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The interplay among gestures, discourse, and diagrams in students' geometrical reasoning

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Abstract This study explores interactions with diagrams that are involved in geometrical reasoning; more specifically, how students publicly make and justify conjectures through multimodal representations of diagrams. We describe how students interact with diagrams using both gestural and verbal modalities, and examine how such multimodal interactions with diagrams reveal their reasoning. We argue that when limited information is given in a diagram, students make use of gestural and verbal expressions to compensate for those limitations as they engage in making and proving conjectures. The constraints of a diagram, gestures and linguistic systems are semiotic resources that students may use to engage in geometrical reasoning.

Keywords Gesture · Geometrical reasoning · Diagrams · Systemic functional linguistics · Semiotics · Conjecture

Principles and Standards for School Mathematics (NCTM, 2000) establish the expectation that students develop reasoning skills and be able to formulate and prove conjectures. Similar expectations are contained in the Standards for Mathematical Practice that are part of the Common Core State Standards (CCSSI, 2010), which include SMP3 "Construct viable arguments and critique the reasoning of others." Since diagrams help to state geometric problems and retrieve related geometric concepts, geometric diagrams are key resources in students' geometrical reasoning. An investigation of students' interactions with diagrams may help us understand how students reason when making and proving conjectures about geometric objects.

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Research has discussed the gap between the physical properties of a diagram and the geometrical properties of a figure (Duval, 1995; Fischbein, 1993; Laborde, 2005; Mariotti, 1995). Duval (1995) has argued that diagrams demand different kinds of grasps or apprehensions. Students may grasp the figure *operationally*, for example when they modify the diagram mentally or physically. Students may grasp the figure *perceptually,* for example when they recognize the properties of the figure by its shape, size or sub-figures. Students may demonstrate *sequential* grasp, for example when constructing or describing a figure. And students may have a *discursive* grasp when they state the mathematical properties represented in the figure. Herbst (2004) has argued that students engage in different kinds of interactions with diagrams, which are arguably tied to the *instructional situations* that frame the mathematical work they are called to do (Herbst, 2006). Some interactions with diagrams involve proximal contact with diagrams, such as in constructing or measuring. Other interactions involve using the diagram to illustrate verbal statements that could be made without the diagram's existence. Yet other kinds of interaction use the visual inspection of a diagram as the source of verbal descriptions while they keep contact distal. These different kinds of interactions with diagrams may engage students in particular ways of thinking. We argue that some interactions may help advance students' reasoning and conjecturing. We are interested in how interactions with diagrams can support students in the work of figuring out whether a conjecture is reasonable.

The building of mathematical knowledge of geometric objects requires that one go beyond the making of empirical statements about figures. But since students' knowledge of mathematical objects is mediated by their representations, their building of knowledge is likely to require more than simple engagement in deductions from definitions and axioms. Herbst (2004) has argued that building geometric knowledge also requires students to make "*reasoned conjectures,*" statements about figures that arise through deduction from the possibilities of a geometric figure instantiated in a diagram. Herbst (2004) further proposes that, in order to engage in making reasoned conjectures, students may have to act on a diagram, creating representations for new geometric objects and anticipating relationships that may then obtain.

To effectively examine students' reasoning through interactions with diagrams, both gestural and verbal expressions need to be observed. Gestures and words create a "multimodal representation" (McNeill, 1998) of the mathematical objects. The importance of language as representation of mathematical understanding has been addressed profusely. This literature has contributed to establish the notion that language not only expresses thought but also generates it (O'Connor, 1998; Sfard, 2001). Gestures are another communication modality that can also be used to generate ideas rather than just express them (Núñez, 2004). McNeill (1992) notes that gestures are "parts of the discourse" that can be seen as a mode of communication, especially in description and explanation (Roth & Welzel, 2001). When students present their conjectures, the use of gestures helps them develop and communicate complex explanations without the need to use formal mathematical language; thus gestures may enable students to engage in arguments about geometric objects before all those objects have been conceptualized formally and represented in formal language. With both gestural and verbal expressions, students can communicate more of their reasoning and thinking to their peers and teachers.

This study identifies forms of interactions with diagrams that are involved in making conjectures in public, through multimodal representations. We explore how students use gesture and language to interact with diagrams and how such multimodal interactions with diagrams may allow them to engage in reasoned conjectures in the context of tasks where conjectures are called for. We ask the questions: (1) How do students draw from gestural and linguistic resources when they reason publicly in a task involving geometric diagrams? (2) How does such reasoning in the context of tasks that put a premium on making reasoned conjectures differ from the reasoning typically found when students do geometric proofs, particularly in regard to their use of gestural and linguistic resources?

1 Theoretical framework

1.1 Learning as participation

According to Lave and Wenger (1991), "learning is an integral and inseparable aspect of social practice" (p. 29). Full participation in the socio-cultural practices of a community contributes to successful learning. The learning-as-participation metaphor (Sfard, 1998) describes learning as participation in a community of practice, realized through communication that uses the semiotic resources within that setting (see also Brown, Collins, & Duguid, 1989).

From this situated learning point of view, the learning of mathematical practices requires becoming a participant of situations where those practices are done (Lave, 1988). Consider the case of the high school geometry class in the USA: To engage in geometrical thinking and learning, students have to engage in situations that involve interactions with diagrams. Such interactions may include working with diagrams verbally (i.e. describing), physically (i.e. drawing), or with gestures. Through the participation in the work with diagrams, students may make conjectures and justify them. Thus, the conjecturing work that we are interested in tracking involves public use of multiple communication modalities in participation in situated activities: How are these modalities used to advance new knowledge claims in public?

1.2 Interactions with diagrams

We use the words *diagram* and *figure* to mean different things. With *diagram* we refer to the sign used in communication, and with *figure* we refer to the mathematical object (or referent) that the sign purports to refer to. The interrelationship between diagrams and geometric figures has been addressed in the mathematics education literature on visual perception and geometrical reasoning. This literature shows, among other things, that the figure to which a diagram refers is problematic. For example, Fischbein (1993) speaks of conceptual and figural properties of a figure. When students are working on geometric problems with diagrams they can access the visualized (perceived) image of those geometric objects as well as the concept of those objects. The relationship between these two can be complicated: the capacity to perceive a figure (through its diagram) has been identified as an obstacle to understanding a figure conceptually (Duval, 1995).

