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Examining the discourse on the limit concept in a beginning-level calculus classroom

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Abstract Existing research on limits documents many difficulties students encounter when learning about the concept. There is also some research on teaching of limits but it is not yet as extensive as the research on student learning about limits. This study explores the discourse on limits in a beginning-level undergraduate calculus classroom by focusing on one instructor's and his students' discourses through a communicational approach to cognition. The findings indicate that some of the limit-related contexts in which students struggled coincided with those in which the instructor shifted his elements of discourse on limits. The instructor did not attend to the shifts in his discourse, making them implicit for the students. The study highlights that the discrepancies among participants' discourses signal communicational breakages and suggests that future studies should examine whether teachers' explicit attention to the elements of their discourse can enhance communication in the classrooms.

Keywords Limits · Commognitive framework · Teacher discourse · Student discourse

1 Introduction

Limit is a fundamental concept of calculus that also presents major challenges for students. There is a rich body of research indicating that students' informal realizations of limit are mainly based on dynamic motion, which can interfere with the representational and formal realizations of the concept. (Bagni, 2005; Tall, 1980; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991) Student difficulties with limits can also be based on difficulties related to the underlying concepts of limits such as real numbers, functions, and infinitely small and large quantities (Parameswaran, 2007; Sierpińska, 1987). The dominance of dynamic and procedural aspects of limits in calculus textbooks and teaching, and students' attitudes towards mathematics are considered among the factors that can contribute

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to student difficulties about limits (Bezuidenhout, 2001; Parameswaran, 2007; Williams, 1991). At this point, it is important to note that existing research on student learning of limits is mainly based on a cognitivist framework, whereas this work uses a communicational approach to cognition to explore student learning of the concept.

According to Sfard (2001), the cognitivist framework is based on the metaphor *learning as acquisition*, which considers learning "as storing information in the form of mental representations" (p.20). It considers understanding as relating new knowledge to prior knowledge by refining existing mental representations. By doing so, it highlights the individual nature of learning, viewing it as the acquisition of necessary mental schemes. This study uses a discursive framework developed by Sfard (2008), which is based on the metaphor *learning as participation* and views learning as change in one's discourse through becoming a participant in a discourse community. Basing learning processes on social foundations, this framework considers discourse as its central unit of analysis in which "the language of *mental schemes*, *misconceptions*, and *cognitive conflict* seems to be giving way to a discourse on *activities*, *patterns of interaction,* and *communicational failures*" (Sfard, Forman, & Kieran, 2001, p. 1, italics in original).

Besides the aforementioned studies, there also exists some research focusing on teaching of limits (e.g., Bergsten, 2007; Nardi, 2007) but such research is not yet as extensive as the research on student learning about limits. This study examines one instructor's and his students' discourses on limits in a beginning-level calculus classroom. The goal of the study is to use the discursive framework to provide further insights about the issues regarding teaching and learning of limits by identifying the instructor's and students' discursive patterns and comparing those to highlight communicational links and failures in the classroom. The study addresses the following questions: (a) what are the characteristics of the instructor's discourse on limits in a beginning undergraduate-level calculus classroom?, and (b) how do the elements of the instructor's discourse compare and contrast with his students' discourses on limits?

In what follows, the basic tenets and terminology of the framework used in the study are introduced. Next, the instructor's discourse on limits is described. This is followed by the examination of students' discourses, which also includes the comparison of their discourses with the instructor's discourse on limit.

2 Theoretical framework

The study is based on Sfard's (2008) *commognitive* framework, which highlights the close relationship between cognition and communication. Sfard formulates thinking as an individualized form of communication and considers cognitive processes and interpersonal communication as facets of the same phenomenon. Given these assumptions, the term *commognitive* combines the terms *cognitive* and *communicational*. From this perspective, developmental transformations are "the result of two complementary processes, that of *individualization of the collective* and that of *communalization of the individual*" (Sfard, 2008, p. 80, italics in original). Although the development of mathematical discourse over history can differ from students' individualization, the collective discourse includes the tools students are expected to use as they develop their mathematical discourse. One of the goals of school learning is to change elements in students' discourses so that they can participate in the historically established activity of mathematics.

For this study, the term discourse refers to the "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Mathematics is then characterized as a specific type of discourse that is distinguishable by its *word use*, *visual mediators*, *routines*, and *narratives* (Sfard, 2008). *Word use* refers to the ways in which participants use words in their mathematical discourse. *Visual mediators* refer to the visible objects created and operated upon to enhance mathematical communication. *Routines* are the collection of meta-level rules characterizing the repetitive patterns in participants' discourse. Finally, *narratives* refer to the set of utterances describing mathematical objects and their relationships that are subject to endorsement or rejection. Axioms, definitions, and theorems are among the endorsed narratives of mathematics.

