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Dissecting success stories on mathematical problem posing: a case of the Billiard Task

Boris Koichu · Igor Kontorovich

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Abstract "Success stories," i.e., cases in which mathematical problems posed in a controlled setting are perceived by the problem posers or other individuals as interesting, cognitively demanding, or surprising, are essential for understanding the nature of problem posing. This paper analyzes two success stories that occurred with individuals of different mathematical backgrounds and experience in the context of a problem-posing task known from past research as the Billiard Task. The analysis focuses on understanding the ways the participants develop their initial ideas into problems they evaluate as interesting ones. Three common traits were inferred from the participants' problem-posing actions, despite individual differences. First, the participants relied on particular sets of prototypical problems, but strived to make new problems not too similar to the prototypes. Second, exploration and problem solving were involved in posing the most interesting problems. Third, the participants' problem posing involved similar stages: warming-up, searching for an interesting mathematical phenomenon, hiding the problem-posing process in the problem's formulation, and reviewing. The paper concludes with remarks about possible implications of the findings for research and practice.

Keywords Billiard Task · Considerations of aptness · Exploration · Interesting problems · Knowledge base for problem posing · Pre-service mathematics teachers · Problem posing · Prototypical problems

1 Introduction

One of the most robust observations from the professional literature on mathematical problem posing is that diagnostic and learning opportunities associated with this activity are, as a rule, much more impressive than mathematical problems produced as a direct result of the effort.

B. Koichu (✉) · I. Kontorovich
Department of Education in Technology and Science, Technion—Israel Institute of Technology, Haifa 32000, Israel
e-mail: bkoichu@technion.ac.il

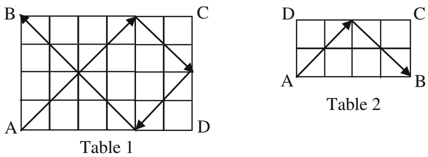
Indeed, problem posing is broadly recognized as a powerful research tool for revealing components of teachers' mathematical knowledge (De Corte & Verschaffel, 1996; Ma, 1999) as well as a means for developing mathematical thinking and creativity (Silver, 1994; 1997; Silver, Kilpatrick, & Schlesinger, 1990; Toluk-Uçar, 2009). At the same time, many studies have reported that a sizable portion of problems generated by students and teachers are ill-formulated or cognitively undemanding (Crespo & Sinclair, 2008; Harel, Koichu, & Manaster, 2006; Ma, 1999; Silver & Cai, 1996; Silver, Mamona-Downs, Leung, & Kenney, 1996).

This observation has its consequences for ongoing research efforts to reveal the nature of mathematical problem posing and its potential in mathematics teacher education and practice. Namely, the existing body of data is suggestive about processes involved in posing not necessarily *interesting* problems (e.g., Crespo & Sinclair, 2008; Silver et al., 1996) as well as about the sources of teachers' difficulties with problem-posing tasks (Harel, et al., 2006; Ma, 1999). The accumulated data, however, include a relatively small number of "success stories," i.e., cases in which problems posed in a particular research setting are perceived, by the problem posers or their peers, as interesting, meaningful, cognitively demanding, or surprising. Consequently, the data are less suggestive about the processes involved in posing such problems.

This paper is drawn from our experience of collecting success stories in a problem-posing situation known as the Billiard Task (see Fig. 1). Silver et al. (1996) have argued that the Billiard Task is rich enough in order to stimulate generation of interesting problems and conjectures and, simultaneously, it is accessible enough as it requires only knowledge of rather basic mathematical concepts. In addition, Cifarelli and Cai (2005) utilized the Billiard Task in a study on exploration strategies of mathematics teachers.

During the last 2 years, we offered the Billiard Task to more than 20 prospective and in-service mathematics teachers and documented how they worked. In this paper, we analyze two success stories from our collection. The stories are chosen to be presented here as particularly transparent and insightful regarding the problem-posing processes the posers went through. The analysis is driven by the following question: What are some of the traits of problem-posing processes that lead the posers, two prospective mathematics teachers, to formulate interesting problems in the chosen problem-posing context?

Imagine billiard ball tables like the ones shown below. Suppose a ball is shot at a $45°$ angle from the lower left corner (A) of the table. When the ball hits a side of the table, it bounces off at a $45°$ angle. In Table 1, the ball travels on a 4×6 table and ends up in pocket B, after 3 hits on the sides. In Table 2, the ball travels on a 2×4 table and ends up in pocket B, after 1 hit on the side. In each of the figures shown below, the ball hits the sides several times and then eventually lands in a corner pocket.



Based on the given situation, pose as many interesting mathematical problems as you can.

Fig. 1 The Billiard Task (adapted from Silver et al., 1996)

2 Theoretical framework

Following Stoyanova and Ellerton (1996), we refer to mathematical problem posing as "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). This student-centered and process-oriented definition is adopted in our paper for analyzing the cases of interest. Two subsections below present our interpretation of the key components of the definition: the notion of *mathematically meaningful problem* and a set of attributes that we consider for characterizing the processes of formulating such problems.

2.1 What counts for a mathematically meaningful problem?

Various sources (e.g., Kilpatrick, 1982; 1985; NCTM, 2000) refer to a *mathematical problem* as a task involving mathematical concepts and principles, for which the solution method is not known in advance by the person(s) engaged in it. Such a conceptualization presumes that a particular task can or cannot be seen as a mathematical problem depending on mathematical background and attitude of its solvers, as well as on conditions under which the task is dealt with (Kilpatrick, 1985).