Laborde (2005) proposes two kinds of properties of a figure—spatio-graphical and theoretical—that may be revealed when students are working on geometric problems with diagrams. Theoretical properties are those necessitated by the definition of the figure while spatio-graphical properties are those that are contingent to specific cases of the figure, as eventuated in choices made when constructing a diagram (e.g., orientation, specific angle values, specific side lengths, etc.). According to Laborde, geometry beginners' identification and interpretation of figures tend to be based on spatio-graphical properties represented in diagrams. For example, students may determine that an angle is 90° by actually measuring its representation in the diagram with a protractor. To promote students' understanding of figures to a theoretical level students need to engage in exploration and justification.

1.3 Students' interactions with diagrams

In his discussion of students' interactions with diagrams in geometry, Herbst (2004) proposes four modes of interactions between the actor, the diagram and the geometrical object (the figure). In *empirical* interactions the actor relies on physical features of a diagram to make a statement about a figure. Within this mode, components of a diagram are identified with components of a figure (e.g., a dot *is* a point, a stroke *is* a segment), as if there was no semiotic mediation or as if the diagram was the figure. On the contrary, in *representational* interactions the actor uses the theoretical properties of a figure to make a statement about a diagram (e.g., to say what the diagram is meant to show; this is often aided by a markup convention that includes hash marks, arcs, arrows, etc.). Within this mode of interaction, components of a diagram are seen as indices or symbols for geometrical objects (components of a figure). While those two modes of interaction describe polar opposite ways of treating the relationship between diagram and figure, Herbst (2004) also identifies other modes of interaction.

Herbst (2004) identifies a *descriptive* mode of interaction and proposes it as characteristic of the role that diagrams play in the situation of "doing proofs" (Herbst & Brach, 2006) in high school geometry classrooms in the United States. Within this mode of interaction, diagrams include two layers: on the one hand, they *represent* the givens of the problem and contain other elements that can represent properties justified through the proof; on the other hand, they rather accurately embody properties that could be read off the diagram, suggesting to the user what could be asserted about the figure. When they are "doing proofs," students use visual perception to hypothesize what could be true (thus interacting with the diagram in the empirical mode). But students are also expected to rely on diagrams only as symbols (using the markings to detect which elements of a diagram signify elements of the figure) at the time of justifying the statements they make (thus interacting in the representational mode part of the time). The descriptive mode alludes to this hybrid mode of interaction.

In order to have students make "reasoned conjectures" and construct mathematical knowledge, Herbst (2004) suggests that students have to interact with diagrams *generatively*. The work of making reasoned conjectures involves students in making hypotheses and predicting what could be true about a figure. *Generative* interactions with a diagram that might support such work include creating objects in the diagram that were not originally given and attributing status of geometric objects to them; they also include prescribing hypothetical (possible) properties of diagrams that rely on those objects. Mathematical arguments can be assisted by those generative actions (e.g., 'if I slide a vertex of a triangle on a line parallel to the opposite side, the height and the base will be constant, so the area will be the same'). An important distinction between the generative and the descriptive modes of interaction is that generative interactions put the agent in proximal contact with the diagram, altering it, unlike the descriptive mode in which contact is distal and limited to perception.

1.4 Reasoning with and through a semiotic system

While reasoning is often considered a cognitive intra-individual process only expressed in writing, the learning-as-participation framework described above encourages us to think of the talk where people make claims and offer arguments as more than just the communication of reasons. We take students' talk and action with semiotic systems as instrumental for the creation of conjectures and arguments. This is, of course, a theoretical assumption that we adopt based on prior scholarship that situates thinking in practice (e.g., Sfard, 2008; see also Lemke, 2009), and use to explore our data, not a claim that we seek to verify in this paper.

To track students' public reasoning in geometry class, it is important to consider the semiotic systems used in mathematical activity. Arzarello (2006) proposes the notion of *semiotic bundle* that collects different types of signs that are used in the events we examine. Further, it is important to examine the dynamic relationship among these different signs (Arzarello, 2006; Arzarello, Paola, Robutti, & Sabena, 2009; Maschietto & Bartolini Bussi, 2009; Radford, 2009), because students think "in and through" (Radford, 2009) these signs. Students' learning and thinking occur when they interact with these signs. In activities where students' reason and interact with diagrams, the semiotic bundle may involve diagrams, gestures, and written or oral language. In the following, we discuss gestures and oral language as components of the semiotic bundle or multimodal representation that students use to convey and generate meanings.

1.5 Gesture as a meaning making symbol system

Studies on gesture in science and mathematics learning suggest that gestures allow students to express as well as develop their imagistic thoughts and spatial reasoning (e.g. Cook & Goldin-Meadow, 2006; Nemirovsky & Tierney, 2001; Nemirovsky, et al., 1998; e.g. Roth & Lawless, 2002b). Gestures can be taken as part of students' explanation and communication, especially when their ideas or concepts are not yet well developed (Goldin-Meadow & Singer, 2003; Roth & Lawless, 2002a; Roth & Welzel, 2001). By using verbal and gestural expressions, students can show their visualization of diagrams more explicitly (Presmeg, 2001).

As Kendon (2004) suggests, the interpretation of gestural and verbal expressions should be contextualized. To investigate the role of gestures in geometric reasoning, it is important to examine how gestures are employed in interactions with diagrams. When students need to make conjectures about a figure, gestures are visible semiotic resources with which they can describe what they are considering in diagrams. Gestures can be used as tools to prescribe what could or should be true about a figure by depicting a diagram in a particular way. In this study we are particularly interested in how students utilize gestures to further their reasoning, and what gestures represent students' geometrical thinking.

1.6 The modality system of language as resource for making statements

(Semantic) modality[1] is a subsystem of language which is used to encode the various degrees of uncertainty that lie between the polarities of asserting and negating a statement (Halliday & Matthiessen, 2004; Martin & Rose, 2003). In systemic functional linguistics (SFL), modality is seen as a main resource for speakers to express attitudes toward propositions and proposals along various spectra that unfold between the two poles of positivity and negativity. While polar statements (e.g., "this is so" or "this is not so") convey definite attitudes toward the meaning being transacted, modality affords a way for speakers to relate to those meanings in less definite ways and so invite the audience to interact. For

[1] The word modality is often used to refer to each of the semiotic systems used in communication; thus the expression "multimodal representation" used above. Modality is also used in linguistics with a different meaning. Particularly, in systemic linguistics, modality is used to name one of the systems with which speakers construct relationships with their audience. This is the sense with which it is used in this section. Elsewhere in the paper we may use modality in the first, semiotic, sense. The context will help clarify what is the usage alluded to.

example, between the polar positive statement “this is an isosceles triangle” and the polar negative statement “this is not an isosceles triangle”, there may be different degrees of probability about the figure, from very possible to less possible—“this has to be an isosceles triangle”, “this would probably be an isosceles triangle”, “this might be an isosceles triangle”. The modality system of language provides semiotic resources to encode that range of possibilities.