Sfard (2008) differentiates between two types of discourse: *colloquial discourses* are non-specialized, everyday discourses whereas *literate discourses* (e.g., mathematical discourse) are "mediated mainly by symbolic artifacts created specifically for the sake of communication" (p. 299). An important feature of word use in mathematical discourse is objectification, which occurs through reification—replacing the talk about processes and actions with states and objects—, and alienation—"using discursive forms that present phenomena in an impersonal way" (Sfard, 2008, p. 295). Through objectification, we identify the commonalities among different processes within a discourse and unify many lower-level phenomena under one name. Objectification increases the effectiveness of mathematical communication and is also a means of formalization, especially through the use of symbolic artifacts. However, it hides the discursive layers that constitute mathematical objects.

The study examines participants' word use on limits in terms of the *degree of objectification*. In her earlier work, Sfard (1991; 1992) elaborated on reification as the transition from operational to structural modes of thinking. Whereas the structural mode of thinking treats "mathematical notions as if they referred to object-like entities" (Sfard, 1992, p. 60), the operational mode of thinking "speaks about *processes, algorithms and actions* rather than about objects" (Sfard, 1991, p. 4, italics in original). In this study, degree of objectification is conceptualized in terms of three descriptive categories to explore the participants' word use. Those categories are *colloquial* (talking about mathematical concepts in the everyday sense); *operational* (talking about mathematical concepts as processes or actions); and *objectified* (talking about mathematical concepts as objects or object-like entities) word use.[1]

Another focus of the study is metarules and the roles they play in mathematical discourse. When learning is realized as becoming a full participant in the discourse of "communities of practice" (Lave & Wenger, 1991), the norms, values, goals, and actions of a community need to be considered when examining the discourse of the community. In the case of mathematical discourse, norms, values, goals, and repetitive actions of a community are considered meta-level rules because they regulate the practices as participants generate and substantiate mathematical meanings although they may not be directly visible in the final product of the community, e.g., the "form" of mathematics.

The term metarule is quite broad since it can signify many different patterns in the activity of participants (Sfard, 2001). For example, it is possible to talk about the metarules regulating participation (e.g., raising hands before speaking, working in groups), or metarules characterizing participants' intentions (e.g., genuinely engaging in mathematical activity versus acting to please the teacher), or the metarules regulating the object-level rules of

[1] Although compatible with the overall assumptions of the framework, this categorization of word use differs from Sfard's (2008), where she presents a four-stage model of the development of word use and examines students' word use over a period of time. In this study, students only agreed to be interviewed once so the focus is on the *description* rather than *development* of their word use. In other words, the study does not make any claims about the development of elements in students' discourses but describes them in the context of the interview sessions.

mathematics (e.g., using the metaphor of motion to compute limits, using graphs to realize functions). The combination of all such metarules constitutes the routines in the participants' discourse. In this study, the focus is on the metarules regulating the object-level rules of mathematics.

Metarules are often tacit since they signify the "regularities observed in those aspects of communicational activities that are not directly related to the particular content of the exchange" (Sfard, 2001, p. 30). Note that the use of metaphors—"the action of 'transplanting' words from one discourse to another" (Sfard, 2008, p. 39)—is a metarule of mathematical discourse. Therefore, the exploration of metaphors behind the different layers of a mathematical discourse is part of the exploration of metarules.

Changes in the metarules of a discourse eventually require changes in the routines as well as word use, visual mediators, and the endorsed narratives of the discourse in a community of practice. For example, changes in mathematicians' values regarding mathematical rigor led to the movement called arithmetization of analysis, leading to a reformulation of "the logical foundation of the number system" (Kline, 1972, p. 989) in a purely arithmetic manner. This then resulted in changes in the metarules regulating different formulations of the same concept. For example, Weierstrass' formal definition of limit—an attempt to formulate limit in more precise terms—replaced the metaphor of dynamic motion in Cauchy's definition with "a static formulation involving only real numbers, with no appeal to motion or geometry" using the metaphor of discreteness (Edwards, 1979, p. 333). Such modifications in the metarules supported the modification of the visual mediators (use of symbolic rather than graphical representations); word use (using words signifying proximity rather than motion); and endorsed narratives (considering limit as a static number rather than a process). As the example indicates, the four elements of mathematical discourse are quite intertwined with each other and the full participation in the discourse on limits requires the orchestration of all these elements.

