is the simplest, being one of the basic problems in parallelograms. Problems 2 and 3 require knowledge of a relatively large number of more advanced theorems and definitions used in the proof (Fig. 1).

3.4 Data collection

The teachers were asked to change three geometry proof problems (Fig. 1), taken from a regular geometry textbook, into investigation problems. They were asked to think aloud during individual interviews, which were videotaped and transcribed. Artifacts such as teachers' drawings and handwriting were collected.

3.5 Data analysis

Data analysis was performed with respect to the types of teacher-generated investigation problems and the types of transformations the teachers performed on the proof problems.

Types of problems The second author of this paper analyzed all the teacher-generated problems to ascertain whether they were formulated clearly and made mathematical sense. All clear teacher-generated problems were divided, according to their degree of openness, into two large categories: investigation-oriented problems and non-investigation problems (see the detailed description provided in "Types of problems formulated by the teachers").

Task transformations For the analysis of problem transformations performed by the teachers when generating investigation-oriented problems, we developed a framework for the analysis of problem transformations with detailed consideration of the changes performed by the

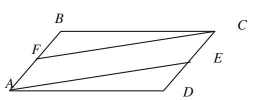
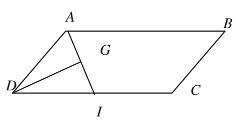
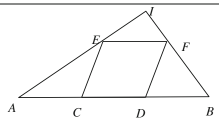
Problem 1	**Problem 2**	**Problem 3**
		
Given: *ABCD* - parallelogram $AE \parallel CF$ Prove: $\angle AED$ and $\angle BFC$ are equal angles	Given: *ABCD* - parallelogram *AI* and *DG* are angle bisectors Prove: a) $\angle AGD$ is a right angle b) AG = GI	Given: *AIB* - triangle *CEFD* rhombus inscribed in the triangle $AC = CD = DB$ Prove: ΔAIB is a right triangle

Fig. 1 Three proof problems presented to teachers for transformation into investigation problems

teachers to the givens and the goals of the initial proof problems. The transformations made by the teachers to all the problems were analyzed and categorized using this scheme (see "Analysis of problem transformations performed by teachers").

4 Findings

4.1 Types of problems formulated by the teachers

4.1.1 Investigation-oriented problems

Investigation-oriented problems were classified in two types according to the degree of their openness: *discovery* problems and *verification* (pseudo-investigation) problems.

Teacher-generated discovery problems A problem was classified as a discovery problem if it was formulated as an open problem that required conjecturing, analyzing a conjecture, and proving. These problems contained expressions like: "Find connections between…" "What will happen if?.." "What can you say about?.." "When?.." "Is it possible?.." (see Examples 1.1, 1.2, and 1.3 in Fig. 2).

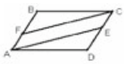
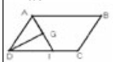
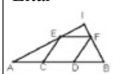
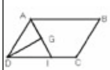
Example #	Problem/ Teacher	Teacher-modified problem	Explanation
	Discovery problems		
1.1	Problem 1 Hadas 	Investigate the connections that were generated [in the parallelogram in which two adjacent angle bisectors were drawn], and prove one or two of the new connections.	The problem allows students *to discover* properties of the geometric object.
1.2	Problem 2 Yael 	[When the givens of Problem 1 are maintained] What happens if [point] I coincides with C?... I can ask ..If ABCD is another quadrilateral?	The students are required *to discover* a special kind of quadrilateral that satisfies the condition "I coincides with C".
1.3	Problem 3 Lital 	Check areas, that is to say, whether there is a connection between them. What is the connection between the areas of the triangles and the area of the rhombus?	The students are asked *to discover* the relation-ship between areas of various geometric figures in the given object. The concept of area which is not mentioned in the original problem was added by Lital.
	Verification problems		
1.4	Problem 2 Riki 	Instead of the angle bisector [AI] I will draw a line here and … this [AD] is going to be equal to this [DI]... AD equals DI. Do we necessarily obtain an angle bisector here [does AI bisects angle DAB]? 	The givens of the original problem have been changed, however from the new givens one can infer that an isosceles triangle is obtained. Thus the problem asks students about a particular property that needs to be verified and proved (AI bisects angle DAB). The teacher-generated problem does not require discovering new properties.