According to Halliday and Matthiessen (2004), there are four types of modality: probability, usuality, obligation, and inclination: probability and usuality state the intermediate degrees of confidence on propositions (the extent to which the speaker asserts what is the case), whereas obligation and inclination show intermediate degrees of commitment to proposals (the extent to which the speaker argues what ought to be the case). Within each category, modality can be expressed in various degrees and these can be realized with lexical choices. For example, degrees of probability, from high to low, can be conveyed with words such as “certainly/probably/possibly/unlikely.” Likewise, usuality can be realized with words such as always/usually/sometimes/never. Students’ use of the modality system is a crucial observable in assessing the nature of their interaction with diagrams.

2 Data collection and method

2.1 Data selection

The data in this study come from a corpus of video recorded lessons in a project that studied mathematical work in high school geometry classrooms (including students who range in age from 13 to 15). A main activity of the project consisted of documenting customary teaching of high school geometry instruction. To do so the project had collected records of intact weekly lessons taught by four teachers (Emma Bello, Megan Keating, Cecilia Marton, and Lucille Vance)[2] in the same large comprehensive high school .We describe those lessons as *intact* (drawing the word from Greeno, 1998) because the lessons involved no special design or negotiation of lesson plan between teachers and researchers; rather the lessons represented customary teaching practice in those classes (see Herbst, et al., 2009). A second activity of the project was to try out short interventions that could be described as engaging students in problems in which they might come to use reasoning to develop new concepts. In these interventions the project’s goals had been to examine how teacher and students negotiated their work together. While this paper does not report on the complete scope of work of the project, it uses records from the two kinds of lessons described. In this paper we focus on students’ use of diagrams, gestures, and language in the reasoning observed in an intervention lesson by comparing it with how students used diagrams, gestures, and language when doing a proof in an intact lesson.

The intervention lesson had been developed by the second author to observe students’ use of hypothetical-deductive reasoning in interaction with diagrams. The intervention was based on a problem that provided a context for making conjectures about the properties of angles formed by parallel and intersecting lines, a topic that had not yet been taught in the class: Given a set of 6 lines the students were asked to specify how many angles they would have to measure in order to know all the angles formed by those lines (see Fig. 1 and Appendix). We hypothesized that when working on this problem, students might be engaged in different kind of reasoning than when they engage in doing proofs in intact lessons. The

[2] All the names of the teachers and students in this paper are pseudonyms.

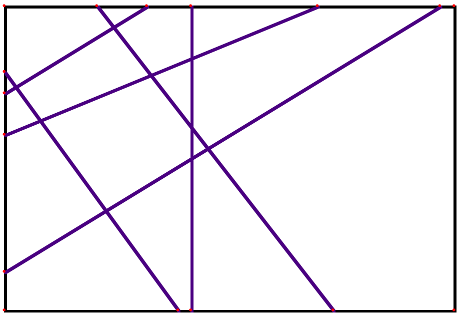


Fig. 1 The diagram provided in the intervention: parallel and intersecting lines

problem gave no information about the relationship between the given lines (e.g., while two pairs of lines appeared to be parallel, nothing was said about them being parallel or not). Thus students would have to engage in making hypotheses and deriving consequences from those. Emma, Lucille, and Megan implemented the intervention in five of their classes.

Inasmuch as we wanted to understand the role of interactions with diagrams and language in the making of reasoned conjectures, our analysis focuses on lessons where students worked on the task above, in which students had the opportunity to use deductive inference to find out the measures of angles. Our examination of a case of an intact lesson in which students engaged in public deductive work with diagrams serves as backdrop for comparison, providing an example of how students interact with diagrams, and use gesture and language when doing proofs. To understand how students' reasoning and justification of conjectures were communicated to the teacher and their peers, we attended to their gesture and language use in students' public interactions with diagrams. Students' public interactions with diagrams could appear in different guises: For example, students might talk at the board and interact with the diagrams drawn on the board or shown on an overhead projector. Or students might talk from their seats referring to the diagram presented on the board. Among the intact lessons gathered in this study, students interacted publicly with diagrams when they were called up to the board to write and talk about their solutions to homework problems. The intact lesson selected in this study represents a typical case in which students interact with diagrams as they present their homework publicly. Video records from five intervention lessons were inspected against the backdrop of that intact lesson.

3 Method

We examined the semiotic bundle (Arzarello, 2006) composed of gestures, diagrams, written and oral languages that students produced and used while they were reasoning with diagrams. To do that, we relied on transcripts and images produced from the video records (Zack & Graves, 2001). According to Arzarello (2006; see also Arzarello et al., 2009), to analyze the semiotic bundle one does synchronic analysis and diachronic analysis. We applied synchronic analysis to investigate the simultaneous relationships among signs–gestures, diagrams, and language–that occurred at the same time. For example, we looked into how gestures were made while students were verbalizing their thinking, and how gestures were made along with diagrams drawn virtually or on the board. We also implemented diachronic analysis to inspect how these signs changed over time. That is, we tracked how gestures evolved as students' reasoning developed.

To capture students' uses of gestures in their interactions with diagrams, we represent those gestures graphically. To facilitate the synchronic analysis of gestures, interaction with diagrams, and language use, the transcript lays out those graphic representations of diagrams and the gestures used to interact with them next to the students' speech. The integration of transcripts and graphic representations of gestures helps us show what diagrams students were referring to, what diagrams they were drawing virtually or on the board, and what specific marks they were adding to the present diagrams.

We further analyzed students' discourse by identifying the markers of modality that students used to express meanings about the diagrams. As noted above, the modality resources of language permit expressing degrees of uncertainty and hence allow space for negotiation. When students make conjectures and justify them, their use of modality in the discourse indicates that they are not stating facts purportedly known as true. Instead, they are proposing ideas for the consideration of others, and as they provide reasons to justify them, they use modality to express their confidence in them.