3 Methodology

This is a case study investigating the discourse on limits in a beginning-level undergraduate calculus classroom at a large mid-western university in the United States. Data sources for the study consisted of eight video-taped classroom observations in which the instructor discussed limits and continuity; a diagnostic survey given to all students at the end of their lessons on limits; four audio-taped task-based interviews on limits; and students' written work. The diagnostic survey, which was taken from Williams (2001), was used to gain information about students' realizations of limit at the end of their instruction. The diagnostic survey was also used to select the students for the interviews with the purpose of having varied responses in terms of students' realizations of limit. The interview sessions were designed to further investigate the characteristics of students' discourses on limits and consisted of six questions. Two of the questions were taken directly from research exploring student learning of limits; four were designed by the researcher taking into account the student difficulties about limits documented in the literature and the instructor's discourse in the classroom. Each student was interviewed individually and the sessions lasted between 55 and 75 min. The class observations and the interviews were transcribed with respect to the participants' utterances and actions.

The instructor's and his students' discourses were analyzed with respect to word use, visual mediators, routines, and endorsed narratives. Word use on limits was classified as colloquial if participants talked about limits in everyday sense (e.g., "what is the speed limit?"); operational if they talked about limits as a process based on dynamic motion (e.g., "the function values get closer and closer to five as x values approach one"); and objectified if they talked about limit as a number, an end-state, or a distinct mathematical entity (e.g., "the limit is five," "this limit does not exist," "what is the limit?").

For the analysis of visual mediators, an inventory of all the visual mediators the participants used was created from the transcripts, which also included the snapshots of everything they wrote and drew. Those mediators were then classified into three categories: written words; graphs; and symbolic representation.

For the investigation of routines, particular attention was paid to the participants' actions. Here, only the metarules regulating the object-level rules that were most relevant to the analyses of word use, visual mediators, and endorsed narratives were reported as routines. For example, some of the routines in the instructor's discourse on limits consisted of repeated mathematical procedures (algebra-based routines) he utilized in the classroom. Some other routines, such as graphing, were repeated actions emerging also from the analysis of participants' word use and visual mediators. Once the routines were identified, they were analyzed in terms of *when* and *how* they were used (Sfard, 2008).

Object-level narratives about limit refer to the definitions, theorems, and facts related to the concept. Although present in the participants' discourses (to the extent that they talked about their realizations of limit as a concept), object-level narratives are not the main focus of the study to avoid restating all the facts about limits that can also be found in a introductory calculus textbook. Instead, the focus is on the meta-level narratives that were most relevant to the participants' word use, visual mediators, and routines (e.g., 'limit is a number' and 'limit is a process').

Next section reports on the instructor's discourse and compares his discourse with the students' discourses on limits. The following pseudonyms will be used for the participants: Dr. Brenner (the instructor); Amy, Jessica, Harry, and Keith (the students).

4 Analysis

4.1 The instructor's discourse on limits

Dr. Brenner discussed limits and continuity during eight 50-min lessons. In those lessons, there was no student-student interaction and few instances of student-instructor interaction. Although the instructor encouraged students to ask questions in the classroom, he did not facilitate any student discussion. Students interacted with the instructor when they asked clarifying questions; when they did not follow his explanations; and when they corrected a few computational mistakes he made in the class.

Throughout the lessons, Dr. Brenner had 775 limit-related utterances: two colloquial; 139 operational; and 634 objectified. Dr. Brenner's overall word use on limits was dominantly objectified (about 82 % of his total utterances) where he referred to limit as a number or a distinct mathematical object. However, there were two mathematical contexts in which he shifted his word use from operational to objectified and vice versa: informal definition of limits and computing limits. In those contexts, Dr. Brenner consistently distinguished the "limiting process" from the end result of that process, which he referred to as a number (if the limit existed). When he described the behavior of functions as the x values approached the limit points, he referred to limit as a process based on dynamic motion:

'As x gets closer and closer to 0, the function values get closer and closer to 1.'
'As x approaches 1, the function values approach or get close to 2.'
'You can think of approaching like this as a limiting process.'
'As x values get smaller and smaller, the function values get larger and larger.'
'If x is less than one, the function values will tend to negative infinity.'

When he reported on the limits of the functions, however, Dr. Brenner's operational word use gave way to objectified word use:

'[As x gets closer and closer to 0, the function values get closer and closer to 1.] So this limit is 1.'
'[As x approaches 1, the function values approach or get close to 2.] The limit is equal to 2.'
'[As x values get smaller and smaller, the function values get larger and larger.] So this limit does not exist.'

Dr. Brenner also introduced the formal definition of limit and used this definition for some proof problems. In these contexts, his word use was entirely objectified and he did not utter any words signifying motion. Instead, he used words signifying proximity:

'ε plays the role of measuring how close we are to L [the limit value] in our function values.'
'The difference of the function values from the limit value should be less than ε.'
'The function values are arbitrarily close to L [the limit value] as long as x is sufficiently close to a.'

Dr. Brenner's word use indicates that he alternated between the metarules of using the metaphor of continuous motion (when talking about the behavior of the functions) and using the metaphor of discreteness (when reporting on the end result of the limiting process) in the context of the informal definition and computing limits. In the context of the formal definition, however, he only used the metaphor of discreteness.