Fig. 2 Teacher-generated investigation problems

Teacher-generated verification problems A problem was classified as a verification problem if it did not require conjecturing but only checking a proposition that had to be proved. In most of these problems students are asked to verify a true proposition, and there is no need to examine under which conditions the proposition is true. The teachers generated problems of this type by substituting the requirement “Prove X is” by asking “Is X true?” (see Example 1.4 in Fig. 2).

4.1.2 Non-investigation problems

Teacher-generated non-investigation problems included proof problems, guidance problems, and computational problems. *Proof problems* required explanations for an argument that was determined a priori to be true (Example, 2.1, Fig. 3). *Guidance problems* gradually led to the solution of the original problem (Example 2.2, Fig. 3). *Computational problems* contained numeric values that required the computation of a segment length, angle size, perimeter, or area of a shape, without drawing conclusions from the computation (Example, 2.3, Fig. 3). The non-investigation problems matched the types of problems in the regular textbooks.

4.1.3 Unclear problems

Problems were classified as unclear in two cases: (1) if the formulation of the problem was not verbally clear; or (2) if the problem did not make mathematical sense or required proving an incorrect statement (Examples 3.1, 3.2, Fig. 4). Participants produced a total of 21 unclear problems, of which 15 were formulated by teachers from group A (users of DGE; Fig. 5) and 17 were formulated for problem 3.

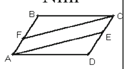
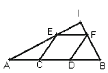
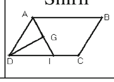
Example #	Problem / Teacher	Teacher-modified problem	Explanation
	Proof problem		
2.1	Problem 1 Nilli 	Instead of parallel segments it is possible to give equal segments [FC, AE]...Now *it must be proven* that AFCE is a parallelogram.	The problem changes the initial givens (parallel segments with equal segments), but the students are required to prove the property of the given object.
	Guidance problem		
2.2	Problem 3 Ronit 	What is the connection between the equality of the segments… and the size of the angles? What are the theorems that lead us to the transition from segments to angles?... The givens say nothing about angles, and this is part of our investigation. Then you arrive at what I usually say: "Mark x and y and calculate the angles."	Ronit does not change the problem formulation but provides a system of prompts for her students when they are solving the proof problem. She considers searching for an appropriate theorem investigation in geometry.
	Computation problem		
2.3	Problem 2 Shirli 	I can give a certain value of an angle and then ask what happens?... Let’s say DAG is 20º or 60º... Compute all the other angles obtained in the sketch.”	Shirli added a numeric value to angle DAG and asked students to calculate the other angles.

Fig. 3 Examples of non-investigation teacher-generated problems

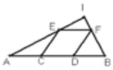
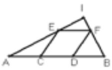
Example #	Problem / Teacher	Teacher-modified problem	Explanation
3.1	Problem 3 Hadas 	Why the diagonals of the rhombus, which are perpendicular to each other, and the right triangle ABI? Then it is formed here by the angles that apparently are swapped and match, they will be here, so here too it is perpendicular.	We found it difficult to understand the precise question Hadas was asking here.
3.2	Problem 3 Lital 	How can we do it so that the area of the rhombus will be largest? This can be investigated with a construction and motion [in DGE], the issue of the area. When is it maximal?	The question that Lital asks cannot be answered based on the givens of the problem: the larger the side of the triangle the larger the area of the rhombus.

Fig. 4 Examples of unclear teacher-generated problems

4.1.4 Distribution of teacher-generated tasks according to their type: group-related and problem-related differences

Table 1 depicts the distribution of teacher-generated problems among the different types. They also demonstrate the differences in the distribution of teacher-generated problems according to the difference by type between the teachers who implement DGEs in their classes (group A) and those who do not (group B). Teachers in the two groups of participants generated a total of 194 problems from the three given ones. The majority of teacher-generated problems were investigation oriented (155 of 194). Only 18 of 194 teacher-generated problems were non-investigation problems, and 21 of 194 problems were unclear. Most of the investigation-oriented problems (139 of 155) were discovery problems.