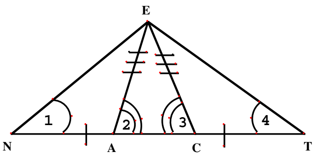
To handle the discourse in which students interacted with diagrams in public, we parsed the transcripts into clauses, and then identified tokens of modality in each clause. We looked at the following indicators: (1) finite modal operators, for example, *must*, *should*, and *might* show degrees of obligation or inclination from high to low; (2) modal adjuncts, for example *always, usually, sometimes, and never* show degrees of usuality, from high to low. In the context of interacting with diagrams, students might say how likely it is that the figure would be what they think it is or they might say what they think should be true about it. In particular, utterances that prescribe that figures *have to be* or *should be* point to an interaction with a diagram aimed at generating necessary geometrical properties of a figure (Herbst, 2004). In contrast with polar statements that describe what the figure is or is not like, students use of modality may show that students' reasoning concerns theoretical properties (Laborde, 2005) of the figures. We hypothesize that students use polar statements (marked in bold in transcript) when stating facts about the diagrams, and they use modal statements (underlined in transcript) when making and justifying conjectures about the figures represented.

4 Data analysis

In this section, we inspect students' interactions with diagrams in the intact and intervention lessons through their uses of gestures and the uses of modality in the language.

4.1 The intact lesson: pointing to facts with gesture and language

In the episode selected from an intact lesson students were called up to the board to present their homework, thus interacting with diagrams publicly. Along with three other classmates, Marcus was asked to present the solution to a homework problem on the board. Marcus presented his diagram and proof (see Fig. 2) without being asked to correct them, which tacitly implies the teacher approved of them.



Statements	*Reasons*
1. $\angle 1 \cong \angle 4$, $\overline{NA} \cong \overline{TC}$	1. *Given*
2. $\overline{EN} \cong \overline{ET}$	2. *2 ≅ base ∠'s → 2 ≅ opp. sides*
3. $\Delta ENA \cong \Delta ETC$	3. *SAS pst*
4. $\overline{EA} \cong \overline{EC}$	4. *CPCTC*
5. $\angle 2 \cong \angle 3$	5. *2 ≅ opp. sides → 2 ≅ base ∠'s*

Fig. 2 Marcus's board work. "pst" refers to postulate

To get a better sense of how Marcus interacted with the diagram, the diagram given in the textbook (Boyd, et al., 1998, p.226) is provided below. As Fig. 3 shows, the labels were given in the original diagram in the textbook, and what Marcus did was to add different marks to highlight the angles and segments that were mentioned in the proof. Specifically, he made the same number of hash marks to indicate the congruence of a pair of segments, and the same number of arcs to show the congruence of two angles.

In Marcus's short presentation of the solution to a homework problem, he drew the diagram and wrote the complete proof before the oral presentation. As Fig. 4 shows, Marcus talked through his proof on the board step by step. When certain parts (angles or segments) of the diagram were mentioned in the proof, he pointed to them; he also traced the triangles with his fingers. When Marcus was referring to the properties of the figure, he dominantly used present tense and polar positive statements (e.g., "is") about the diagram. For example, "$\overline{NA}$ is congruent to $\overline{TC}$", or "$\overline{EN}$ is equal to $\overline{ET}$" or "because two congruent base angles give you two congruent opposite sides". Some of the statements were given in the statement of the problem, e.g. $\overline{NA}$ is congruent to $\overline{TC}$; other statements were inferred from the given, e.g. $\overline{EN}$ is equal to $\overline{ET}$. All those statements about figures were polar, thus providing no invitation for negotiation (Martin & Rose, 2003). Marcus also expressed modal meanings when he was generating new statements from previous ones (e. g., "you can say, angle 2 and angle 3 are congruent"). We take the modal "can" to show some degree of uncertainty as to the relevance for the proof of making the statement that follows, the statement about the figure is in the positive polar form ("angle 2 and angle 3 are congruent"). Therefore, when Marcus interacted with the diagram, he did not have to negotiate the information in the diagram. Instead, he stated the facts about the diagrams from the given. When he was inferring information from the given or postulates, he for the most part stated them as facts.

Given: $\angle 1 \cong \angle 4$

$\overline{NA} \cong \overline{TC}$

Prove: $\angle 3 \cong \angle 2$

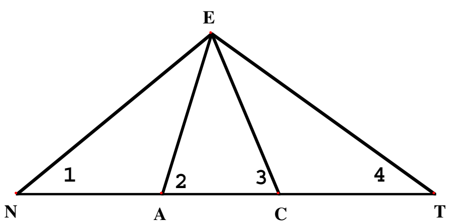
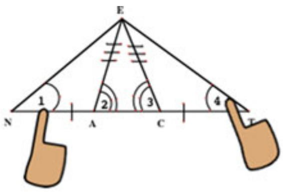


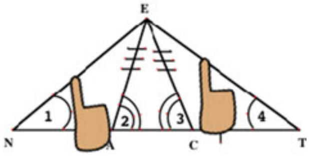
Fig. 3 The book problem Marcus was working from (drawn according to Boyd, et al., 1998, p.226)

Alright! The given for 38[#] was, uh,
angle 1 **is** congruent to angle 4,
which **is** uh, two base angles (points to the angle 1 and angle 4 on the board respectively)[##] of the triangle N-E-T,



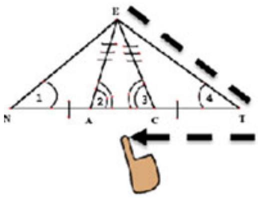
a. Marcus points to angle 1 then to angle 4

and then $\overline{NA}$ **is** congruent to $\overline{TC}$,
so from there, I put $\overline{EN}$ **is** equal to $\overline{ET}$,
because two congruent base angles **give** you two congruent opposite sides,
which <u>would be</u> these two (points to segment $\overline{ET}$ and $\overline{EN}$ respectively),