Three types of visual mediators were identified in Dr. Brenner's discourse: written words; graphs; and symbolic representation. Reporting the types of visual mediators does not give much insight about Dr. Brenner's discourse, so their discussion also incorporates the metarules regarding when and how he used those mediators.

Written words correspond to what Dr. Brenner wrote on the board besides mathematical symbols when he talked about limits. What will be highlighted here is the difference between the metarules regarding his written and spoken words in particular contexts. It was mentioned that Dr. Brenner used a combination of operational and objectified utterances in the contexts of the informal definition of limit and computing limits. The analysis of his actions also revealed that his operational utterances occurred when he used the words verbally whereas his objectified utterances consistently occurred when he wrote the words on the board. In other words, Dr. Brenner did not write his operational utterances on the board when talking about the informal definition of limit and computing limits. The context in which his written and spoken words were most consistent with each other was the formal definition of limit. These findings suggested the following meta-level rules in his discourse: writing words on the board when his utterances were formal; and using the words only verbally when his utterances were informal.

Dr. Brenner drew graphs in three different settings: (a) when he computed the limit of a function, (b) when he explained a particular definition, theorem or fact about limits, and (c) when he solved a problem that specifically asked to draw the graph of a given function. Dr.

Brenner used a graphical approach for 16 of the 64 limit computation problems on which he worked. He used algebraic and graphical approaches for five of the problems to address students' confusions. In those cases, the graphs were drawn only after the limits were initially computed by an algebraic method. The remaining 43 problems were solved using only an algebraic approach. In the context of computing a limit, Dr. Brenner's primary visual mediators were symbolic rather than graphic as he frequently used algebra-based routines such as plugging in (when the functions were continuous), using the limit laws (when finding the limit of a sum, difference, product or quotient), changing variables (the method of substitution), cancelling out the common factor (when the numerator and the denominator of a rational function were both equal to zero at the limit point), etc.

Dr. Brenner also used graphs when explaining a definition, theorem or fact about limits; and when solving a problem that specifically asked for the graph of a function. There were 21 such graphs identified throughout the eight lessons. Twenty of those were utilized to explain limit related facts and one was drawn as the solution of a problem about graphing a function. In the cases where Dr. Brenner utilized graphs to explain the endorsed narratives about limits such as definitions and theorems, the graphs were mostly drawn when he introduced them for the first time. In the remaining instances, he drew graphs for further elaboration after realizing the students were not clear about the mathematical ideas he communicated previously.

Mathematical symbols were the primary visual mediators Dr. Brenner used both in the context of computing limits and also in other limit-related contexts. He solved most of the limit computation problems by means of algebraic manipulations and represented definitions, theorems and facts about limits using mathematical notations consisting of symbols (as well as written words).

In summary, graphing was not a main metarule when Dr. Brenner determined the limits of functions. He used graphs more often to introduce a definition, theorem, or fact about limits rather than to compute limits of functions. In his discourse, graphing was a metarule that mainly served as an aid for teaching ideas related to limit and for addressing students' confusions about the algebraic solutions of the problems. Dr. Brenner's dominant metarule throughout the eight lessons was using symbolic representation.

The routines in Dr. Brenner's discourse were used as he substantiated and endorsed two meta-level narratives about the limit concept: 'limit is a number', and 'limit is a process'. He endorsed 'limit is a number' through the metarules of (a) using objectified utterances, which were also written on the board, (b) using the metaphor of discreteness, and (c) using symbolic representations and graphs (to discuss limit-related definitions and theorems) as visual mediators. He endorsed 'limit is a process'—though not as frequently as the former narrative—through the metarules of (a) using operational utterances, which were *not* written on the board, (b) using the metaphor of continuous motion, and (c) using graphs as visual mediators when computing the limits of some functions.

The consistencies in his word use and routines indicated that, as a mathematician, Dr. Brenner was aware of the distinct metarules behind different realizations of limits. However, the shifts in metarules and associated elements of his discourse such as word use, routines, and endorsed narratives remained implicit for students, supporting the tacit nature of metarules.

4.2 Comparison of students' discourses with the instructor's discourse on limits

4.2.1 Students' responses to the diagnostic survey

The results of the diagnostic survey given to all 23 students in Dr. Brenner's class at the end of their discussions on limit and continuity indicated that students mainly considered limit as a process rather than a number. The survey was taken from Williams (2001) because his classification of different views of limit is widely endorsed in the literature. The purpose of the survey was to gain some information about students' realizations of limits at the end of their instruction and to select the students for the interviews.

The first question of the survey included six statements about limits and asked students to decide whether the statements were true or false. The second question then asked them which of the six statements best described their understanding of limits. The third question asked students to describe, in their own words, what they understood a limit to be. The final question asked students to give a rigorous (formal) definition of limit, if possible.