We found that teachers from group A generated overall fewer problems than did the teachers from group B; and fewer investigation problems in particular (see Table 1). However, teachers from group A generated more discovery problems than teachers from group B, while almost all verification problems were formulated by teachers from group B. Additionally, we found that teachers from group A generated more unclear problems (mostly for problem 3) than teachers from group B, whereas almost all non-investigation problems were formulated by teachers from group B (see Table 1).

Analysis of the teachers' discourse with the interviewer lends additional data to our findings. None of the teachers in group A asked about the meaning of the inquiry problem when presented with the task of problem transformation. By contrast, three of the five teachers from group B asked the interviewer, "What is an investigation task?" and asked to be given an example of an inquiry problem. When they were provided with an example of an investigation problem, in some cases they paid attention to the openness of the example rather than to the degree of discovery embedded in it. Moreover, the teachers from group B at times did not see a reason for using investigation problems in the classroom. The following excerpt from an interview with a teacher from group B exemplifies this attitude:

> Nilli: I have never transformed a problem into another problem. This problem (#1) is very simple…. Excuse me for saying so, but I cannot see what the meaning of investigation is…. If we'd like to make this an investigation problem, I don't know

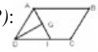
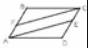
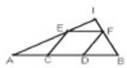
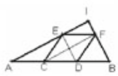
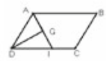
1. No Change in givens	**Static changes in givens**		**4. Dynamic changes in givens**
	2. Changing the initial givens by the initial goal	**3. Adding or removing an object / Reducing given properties**	
A. Focusing on the initial goal			
1A (Riki, P. 2):  I would ask them to draw the angle bisectors [AI, DG] and find the angle [AGE]. What can you say about the triangle that we receive here?	***Not applicable*** *The initial goal is changed.*	*3A (Nilli, P. 1)*  I can play with the figure [ABCD]. Is it true in a trapezoid? I would give them several trials – in a parallelogram, in a trapezoid, and in any quadrilateral.	***Non-sense*** *When properties are added the givens are changed.*
B. Replacing the initial goal by the initial given			
Not applicable *The initial given is unchanged.*	*2B (Riki, P. 1):* We can start backwards: given that the two lines are parallel [AE, CF] and the angles E and F are equal, what kind of quadrilateral should be given [ABCD]?	*3B (Dana, P. 1):* What kind of quadrilateral [AFCE] is constructed by bisectors of opposite angles? For a trapezoid or for a rectangle which type of quadrilateral [ABCD] should we choose?	4B (*(Nilli, P. 2)*: For which figures is the angle bisector a diagonal?
C. Focusing on a new goal			
1C (*(Tamar, P.3)*: Is it possible that these two triangles [FDB, ECA] are congruent? 	***Not observed***	3C (*(Nilli, P. 3)*: What can you say about the areas of parallelograms AEFC and BFED and the area of the rhombus?	4C *(Yael, P. 2)*: I can play with the parallelogram [ABCD]. I can check what happens with the triangle [ADI]. I can change it to a rectangle, a rhombus, or a square.
D. Undetermined goal			
1D (*Ronit,, P. 2)*: In parallelogram ABCD draw two adjacent angle bisectors. What can you say? Let them say all they can about what they have obtained.	***Not observed***	*3D (Hadas, P. 3)* Construct the diagonals in the rhombus [EDCF] and see additional properties. 	4D (*Ronit,, P. 2)*: What happens if angle ADC is 60^{o}? What happens? 

Fig. 5 Problem-transformation framework: examples of teacher-generated investigation-oriented tasks

why [I would do this]. Sometimes we just complicate [the situation]. What we are doing as investigation, to my mind, should enlighten and scaffold.

The interviews in our study appear to have been a source for the participants' own learning about investigation-oriented problems in geometry. This learning is mainly observed in teachers from group B. It is reflected in the numbers of teacher-generated discovery problems produced for problems 2, and 3 which were greater than the number teacher-generated discovery problems produced for problem 1 (see Table 1). As described below, the differences in investigation problems generated by teachers from groups A and B, also reflect learning that occurred in teachers from group B.