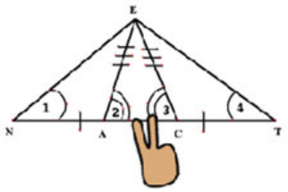
b. Marcus points to segment $\overline{ET}$ and $\overline{EN}$

and then from there,
you <u>can</u> say, triangle E-N-A **is** congruent to triangle E-T-C,
because of the side-angle-side postulate,
right there (traces angle *ETC* from segment $\overline{ET}$, to vertex T and then to segment $\overline{TC}$),



c. Marcus traces segment $\overline{ET}$, angle *ETC* and points to vertex *C*

and then from there,
you <u>can</u> say,
$\overline{EA}$ **is** equal to $\overline{EC}$,
because of CPCTC,
and then, uh,
you <u>can</u> say, these two angles (points to angle 2 and angle 3 simultaneously), angle 2 and angle 3 **are** congruent,
because two congruent opposite sides **give** you congruent base angles.



d. Marcus points to angle 2 and angle 3 simultaneously

Fig. 4 Marcus's presentation of his proof. [#] Indicating the number of the homework problem in the textbook. [##]The gestural actions are described in parentheses

4.2 The intervention lesson

In the intervention lessons we saw a variety of uses of the diagram, gesture, and language. These are described in the following subsections.

4.3 The intervention lesson: extending lines outside the given frame

In the intervention lesson, in order to get more measurements of angles without actually measuring them, students had to apply known properties, such as the angle sum theorem of a triangle. Students used gestures that depicted a virtual diagram onto the given diagram in order to propose some hypothetical situations outside the given frame.

Hypothesizing a triangle In Megan's second period, Audrey proposed that two of the given lines would intersect outside the given frame, thus forming a triangle. She first identified a possible triangle by pointing to two vertices in that hypothesized triangle (see Fig. 5a and b) and pointing to a spot outside the given frame (see Fig. 5c). This virtual spot indicates the third vertex formed by the continuation of two line segments. Based on this virtual diagram, Audrey could get the measurement of the third angle in the virtual triangle by applying the angle sum theorem. Figure 5 shows how she explained it.

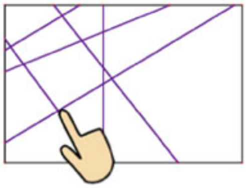
Although the intersecting point was not shown on the given frame, and whether the two lines would intersect outside the given frame was not stated in the given activity, Audrey and her group mates made the assumption that a virtual triangle existed. To justify this assumption, she virtually gestured a triangle: She first pointed out that each of the known angles "is eighty." The present tense suggests that she obtained the measurements of the two angles by measuring with protractor, so the angle "is" eighty in an empirical sense (she later corrected herself, saying that one of the angle measurements was sixty degrees).

Based on the two measurements, the third angle of the virtual triangle "has to" equal forty degrees, because the angle sum of a triangle "has to equal to one eighty." Therefore, the third angle is highly "obliged" (Halliday & Matthiessen, 2004) to be 40°. She gestured to show an intersecting point, formed by the two extended line segments, as the third vertex of a triangle. Audrey switched the verb from present tense "is" to "has to", indicating that she was no longer measuring but inferring. Her gestures first pointed to existing objects (two intersections in Fig. 5a and b), then created a virtual object (by pointing to the place where the two lines would meet; Fig. 5c). Then she used those three objects to create a virtual triangle, gesturing the outlines of the triangle (Fig. 5d), with which she could warrant considering a property relevant (angle sum theorem), which she used to make an inference (the third angle has to have a certain measure) about the virtual angle at the virtual vertex of the virtual triangle.

Conjecturing two lines parallel Students used their gestures to show that two lines were parallel, and to virtually indicate that the two parallel lines would extend outside the screen. In Megan Keating's third period class, a group of students came up with the conjecture that those two lines were parallel and further justified the conjecture.

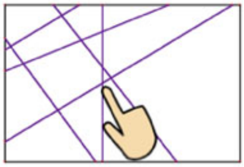
In this episode, Collin and Anthony justified their conjecture through an argument that resembles a proof by contradiction. Collin first stated that the two lines should be parallel by moving his open palm along outside the screen (see Fig. 6a). Then, to prove that these two lines are parallel, he assumed that the lines would intersect at a certain point if the statement were false. He proposed the possibility that the extended two lines "*would not* be parallel." Based on this assumption, these two lines would intersect at some point and then "you'*d* get a triangle." A hypothetical intersecting point was positioned outside the screen (see Fig. 6b). Instead of gesturing with an open palm, Collin narrowed the space between his thumb and index finger. This variation of gestures indicates his differentiation of the notions of parallelism and incidence. After making the assumption that the two lines intersected, Collin attempted to get the measurement of the third angle in the virtual triangle formed by the two lines. However, with the angle sum theorem and the known measurements of two other angles in the virtual triangle, it is impossible to have a third angle anywhere. Anthony proclaimed "there *can't be* another point down here", indicating that the two lines cannot be intersecting anywhere, and thus should be parallel. His gesture, by pointing a spot (Fig. 6d)

If you…alright
You know that this (points to the up left vertex of the triangle) is eighty,



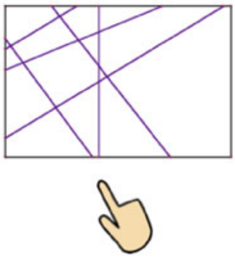
a. Audrey points to the up left vertex of the triangle

and this (points to the up right vertex of the triangle) is eighty,



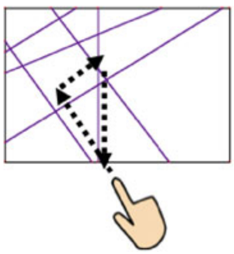
b. Audrey points to the up right vertex of the triangle

So this (points to the virtual third vertex, outside the screen) has to equal a hundred…uh I mean one eighty



c. Audrey points to a spot outside the screen

[Megan: the triangle] the triangle (traces loosely around the triangle, and stops at the spot outside the screen) has to equal one eighty. So eighty (points to the top left angle in the triangle) plus eighty (points to the top right angle in the triangle) and then you have to add forty in here (writes “40” at the virtual third angle)

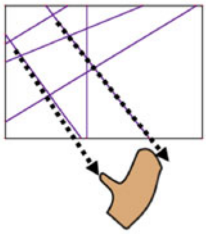


Multimodal interactions with diagrams 42

d. Audrey traces loosely around the triangle, and stops at the spot outside the screen (Dotted lines added indicating the tracing path; arrows indicate the directions of tracing movements)