The statements in the first question of the survey represented different—and often related—views of limit, some of which were difficulties addressed by research on student learning, e.g., the dynamic view (Bezuidenhout, 2001; Tall & Vinner, 1981) and related views such as considering limit as unreachable (Tall & Schwarzenberger, 1978), as a bound (Cornu, 1991), or approximation (Parameswaran, 2007). Table 1 shows the statements, the corresponding views of limit, and student responses to the first question of the survey.

Statement 1 was considered as true by 20 of the students and reflected a dynamic view of limit based on the metaphor of motion. This statement addresses limit as a process since it also mentions limit as a means for *describing* the behavior of a function rather than talking about it as a specific number. Statement 1 was chosen by 11 students as best describing their views of limit.[2]

The second popular statement that was chosen as true by 16 students was Statement 3, which reflects a formal view of limit based on the metaphor of discreteness. However, this statement was chosen by only one student as best describing his view of limit. Some reasons why the majority of the students chose the statement as true may be because the instructor introduced the formal definition of limit in the class and he uttered phrases similar to those in Statement 3.

Students' responses to Statements 4 and 5—which are connected to the dynamic view—indicate that the majority of students considered limit as unreachable or an approximation, possibly due to their focus on the process of the function values "getting close" to the limit value rather than the end result of that process. Statement 4 was considered as the best statement by six students whereas Statement 5 was considered as best by one student. Although a dynamic view related sentence, Statement 6 was considered to be false by the majority of students and was chosen as the best statement by only three students. Students' rejection of this statement as true may be due to the fact that the instructor never used plugging in values that are successively closer to a number as a routine in the classroom.

Statement 2 reflects the view of limit as a bound; it was rejected by the majority of the students to be true, and was not chosen as the best statement by any of the students. That the instructor used many examples in which the limit value was attained and was not necessarily the maximum value of a function may be some reasons why students considered this statement as false.

The third question in the diagnostic survey asked students to describe what a limit is. One student did not provide a response for this question and one wrote that limit was not a clear idea to her. All of the remaining students used the metaphor of continuous motion when describing limits. Nine of those students objectified limit as a number, a value, or a point and distinguished the limit process from the end result of that process by providing responses such as "as x approaches some number s, the limit is some number L," and "a limit is a value that the function approaches as x approaches a certain value." The remaining students' word use was operational as they only realized limit as a process. Their responses were of the form

[2] One student chose "none" as her response to the second question. Therefore, 22 students chose one of the six statements as best describing their views of limit.

Table 1 Students' responses to the first question of the diagnostic survey

Statements (Williams, 1991; 2001)	View of limit (Williams, 1991; 2001)	Number of student responses (n=23)	
		True	False
1. A limit describes how a function moves as x moves toward a certain point.	Dynamic-theoretical	20	3
2. A limit is a number or point past which the function cannot go.	Boundary	6	17
3. A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.	Formal	16	7
4. A limit is a number or point the function gets close to but never reaches.	Unreachable	13	10
5. A limit is an approximation that can be made as accurate as you wish.	Approximation	12	11
6. A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.	Dynamic-practical	9	14

"as the function's x values become closer and closer to s, the function becomes closer and closer to L," "as x gets closer to s, the function gets closer to L." Note that the objectified utterances refer to limit as a noun ("limit is…"), whereas the operational utterances describe the behavior of the function values getting close to the limit value, which is different than those values being equal to the limit value.

Only five students provided a response for the fourth question, which asked students to provide a formal definition of limit. One of those students wrote the limit notation $\lim_{x \to s} f(x) = L$ as his response. The other students used operational words that defined limit as a process (e.g., "As x approaches a number, y approaches a number," "Limit describes the function's y-values as x-values approach another number").

The findings of the diagnostic survey suggested that, unlike the instructor, the majority of the students only endorsed the narrative 'limit is a process' and did not objectify limit as a number at the end of their instruction. The students did not provide a description of limit based on the metaphor of discreteness but mainly used the metaphor of continuous motion as a metarule when talking about the concept. In addition, they did not show signs of recognizing the differences between the informal and the formal definition of limit with respect to the distinct metarules on which they are based. In fact, 18 of the 23 students did not verbalize their realizations of the formal definition in the survey. The remaining students' responses about the formal definition were based on the metaphor of continuous motion suggesting that they did not attend to the instructor's shifts in word use and metarules in the contexts of the informal and formal definition.

The diagnostic survey helped gain some information about students' word use and endorsed narratives about limit and the metarules behind them but the findings must be interpreted with care. First, most of the views about limits in the survey were directly presented to the students; they had the opportunity to express their realizations of the concept only in the last two questions. Second, the context of the diagnostic survey was not suitable for an in-depth exploration of students' repetitive actions in the form of routines, and the visual mediators they used. A detailed analysis of students' discourses on limits was possible in the interview sessions, which is reported in the next section.