Table 1 Distribution of teacher-generated problems by type

			Distribution by groups of teachers			Distribution among the problems and by groups								
			N	N_A	N_B	N_1	N_{1A}	N_{1B}	N_2	N_{2A}	N_{2B}	N_3	N_{3A}	N_{3B}
Teacher-generated problems	Total		194	91	103	60	32	28	70	28	42	64	31	33
	Investigation-oriented problems	Total	155	75	80	49	32	17	63	26	37	43	17	26
		Discovery	139	74	65	43	31	12	56	26	30	40	17	23
		Verification	16	1	15	6	1	5	7	–	7	3	–	3
	Non-investigation problems	Total	18	1	17	9	–	9	5	1	4	4	–	4
		Proof	9	–	9	6	–	6	2	–	2	1	–	1
		Guidance	6	–	6	2	–	2	1	–	1	3	–	3
		Calculation	3	1	2	1	–	1	2	1	1	–	–	–
	Unclear problems	Total	21	15	6	2	–	2	2	1	1	17	14	3

N_A number of problems generated by MTs from group A, N_B number of problems generated by MTs from group B, N_i number of problems generated by MTs for problem i, N_{iA} number of problems generated by MTs from group A for problem i

Teachers in group A (DGE users) clearly knew from the beginning of the interview what investigation problems are and based their problem generation on previous experience. They started enthusiastically formulating investigation-oriented problems for problem 1 and produced a similar number of investigation-oriented problems for problem 2. They then had some difficulty in formulating investigation problems for problem 3, manifested in the number of unclear teacher-generated problems (see Table 1). By contrast, teachers in group B, generated a relatively small number of problems for problem 1, but subsequently doubled the number of investigation problems, when transforming problem 2. Moreover, in contrast to teachers from group A, teachers from group B clearly formulated investigation problems when transforming problem 3: teachers who use DGE formulated a larger number of verbally awkward problems. The awkwardness was manifested in the length of the problem, repetitiveness, a lack of clarity in formulation, and at times an absence of continuity in the language. By contrast, the teachers who do not use computers formulated short problems, of one or two lines, expressed clearly and purposefully. Interestingly, teachers from group B solved problems 1, 2, and 3 before transforming them into investigation ones, whereas teachers from group A transformed the problems almost without solving them.

4.2 Analysis of problem transformations performed by teachers

Investigation-oriented problems (N=155) formulated by the teachers were additionally analyzed from the perspective of the transformations that teachers employed when modifying the given problems. Based on analysis of the literature review on problem-posing related to problem transformation (see "Problem transformation as a problem posing activity"), we devised a framework for the analysis of problem transformation from two main perspectives: *Changing the givens* and *Opening up the goal* (Fig. 5). Below we explain the framework and then analyze transformations performed by the teachers using this framework.

4.2.1 Changing the givens

This category refers to the changes applied to the objects or properties given in the initial problems that teachers made when transforming them into investigation-oriented problems. *Not changing the givens* led teachers to investigation-oriented problems in which the givens were identical with those in the original problem (Fig. 5, column 1).

All the problem transformations that resulted in investigation-oriented problems, in which the givens differed from those in the original problem, were classified either as static or dynamic changes of the givens. This distinction was based on the following analysis of dynamic behavior of geometric figures in DGE: *Dynamic changes* are those that can be obtained by dragging in DGE, while *static changes* are those that cannot be obtained by dragging. Dragging (and thus dynamic change) does not change any of the critical properties of the figure constructed in DGE (see distinction between figure and drawing by Laborde, 1992). For example, by dragging a rectangle, it can be transformed into a square but cannot be transformed into a parallelogram, which is not a rectangle. Static changes in DGE usually require additional construction without changing the given figure, or constructing a new figure.

Consequently, problem transformations, which included adding or removing a geometric object to or from the given one (e.g., drawing or deleting points, segments, rays, axes, and circles) or removing some properties of the given figure, were characterized as static changes. Changes of the givens by the goal—when the goals of the initial proof problem became givens in the teacher-generated investigation problems—were also characterized as static changes (Fig. 5, columns 2 and 3).