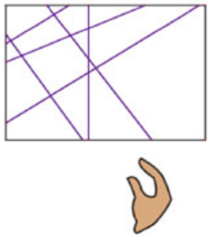
Fig. 5 Audrey proposes a virtual triangle

Collin: We found out that since
If you extended these two lines (use his thumb and right index finger to trace, and virtually extends the two lines outside the screen)



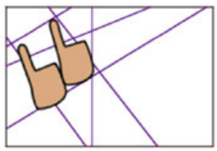
a. Collin traces the two lines downward and stops at a spot outside the screen

You'd eventually,
If they would not be parallel,
you get a triangle (hand rests on a spot that might be the intersection of the two lines)



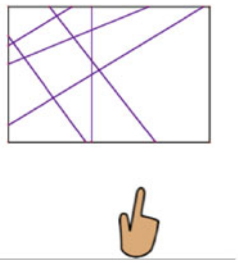
b. Collin narrows the space between his thumb and index finger, and rests the wrist at a spot outside the screen

and then, to find the measure of that final angle,
You'd add those two together (points to the two interior angles formed by the two lines and an upper transversal),
and subtract that from one eighty,
and one ten plus seventy **is** equal to one eighty,
so…



c. Collin respectively points to the left and right interior angles formed by the two lines and an upper transversal

Yuri: I was just going to say that if the lines are parallel
Anthony: There can't be another point down here somewhere (points to a spot far from the screen)



d. Anthony points to a spot far from the diagram

Fig. 6 Collin and Anthony prove two lines being parallel farther than the one that Collin had, insisted on the impossibility of an intersecting point by two parallel lines.

Collin's gestures represented a hypothetical object (a possible intersecting point of the two lines). The gestures enabled the students to consider a property of the possible figure (the angle sum theorem in a hypothetical triangle). The measurements of two of the angles were obtained empirically, but then the students shifted from reporting to inferring, concluding with strong conviction that it was impossible for the lines to intersect one another.

4.4 The intervention lesson: reasoning about parallelism

Gestures play a dynamic role in students' reasoning about two parallel lines in the following segment. In Megan's third period class, Yuri elaborated and justified a conjecture about two lines being parallel. He used gestures to sketch a virtual diagram, and to explain that corresponding angles would be congruent if two lines cut by a transversal were parallel.

To justify the conjecture—if two parallel lines are cut by a transversal, corresponding angles are congruent—Yuri used various gestures to construct a specific diagram. First he placed his palm horizontally to virtually introduce a transversal (see Fig. 7a). Secondly, he "drew" a 90-degree angle (see Fig. 7b), indicating a vertical line perpendicular to the previous line. Adding another line verbally ("then you have two lines", 65), Yuri created a pair of lines intersecting with a transversal as an example to illustrate the conjecture. Specifically, this example consisted of two parallel lines perpendicular to the transversal (see Fig. 8a). To show the parallelism between the two lines Yuri also used his right thumb and index finger to show the constant distance between the two parallel lines (see Fig. 7c and d), no matter at what angle they intersect with the transversal. Figure 8b shows Yuri's gesture, representing parallelism, was inscribed on the virtual diagram.

However, Yuri was speaking from his seat and thus his example was not visually available to his peers and the teacher. The teacher then asked him to draw his virtual diagram on the board. He drew two pairs of parallel lines, and marked two pairs of corresponding angles on the board. He also used a variety of gestures to show that the parallel lines would be slanted in certain way that would make the corresponding angles congruent.

Yuri first drew two sets of parallel lines intersecting with each other (see Fig. 9a, b and c), stating that each set of parallel lines "*are* parallel" (line 75, 77). Since he had the visual evidence (the drawn diagram and gestures) to support his conjecture, he further expressed a high degree of certainty about the claim: with the diagram on the board (see Fig. 9b), he first put four arcs on the intersecting angles formed by two pairs of lines (see Fig. 9c), and proposed that all the interacting angles "*have to* be equal" (78).

He then further explained in more detail: He pointed to the horizontal pair of lines and claimed, "both of these lines *have to* be slanted at same angle" (79), because they "*can* either be horizontal or slanted" (80,82). He positioned his palm to show the orientation of each individual set of parallel lines: First, his palm was placed horizontally representing the horizontal pair of parallel lines (see Fig. 9e), then he swung the palm upward to simulate the motion of parallel lines (see Fig. 9f). Similarly, his palm latter represented the vertical pair of lines (see Fig. 9h), and the swing of the palm was downward (see Fig. 9i).

This shows that Yuri conceived a pair of lines as a unit that "have to" be oriented in the same direction due to their parallelism. The same conception of parallel lines was

59 Megan: yeah,
60 Yuri: then one minute you can figure out is you see that,
61 if you have one line (palm placed horizontally),

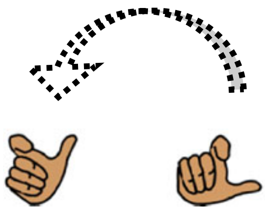
a. Yuri shows his right lower palm horizontally to virtually create a horizontal line (dotted line added indicating its virtual property)

62 I mean if you have one angle that is,
63 like, let's say 90 degrees (sketches virtually a 90-degree angle),

b. Yuri virtually creates a 90-degree angle

64 Megan: yeah,
65 Yuri: then you have two lines,
66 and if both of the lines are parallel,
67 then you can tell that your…You have to have exact same angle,
68 because they have to intersect [at the same point]
69 Yakim: [Yeah, because the] transversal is like...
70 Yuri: because both lines (uses his thumb and index finger to show the constant distance relationship between two lines), [both parallel lines,] (swings his right wrist)

c. Yuri uses his thumb and index finger to show the constant distance relationship between two lines that are parallel



d. Yuri swings his wrist to show that the distance between two parallel lines remains the same even if they both are slanted at different angle (dotted arrow refers to the direction of gesture movement)

71 Megan: [go draw] what you are talking about, draw it up there!

Fig. 7 Yuri virtually draws a diagram with gestures at the seat

also applied when he was talking about the second pair of parallel lines (84–87) that the set of parallel lines “have to” be slanted at the same angle.

With his palms swinging to represent the slanted orientations of sets of parallel lines, he claimed that when the two pairs of lines intersected, “you’re always going to get the same angle” (88) (see Fig. 9k). This word choice (“always”) indicates that there is high degree of usuality in the situation that two pairs of intersecting parallel lines form at identical angles.