4.2.2 Students' responses to the task-based interviews

The interview sessions provided further information regarding four students' realizations of limit. Those students were the volunteering participants among the candidates identified for the interview sessions based on the diagnostic survey. Amy was selected because she showed many signs of the difficulties based on dynamic motion highlighted by research about limits. She did not provide responses that signified limit as a number in the survey. Keith was selected because he was the only student who chose the formal view as best describing his realization of limit. Jessica and Harry were selected because they alternated between referring to limit as a process and a number in different parts of the diagnostic survey. Although some of the students objectified limit in the diagnostic survey, all students struggled whether to talk about limit as a process or a number during the interviews. For example, there were many instances in which the students computed limits of functions and used the equal sign to write limits as being *equal to* particular numbers. The following excerpts demonstrate some of their responses when they talked about the question shown in Fig. 1:

> Interviewer: So what can you say about the limit of this function as x approaches 0 [shows $\lim\limits_{x\to 0} f(x)$]?
>
> Amy: It gets close to 1 [writes $\lim\limits_{x\to 0} f(x) = 1$].
>
> Harry: So this is limit as x approaches 0 of f of x [shows $\lim\limits_{x\to 0} f(x)$]. The y value is 1 [shows the y value of the function at $x = 0$].
>
> Interviewer: So what is the limit of that function?
>
> Harry: It is approaching 1; it gets closer and closer to 1 [writes $\lim\limits_{x\to 0} f(x) = 1$].
>
> Interviewer: What can you say about the limit in this case [shows $\lim\limits_{x\to 0} f(x)$]?
>
> Jessica: 1 [writes $\lim\limits_{x\to 0} f(x) = 1$]. But limit is not only a number.

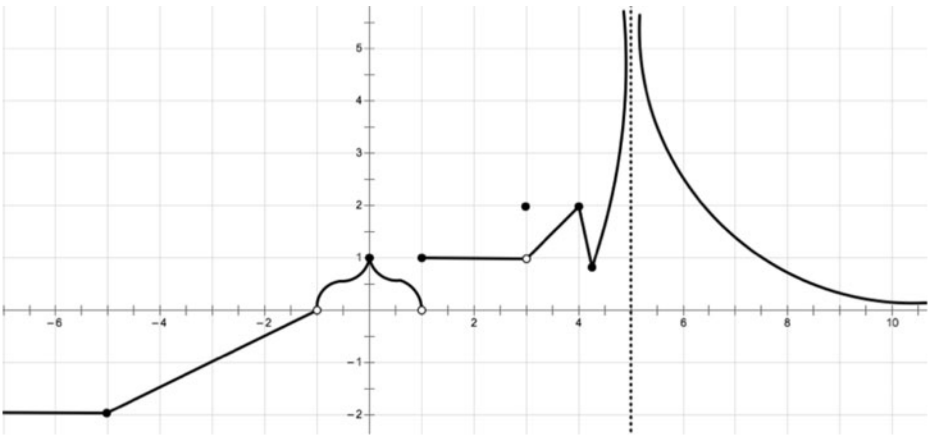

Fig. 1 An interview session question where students were asked to find limits from a graphical representation of a function

Interviewer: What is it?

Jessica: Limit describes a process; it is not only a number. It describes all the process that this point [shows x in $\lim_{x\to 0} f(x)$] moves to.

Interviewer: What can you say about the limit of this function [shows $\lim_{x\to 0} f(x)$]?

Keith: It is going to equal 1 [says this hesitatingly].

Interviewer: Why?

Keith: The limit from the right approaches 1 and the limit from the left approaches 1 so the limit would be 1.

Note that instead of talking about the process of the *function values* approaching the limit value, the students talked about the *limit* approaching a number. Regardless of function representation, graphical, tabular, or algebraic, such mixed utterances about limits were common in students' discourses throughout the interviews. At times, they referred to limit as a number; often, they talked about it as a moving variable. Consequently, the students endorsed the narrative 'limit is a process' more consistently than 'limit is a number'. Even during the instances they endorsed the narrative 'limit is a number', students' utterances were based on the metarule of using the metaphor of dynamic motion. Although Dr. Brenner also shifted his word use, metarules, and endorsed narratives when computing limits, he clearly distinguished the limit process from the limit value, which was not the case for the students during the interviews. Consistent with the findings of the diagnostic survey, objectification of limit as a number was challenging for the students in the context of the interviews.

Since Dr. Brenner introduced it in the class, students were also given the instructor's formal definition of limit and were asked to talk about it during the interviews. Figure 2 shows the question about the formal definition of limit in the interview sessions.