4.2.2 Opening up the goal of the problem

All the investigation-oriented problems were formulated as open problems. At times, when the goal of a new problem was general and did not direct students to an object or a property that had to be investigated, we called it an *undetermined goal* (Fig. 5, row D). In the majority of teacher-generated problems, however, the *goal was determined* by a teacher who directed the students towards examining particular geometric properties. Within the group of "goal-determined" problems, we distinguished among three types of problems: (a) problems *focusing on the initial goal* (Fig. 5, row A) in which students are asked to discover and investigate properties that had to be proven in the proof problem; (b) problems with *the initial goal replaced by the initial given* (Fig. 5, row B), wherein the givens of the initial proof problem were used as the goal in a teacher-generated investigation problem; and (c) problems *focusing on a new goal* (Fig. 5, row C), in which the goal referred to new concepts or new geometric objects that were not part of the initial proof problem. Figure 5 contains examples of the teacher-generated investigation problems obtained by making various transformations.

4.2.3 Distribution of the types of teacher-generated problems among transformations

We identified the transformations teachers made when modifying the problems on the basis of the final product produced by teachers at the interviews. We could have analyzed the process of problem generation by examining the transcripts of the interviews covering the stage of problem formulation, but in many cases the teachers preferred not to talk aloud when transforming the tasks. Nevertheless, the teacher-generated problems reflect the type of transformation the teachers performed. Table 2 presents the distribution of teacher-generated problems

Table 2 Distribution of teacher-generated problems by type of transformation

Changing the givens / Opening up the goal		1. No change	Static changes: 2. Changing the initial givens by the initial goal	Static changes: 3. Adding or removing an object/reducing given properties	4. Dynamic changes	Total no.
A. Focusing on the initial goal	N	17	–	17	–	34
	N_A	7	–	10	–	17
	N_B	10 (2 ver)	–	7 (3 ver)	–	17 (5 ver)
B. Replacing the initial goal by the initial given	N	–	6	6	6	18
	N_A	–	1	4 (1 ver)	3	8 (1 ver)
	N_B	–	5	2	3	10
C. Focusing on a new goal	N	35	–	24	8	67
	N_A	6	–	14	8	28
	N_B	29 (10 ver)	–	10	–	39 (10 ver)
D. Undetermined goal	N	8	–	12	16	36
	N_A	6	–	9	7	22
	N_B	2	–	3	9	14
Total no.	N	60	6	59	30	155
	N_A	19	1	37 (1 ver)	18	75 (1 ver)
	N_B	41 (12 ver)	5	22 (3 ver)	12	80 (15 ver)

y (*x ver*) *x* of *y* problems generated by MTs are verification problems, N_A the number of problems generated by MTs in group A, N_B the number of problems generated by MTs in group B

according to the transformations made. The most frequent transformation of the "opening the goal" type was generating a new goal: 67 of 155 of teacher-generated problems were obtained in this way. Interestingly, we discovered that teachers in groups A and B arrived at tasks of this type in different ways. Teachers in group B discovered new properties of the given objects while solving the given problems, whereas teachers in group A usually observed the new properties in the DGE, and then used the discovered "new properties" as a new goal in the problem.

Transformations 1 (using the same givens) and 3 (adding or removing objects in the givens) were the most frequent among the transformation of givens, producing 119 (60+59) of 155 problems (Table 2). Therefore, transformations C1 and C3 yielded a large number of problems. Transformation C1, leaving the given objects unchanged and requiring solvers to focus on a new goal, was the most frequent type of problem transformation. Thirty-five of 155 inquiry-oriented problems were formulated in this way. Among these problems, ten were verification ones, all formulated by group B teachers, and 25 were discovery problems formulated by the teachers in both groups. The majority of tasks with this type of transformation were generated by teachers in group B (29 of 35 problems). Transformation C3 was the next most frequent one (24 of 155 problems). In this type of transformation the teachers modified the initial problem by adding or removing some objects to or from the givens (i.e., making static changes in the given figure) and focusing on a new goal. In this way, the participants produced discovery problems only. Transformations of type C4, which involved dynamic changes in the givens and focused on a new goal, were less frequent (only eight teacher-generated problems were obtained in this way). This transformation was produced solely by teachers in group A.