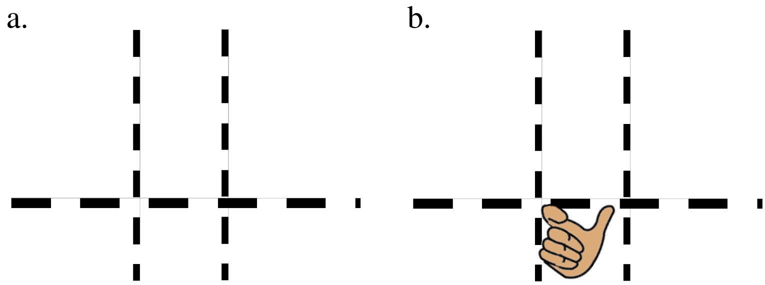


Fig. 8 **a** This diagram is sketched based on Yuri's description with gestures. (To signify its virtual character, the diagram is drawn with dotted lines). **b** Yuri's gestures make his virtual diagram

The conjecture about the corresponding angles congruent in parallel lines cut by a transversal is made and justified with high certainty.

Yuri used his arm to represent the mathematical object (a set of parallel lines). In addition, he considered the property of that object (parallel lines have the same direction) and swung his arm to represent and demonstrate this property. He then made the conjecture that when two sets of parallel lines intersect, they form identical angles. This shows that gestures enable students in conjecturing possible properties of diagrams and reasoning through gesturing.

As the preceding description shows, in the intervention lessons students were able to state conjectures or provide justifications with different degrees of commitment. Particularly, the degrees of commitment increased when students made justifications about their claims with gestures. They made more frequent use of the modality system in order to make assumptions and provide justifications for possible facts.

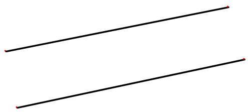
5 Discussion

In this study we investigated the semiotic bundles (Arzarello, 2006) in students' geometrical reasoning, through their simultaneous use of diagrams, gestures, and language. We also examined how students utilized these semiotic resources in developing their geometrical reasoning. In what follows we highlight the features of students' interactions with diagrams in a task that encouraged them to make reasoned conjectures and contrast those features with those of students interaction with diagrams in an intact lesson.

First, we found that the nature of given diagrams plays an important role in students' interaction with diagrams. In the intact lesson, the labels for vertices and angles in the given diagram implicitly point at what elements are likely to be needed when producing the proof (Herbst, 2004). In contrast, the diagram given in the intervention lesson did not include labels and consequently students had to take responsibility for determining which elements might be involved in a conjecture, and how the selected elements might be referred to in the conjecture.

Second, gestures and linguistic resources were employed differently in the intact and intervention lessons. In the intact lesson, the student only used what McNeill (1992) calls deictic gestures, to point to objects that were visually available on the diagram and draw the audience's attention. At the same time, we observed a dominant use of present tense in statements of facts about the diagram; polar statements

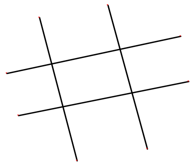
75 Yuri: so we meant these **are** parallel (draws two lines with a distance in between)



a. Yuri draws two lines that have certain distance in between on the board

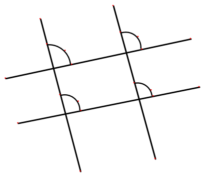
76 Megan: those are parallel, yeah.

77 Yuri: and these **are** parallel (draws another pair of lines that intersect the previous pair)



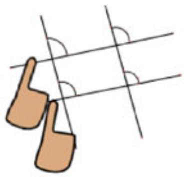
b. Yuri draws another two lines intersecting the previous pair

78 then and we can tell that all these (draws fours marks on the four intersecting angles formed by the two pairs of lines previously drawn) have to be equal,



c. Yuri marks four angles formed by the two pairs of lines

79 because these, we know that both of these lines (points to the first pair of lines) have to be slanted at same angle, right?



d. Yuri respectively points to the lower and upper lines of the first pair

80 The entire line can either be, like, completely horizontal (places his right palm horizontally) or....

e. Yuri shows his right palm horizontally

Fig. 9 Yuri gestures with the diagram to illustrate parallel lines make the corresponding angles congruent

expressed no ambiguity or uncertainty that might risk a conjecture or invite argument from others.

In contrast, in the intervention lessons, in addition to gesturing deictically, students utilized gestures to express ideas that involved imaginary objects, the elements of figures that were not visually available, uses of gesture that could be classified as

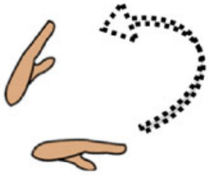
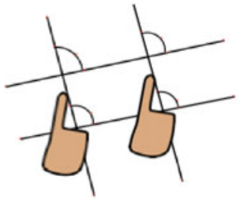
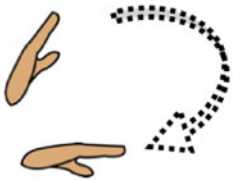
81	Megan:	Ok,	
82	Yuri:	slanted. (swings his right palm)	 f. Yuri swings his right palm (dotted arrow refers to the direction of gesture movement)
83	Megan:	Ok, I can go with that	
84	Yuri:	and these lines (points to the two lines of the second pair),	 g. Yuri respectively points to the two lines of the second pair
85		and these lines <u>have to</u> be either, completely vertical (holds his right palm vertically) or slanted (swings his right palm),	 h. Yuri holds his right palm vertically  i. Yuri swings his right palm(dotted arrow refers to the direction of gesture movement)

Fig. 9 (continued)

iconic in McNeill's (1992) system. These gestures represented virtual mathematical objects (e.g., points of intersection) or mathematical relationships (e.g., parallelism of lines). Students also gestured to animate the diagrams. Gestures were extensively employed to represent the objects that had not been represented in the diagrams, and so to represent the students' conception of figures. Gestures further invited students to consider possible properties and make inferences. With the aid of gestures,

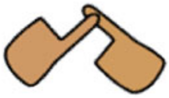
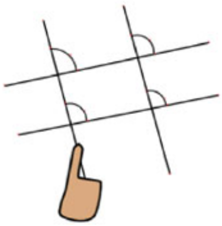
86		but they, they both have to be the same... (uses his thumb and index finger to show the constant distance relationship between two lines),	 j. Yuri uses his thumb and index finger to show the constant distance relationship between two lines
87		they both have to be slanted on the same angle [Megan: Okay]	
88		Therefore, when they intersect (crosses his two index fingers), you're always going to get the same angle	 k. Yuri crosses both his index fingers to show the intersecting relationship of two intersecting lines
89	Yakim:	Yeah, the opposite angles, but not…	
90	Yuri:	No, these…. (points to one of the intersecting angles he marks earlier)	 l. Yuri points to one of the intersecting angles he had marked earlier
91	Megan:	Those are corresponding	

Fig. 9 (continued)

the uses of modality allowed students to make and justify conjectures with high degree of certainty thus contributing to a collective argument. Modality was also used to craft a proof by contradiction: Along with gestures that pointed to a hypothetical mathematical object, modality was useful to state as a possibility something that was eventually shown to be not true. Hence, gestures played a crucial role in engaging students in reasoned conjecturing.