Dr. Brenner's word use on limits was completely objectified and was based on the metaphor of discreteness in this context. Unlike the instructor, the students' word use was operational and was based on the metaphor of continuous motion in the context of the formal definition. Although they uttered phrases like "arbitrarily close," and "sufficiently close," they talked about the *limit* being arbitrarily close to a number rather than the *function values* being arbitrarily close to the limit value. Moreover, when asked to compare and contrast the formal definition with the informal definition, all students mentioned they were similar in that they both described the "approaching process" and so did not attend to the distinct metarules on which these two definitions are based. The students considered the definitions different in that the informal definition was easier or more intuitive whereas the formal definition was used for

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Let a function $f(x)$ be defined on interval about x_0 (except possibly at x_0). We say $\lim_{x\to x_0} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

a) Please explain in your words what this definition means to you.
b) How is this definition similar/different than the informal definition of limit you used in your class?
c) In your opinion, what is the purpose of this definition?

Fig. 2 An interview session question where students were asked to talk about the formal definition of limit. The definition was stated exactly as the instructor introduced it in the class proofs, which the students identified as its main purpose. The students struggled moving from a discourse on limits based on the metarule of using dynamic motion to one based on the metarule of using a static realization of the concept during the interviews.

Note that one of the metarules in Dr. Brenner's discourse was to write his objectified utterances about limits on the board. Dr. Brenner's discussions of limit as a process through operational word use only took place when he communicated his ideas verbally as he explored the behaviors of functions. The students, however, talked about the instructor's investigation of the behavior of the functions as x values approached the limit point as a definition of limit. For example, Amy said finding limits in this manner "fit the definition he [the instructor] presented to us in class," whereas the instructor referred to limit as a number when he introduced the informal definition and did not leave a written record of his operational utterances. Students' responses to the diagnostic survey and interviews indicated that—despite the frequency of Dr. Brenner's objectified utterances (82 %) he wrote on the board—the students chose to focus on his operational utterances he communicated verbally when forming their realizations of limit.

Another distinction between students' and the instructor's discourses was regarding the metarules behind their use of visual mediators. Throughout the eight lessons, Dr. Brenner's primary visual mediators were symbolic whereas the students' primary visual mediators were graphical. The students used some symbolic representations as visual mediators, especially when they used the limit notation. Yet, they used graphs more often than symbolic representation in the context of the interviews. Dr. Brenner used graphs when computing limits but he mostly used an algebraic approach when working on the limit computation problems. He mainly used graphs to communicate limit-related definitions and theorems, and to address students' confusions. In contrast, students used graphs to make sense of the functions when computing limits. In other words, unlike the instructor, they needed to visually represent the functions before talking about their limits.

During the interviews, the routines in students' discourses were used as they substantiated and endorsed two meta-level narratives about the limit concept: 'limit is a number', and 'limit is a process'. They endorsed 'limit is a number'—though quite infrequently compared to the latter narrative—through the metarules of (a) using some objectified utterances, (b) using the metaphor of continuous motion, and (c) using the limit notation (to write the limits as equal to particular numbers) as visual mediators. They endorsed 'limit is a process' through the metarules of (a) using operational utterances, (b) using the metaphor of continuous motion, and (c) using graphs as visual mediators to make sense of the functions when computing limits.

In summary, some of the contexts in which students struggled during the interview sessions coincided with the contexts in which the instructor's shifts in word use, metarules, and endorsed narratives took place. However, there was no evidence in Dr. Brenner's discourse that he considered attending to the shifts in his word use, metarules, and endorsed narratives as pedagogically relevant in the classroom. As a result, those elements of his discourse remained implicit for the students in the study.

That the students relied more on an intuitive rather than a formal definition when realizing limits may seem only natural. The important issue for this study was not whether students could prove statements about limit or provide an explanation of limits using the symbolism inherent in the formal definition, but whether they realized the elimination of motion from the discourse on limits in the formal definition. Existing research on learning of limits identified many difficulties associated with students' sole reliance on continuous motion (Bezuidenhout, 2001; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). Some of those difficulties were also visible during the interviews. For example, Amy's lack of awareness of the metaphor of discreteness and her overreliance on dynamic motion supported her realization that "only continuous functions have limits." During the interview, she was not able to accurately solve the problems in which functions were discontinuous but had a limit at the given point. Keith, on the other hand, initially had difficulties finding the limits of continuous functions. He argued that "moving towards" a value "suggested" that the limit can be "approached but never reached." So he was uncomfortable finding the limits of continuous functions since they attain their limit values. Such difficulties as well as responses to the diagnostic survey and the interviews suggest that students' realizations of limits cannot be reduced to a language problem; they require the coordination of all the elements in the discourse on limits.