Group A teachers performed problem transformations with changes in givens more frequently than did teachers in group B. Of 75 inquiry-oriented problems formulated by teachers from group A, 56 incorporated changes in givens of which 55 were discovery problems. This is in contrast to 39 of 80 inquiry-oriented problems formulated by teachers from group B using this type of transformation. Additionally, transformations with an undetermined (the most open) goal were performed by the teachers from group A more frequently than by teachers from group B (22 vs. 14 problems).

5 Discussion and concluding remarks

In this section, we analyze connections between the framework for analysis of transformations of proof problems into investigation problems devised in this study and problem-posing strategies described earlier in the research literature (Table 3). We discuss our research findings as related to teachers' experiences when teaching geometry with DGE. Through interpretations and explanations of our findings, we arrive at research hypotheses regarding teachers' learning through teaching with DGE and formulate research questions for future research.

5.1 Remarks about problem-posing strategies

All the task transformations in the present study are systematized with respect to changes performed by the teachers in the givens and goals of the initial problems. Some teachers simply opened up the problem goal or manipulated the goal of the problem, leaving the givens unchanged. Focusing on a new goal without changing the givens was used mainly by teachers from group B when solving the initial problems and when producing a proof revealing new properties of the given objects (see Table 3, cell C1). Such a strategy in research literature is considered an instance of the chaining problem-posing strategy (Hoehn,

Table 3 Relationship between the strategies defined in the present study and known problem-posing strategies in the research literature

Changing the givens	1 No change	Static changes			Dynamic changes
		2 Role change	3 Constructive change		4 Dragging change
Opening the goal		The initial given is replaced by the initial goal	Adding or deleting object(s)	Reducing given properties	Adding new properties
A. Focusing on the initial goal	Simple opening up	Not applicable. The initial goal is changed	Constructing an additional object	What if not?	Nonsense When properties are added the givens are changed
B. Replacing the initial goal by the initial given	Not applicable	Symmetrical change	Deleting an object	Construction of a new figure	Special cases What if yes? Dragging an initial figure
C. Focusing on a new goal	Goal manipulation, chaining	Not observed			
D. Undetermined goal	Find properties	Not observed			

1993; Silver et al., 1996). Another transformation of proof problem with no change in givens and undetermined goals resulted in the “find properties” tasks (Hoehn, 1993), in which objects for investigation must first be discovered by the solvers.

In several cases, teachers switched the givens and the goal of the problem. This transformation (2B) is described by Brown and Walter (1993) as a “symmetrical” one. From a logical point of view, in these cases teachers asked to examine the correctness of the statement in reverse to the given one. Since Brown and Walter (1993) describe the “What if not?” strategy as including changes in given objects that are usually obtained by reducing the properties of these objects, in many cases changes identified as being static were analogous to those obtained using the “*What if not*?” strategy (Table 3). By contrast, dynamic changes (which can be obtained by dragging in DGE) can lead to a transformation that adds a property to a figure (e.g., a rectangle can be transformed by dragging into a square). Thus we suggest describing dynamic changes as a “*What if yes*?” strategy (Table 3).

The teachers rarely used “What if not?” and “What if yes?” strategies systematically, and tended to change the given problems accidentally. In some cases, however, when teachers “played with the figures,” the task transformations were performed systematically. As Yael comments in the excerpt below, the teachers often mentioned in passing that they were “playing” with the (given) figure and its properties.

> Yael: Inside the parallelogram [ABCD—problem 1, Fig. 1] I see… triangles [EDA and FBC] and a quadrilateral [AFCE]… I say OK and then I ***play with the parallelogram***, transforming it into different things. What happens with the quadrilateral that is inside; what happens to the triangles?