The different uses of modality expressions and gestures in the intact and intervention lessons can be related to the different characteristics in the given diagrams and more generally to differences in the task students were engaged in. In a usual proof task, diagrams include all the objects needed for students to work with and students are expected to read objects off the diagram, without any further alterations of diagrams (Herbst et al., 2009). The diagram given in the intervention lesson was to some extent incomplete; the task required students to make hypotheses and to conceive possible objects and relationships. Students used gesture and modality to support that activity. As Fischbein (1993) suggests, promoting conflicts between figural and conceptual aspects of diagrams could help students develop "figural concepts;" in the intervention lesson the constraints of the diagram invited students to create signs (e.g. virtual points made with joining fingers) to point to possible objects (e.g., a putative intersection between two lines).

The intervention lesson engaged students in interactions with diagrams that are not often seen in customary geometry class, an interaction that we would describe as *generative* (Herbst, 2004). Generative interactions involve creating new signs to complement a diagram, so that students can "think with" the diagram (Herbst, 2004) and predict what the figure "should be." This kind of interactions with diagrams was observed in the intervention lessons, which seems to suggest that the task was useful to promote students' development of reasoned conjectures. Our work thus shows how gestures and uses of modality in language may be involved in making that work possible, particularly helping students relate to diagrams that can't just be described. Gestures depict certain "hypothetical phenomena" (Pozzer-Ardenghi & Roth, 2005) that could possibly be true in the figures represented by the diagram. Gestures complement the limitations of static diagrams and provide dynamic elements to support students' reasoning. Gestures' role goes beyond the kind that complements speech, but serves as a mediation tool that encourages mathematical reasoning. Through modality expressions, students stated plausible claims with different degrees of certainty toward the diagrams. Therefore, modality can be seen as a tool to express the reasonableness of the statements and to involve others in the process of argument about these reasonable statements.

6 Conclusion

This study identifies gestural and linguistic resources that support forms of interactions with diagrams that are involved in making reasoned conjectures. Although reform documents in mathematics education stress the importance of conjecturing and proving, it has been argued that students in customary geometry classes usually have limited opportunities to make reasoned conjectures about figures (Herbst, 2004, 2006). This study shows that the nature of diagrams could play a role in students' geometrical reasoning and consequently making reasoned conjectures. The constraints of diagrams may enable students to use particular gestures and verbal expressions that, rather than reporting on known facts, permit students to make hypothetical claims about diagrams. It is to be expected that if a diagram did not include signs to represent all the objects that could be talked about, students' allusions to those objects, were they to occur, might be more conjectural than factual. Our report, however, shows that gestures as well as modality expressions can be mediation tools available to compensate the semiotic limitations of diagrams (e.g., their lack of elements drawn or labeled), and could be especially important in enabling students to engage in such conjecturing.

The analysis of gestures and modality expressions highlight the importance of forms of semiotic resources available in mathematics activities, especially in students' reasoning with diagrams. Different uses of gestures in the context of geometrical thinking identified in this study help examine students' conceptions of geometrical properties. More research on uses of gesture in geometric classrooms could contribute to understanding students' mathematical reasoning. Likewise the study suggests that paying attention to the uses of linguistic modality can be key in assessing whether a task engages students in making reasoned conjectures.

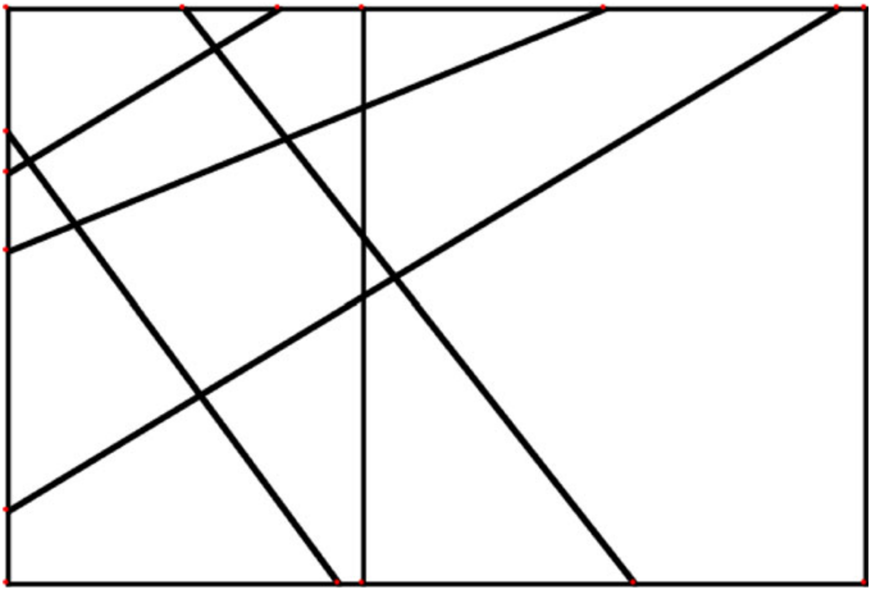
Appendix: activity worksheet of the intervention lesson

Worksheet on angles formed by intersecting lines

Name:

Date:

Period:



There are six lines on the paper and some of their intersections are not visible.

1. Would it be possible for somebody to determine the measures of all the angles formed by those lines, considering that not all angles can be measured? Explain.
2. What is the total number of different angle measures that one would need to determine? Explain.
3. How many of those angle measures would be impossible to find unless one could extend the lines beyond the screen limits? Explain.
4. What is the minimum number of angles that one would have to measure before being able to say "I know all the angle measures"? Explain.

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