5 Discussion and conclusion

This study explored the characteristics of one instructor's discourse on limits and compared his discourse with those of the students. The findings indicate that the elements of the instructor's discourse were mostly consistent with each other except for the contexts of the informal definition of limit and computing limits. In those contexts, he shifted his metarules, word use, and endorsed narratives to substantiate the narratives 'limit is a number' and 'limit is a process', accordingly. The analysis of the students' discourses revealed that although they used similar visual mediators, words, and metarules to endorse particular limit-related narratives, their utilization of these features was not as coherent as the instructor's. The students' use of the same features of discourse as the instructor to substantiate a different narrative ('limit is a process' rather than 'limit is a number') suggests a possible miscommunication in the classroom regarding the discourse on limits. This is also supported by the students' consideration of their approaches to limit as compatible with those of the instructor (e.g., see Amy's comments in Section 4.2.2) despite their struggles with objectifying and coping with the interplay between dynamic and static aspects of limits.

That the instructor's and the students' realizations of limit differed from each other may be a common finding in mathematics education research. The strength of Sfard's (2008) framework is to help elaborate when, how, and in what ways they differ, highlighting the communicational breakages in the participants' discourses. Metarules are among the elements of mathematical discourse that can remain tacit in teachers' and students' discourses (Sfard, 2001; 2008). The exploration of discourse with a focus on metarules enables the identification of the patterns, inconsistencies, and incompatibilities in teachers' and students' discourses as well as the contexts of miscommunication in the classrooms. Consistent with Sfard's (2001; 2008) arguments, the results of the study support the hypothesis that it may be possible for teachers to enhance classroom communication by explicitly attending to the metarules and other elements of their discourse.[3] If students are to participate fully in the discourse on limits, they need to command all features of the discourse. What is mainly gained from using Sfard's (2008) framework is that the elements of discourse have a complex and intertwined relationship and students' realizations of limits are dependent on the orchestration of all of those. The framework allows a rich description, taking into account the contextual, communal, dialogical, and intricate dynamics in teachers' and students' discourses, which cannot be explored—at least, to the same extent—through the approaches that attribute issues of teaching and learning merely to cognition.

The findings do not suggest all students of calculus should learn about the formal definition and limit-related proofs. Instead, they highlight the conceptual challenges surrounding the

[3] Testing this hypothesis is beyond the scope of the study and remains to be examined by future studies. However, there is some evidence indicating that drawing students' attention to metarules can support their learning of the object-level rules of mathematics (Kjeldsen & Blomhøj, 2012).

interplay between the dynamic and static aspects of limit resulting in different realizations of the concept. The results also do not suggest that the dynamic view of limit should be discarded from the classroom discourse since it can be the most useful tool with which to initially realize the concept. In fact, both the informal and the formal definition of limit are contexts in which it is possible to talk about limit as objectified: a number. However, in this study, students' overreliance on the dynamic view supported their realization of limit as a process but not as a number. It seemed necessary that the dynamic view needed to be challenged at some point in order to support the objectification of limit in students' discourses.

It should be noted that the metarules highlighted in the study do not capture all of the metarules in the class and should not be considered as an exhaustive list. Here, the focus was only on the metarules regulating participants' object-level rules about limits; the metarules such as those regulating participation or characterizing the participants' intentions were not the explicit focus of this work. Second, the characteristics of the discourse mentioned here are not assumed to generalize to other classrooms. The formulation of discourse as reflecting a community's norms, values, and goals requires the consideration of discourse as contextual. Although it may be possible to find similar discursive patterns in other calculus classrooms, this is not a claim of the study since it reports on a particular community of mathematics. The conjecture here is that each community may have its own characteristics of mathematical discourse, yet the incompatibility of participants' use of the elements of the discourse may lead to miscommunication. Third, although this study describes features of students' discourses on limits, it does not address the development of those features. The future trajectory of this work is to investigate how students develop their discourse on limits and what factors contribute to such development.

The final remarks concern teaching of limits. The limit literature is quite rich in terms of student learning and the conceptual obstacles regarding limits; we need more research addressing teaching of limits in relation to the students' discourses, which was the main focus of this study. Conducted as a case study, this work cannot address typical patterns in teaching of calculus; yet, it points to some interesting issues to consider. For example, contrary to what the research suggests about teaching of limits, Dr. Brenner did not perpetuate the dynamic view in his class, as evidenced by his discourse on limits. He worked with different types of functions besides polynomials; introduced the formal definition and worked on some proofs; differentiated between the informal and formal aspects of limit basing them on distinct metarules and represented them differently. He dominantly used the metaphor of discreteness signifying limit as a number. The students' discourses, however, were still dominated by a dynamic approach as they realized limit as a process. This observation raises further questions about student learning such as "what is the role of teachers' discourse in students' discourses on limits?," "do we see changes in students' discourses on limits when teachers explicitly attend to the metarules and other features of their discourse in the classrooms?," and "what are the factors—besides teachers' discourse—that impact student learning on limits?" Future research should address such questions to investigate teaching of limits and to connect students' discourses with their sources of learning.

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