5.2 Contributions, hypothesis, and further questions

By analyzing the differences and similarities between the types of problems generated by the teachers from the two groups and by comparing the problem transformations they performed, we attempt to understand what teachers learn from working with DGE in their classes. Teachers from group A appeared to be more familiar with investigation problems, while for teachers in group B problem transformation assignments, as well as the construct of investigation problems, appeared to be new.

As we have presented, teachers from group B questioned the interviewer as to the meaning of investigation problems while teachers from group A did not have any difficulty with the problem transformation assignment. The change that occurred in the number of investigation problems formulated by the teachers from group B for problem 1, in contrast to those formulated for problems 2 and 3, demonstrate that teachers from group B learned the construct of investigation problems. However, as it was noted in the research findings, teachers pay more attention to the openness of a problem, rather than to the potential that it opens for a discovery activity. Moreover, 50 % of the transformations performed by group B teachers were done by simple opening of the goals without changing the givens. Use of the chaining strategy based on a solution to the proof problem demonstrated that teachers from group B tended first to solve a problem (an activity that was usual for teachers from group B) and only then to transform a proof problem into an investigation one.

An additional indication of the newness of the task transformation assignment for teachers from group B can be seen in the number of non-investigation problems generated by these teachers when transforming the proof problems: Overall they produced 17 non-investigation problems, nine of which were for problem 1, four for problem 2, and four for problem 3. Teachers from group A produced only one non-investigation problem. Additionally, teachers from group B generated 15 verification (investigation-oriented) problems in contrast to one verification problem generated by a teacher from group A; that is, teachers from group A formulated more open problems. Unlike their counterparts from group B, teachers from group A performed more changes to givens (both static and dynamic) in the problems. Teachers in group A also performed more dynamic transformations than teachers from group B did.

These findings lead to the following hypotheses: (1) Through the use of DGEs in their classes, teachers develop their understanding of geometry investigation tasks, especially discovery problems, and have no difficulty in transforming proof problems into investigation discovery problems. At the same time, these teachers are less accurate when generating investigation tasks and tend to formulate a larger quantity of unclear problems. (2) Consistent with several studies (Marrades & Gutierrez, 2000; Yerushalmy et al. 1990) which maintain that working with DGE encourages investigation of geometric shapes by means of dragging, which changes the shapes dynamically, we suggest that working with DGE supports dynamic transformations of a problem. (3) Problem transformation assignments, as well as the meaning of investigation problems, are teachable. However, relatively short interview experience is not sufficient to develop teachers' understanding of the distinctions between verification and discovery problems and the potential of discovery problems for teaching and learning geometry. Consequently, we ask: What are the effective ways to train teacher to use DGE in their teaching practice and, in particular, in which ways teachers can develop their expertise in creating discovery problems?

Finally, our observations of the transformations performed by group A teachers to problem 3 showed that problem transformations can be problem-dependent. Of the 15 unclear problems generated by group A teachers, 14 of them were transformed from problem 3. Our explanation is related to the differences in the construction of the figure in the DGE and the drawing of the figure on paper. Leikin (2004) argues that when a proof problem is transformed into an investigation problem, construction in DGE according to the order of givens in the problem leads to either a robust or a soft construction of the figure (Healy & Hoyles, 2001). In contrast to problems 1 and 2, for which construction according to the order of givens leads to robust construction, problem 3 requires re-ordering the givens for the robust construction that allows investigation by dragging. During the interview, teachers were asked to draw a sketch that does not affect the ways of reaching a solution. Teachers from group B solved the problem when transforming it and, therefore, problems formulated by teachers in group B were clearer than those formulated by teachers from group A.

We suggest that the framework for analysis of problem transformations introduced in this study can serve as a model for the design of investigation problems. The model can be used both by instructional designers for developing instructional materials, for example, and by teacher educators for training teachers in creating investigation problems (for the importance of such training, see Leung and Silver, 1997). Use of the model can lead to the design of a rich collection of problems.

Due to the small research sample and the qualitative nature of the investigation, further study is recommended that will examine our hypotheses. We have attempted to demonstrate herein the descriptive and explanatory power, of the framework for analysis of problem transformations and of the types of teacher-generated problems. We suggest that new research be conducted in order to demonstrate the replicability of the model and its predictive power.

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