The descriptor "meaningful," utilized by Stoyanova and Ellerton (1996) in their definition, amplifies the extent of ambiguity embedded in many attempts to operationally determine whether a particular product of one's problem posing is worthwhile. In addition, one can argue, in line with Crespo and Sinclair (2008), that the descriptor meaningful belongs to the rarefied discourse of mathematicians rather than that of learners. Crespo and Sinclair (2008) suggested that the learners' normative understanding of what qualified as a worthwhile problem may develop around the notion of "mathematically interesting" or "tasty" which, in turn, is not extraneous to the notion of "a beautiful problem." Consequently, we treat in this paper the descriptor meaningful in the manner that has been developed in past research for treating such problems' descriptors as "interesting" or "beautiful."

Namely, the professional literature acknowledges that an agreement about what constitutes a mathematically interesting or beautiful problem is elusive (e.g., Wells, 1990), but offers quite stable lists of general characteristics of such problems and their solutions. They include: simplicity, brevity, clarity, elegance, fruitfulness, mathematical deepness and complexity, cleverness, cognitive demand, novelty, and surprise (compiled from Brinkmann, 2009; Crespo & Sinclair, 2008; Dreyfus & Eisenberg, 1986; Koichu, Katz, & Berman, 2007; Wells, 1990).

Following Goldin (2002) and Koichu and Berman (2005), we argue that such general characteristics can be seen as instantiations of one's internal multiply encoded cognitive/affective configurations, to which the holder attributes some kind of truth value. In accordance with this view (cf. also Crespo & Sinclair, 2008, for a compatible view), we operationally consider a posed problem *mathematically meaningful* (or *interesting* or *beautiful*) if it is evaluated as such by the poser of the problem, its readers or solvers, and if one's argumentation underlying the evaluation involves some of the aforementioned general characterizations.

2.2 Attributes of problem posing[1]

The set of cognitive and affective attributes for characterizing problem posing is adapted from a five-facet framework developed in our earlier research (Kontorovich & Koichu, 2009; Kontorovich et al., 2012). Three attributes are particularly relevant to the forthcoming

[1] This section is an abridged and modified version of a section in Kontorovich, Koichu, Leikin, and Berman (2012).

analysis: *mathematical knowledge base*, *problem-posing strategies*, and *individual considerations of aptness*.

2.2.1 Mathematical knowledge base

Problem posing is a natural companion of problem solving (Kilpatrick, 1987; Silver et al., 1996); therefore, mathematical knowledge bases needed for these activities may intersect. In line with Schoenfeld's (1985) model of problem solving, it can be assumed that mathematical knowledge base needed for posing problems includes the knowledge of mathematical definitions, facts, routine problem-solving procedures, and relevant competencies of mathematical discourse and writing. In addition, problem posing relies on one's knowledge of mathematical problems that can serve as prototypes (cf. Kilpatrick, 1987, for the role of *association* and *analogy* in problem posing). Recalling and using a system of prototypical problems rely, in turn, on three components of the problem-solving ability of the posers, which were pointed out by English (1998): the ability to recognize the underlying structure of a problem and to detect corresponding structures in related problems, the ability to perceive mathematical situations in different ways, and the ability to favor some problems over others in routine and non-routine situations.

2.2.2 Problem-posing strategies

We refer to problem-posing strategies as systematic approaches to analyzing and transforming conditions of a given problem-posing task and to question generating. This definition embraces cognitive processes identified by Silver et al. (1996) and Cifarelli and Cai (2005) in the context of the Billiard Task.

Silver et al. (1996) introduced, individually or in pairs, 71 mathematics teachers to the Billiard Task and asked the subjects to write down as many questions appropriate to the task as they could for 10 min. Following this, the teachers were asked to solve some of their own problems for 30 min and then to generate additional ones for 15 min. The analysis of the written responses enabled the researchers to conjecture that several cognitive processes were likely to be involved in the teachers' problem posing in the chosen context. They included:

- *Constraint manipulation*, i.e., posing new problems by systematic manipulations with the task conditions or implicit assumptions (The famous *what-if-not* strategy introduced by Brown and Walter (1983) is a particular case of this strategy.)
- *Goal manipulation*, i.e., posing new problems by manipulations with the goal of a given or previously posed problem where the assumptions of the problem are accepted with no change
- *Symmetry*, i.e., posing a new problem by symmetric exchange between the existing problem's goal and conditions
- *Chaining*, i.e., expanding an existing problem in a way that a solution to a new one would require to solve an existing one first

Cifarelli and Cai (2005) explored how two prospective mathematics teachers formulated and solved their own mathematical problems based on two given situations, one of which was the Billiard Task. The students worked with software designed to draw different paths of the ball depending on the chosen table dimensions. In addition, the software presented "the number of bounces, the length of the path, and the corner in which the ball finished" (p. 306). The teachers were given as much time as they wished to complete the task and actually worked on it about 70 min. The analysis by Cifarelli and Cai (2005) was focused on the ways by which the teachers made sense and explored the relationships between the aforementioned parameters of the ball's paths. Two reasoning patterns, which are particularly relevant to the concerns of our paper, were identified:

- *Data-driven reasoning*, i.e., relying on the need to generate many concrete examples and attempting to determine new relationships based on them
- *Hypothesis-driven reasoning*, i.e., formulating hypotheses and projecting ideas into new situations, even if the poser could not be sure of where the results would lead

2.2.3 Considerations of aptness

The word "aptness" means fitness, suitableness, appropriateness, and also connotes one's propensity to capture particular features of a situation as important. In the chosen context, considerations of aptness are the poser's comprehensions of explicit and implicit requirements of a situation within which a problem is to be composed; they also reflect her or his assumptions about the relative importance of these requirements (Kontorovich & Koichu, 2009; Kontorovich et al., 2012). Types of considerations of aptness relevant to the concerns of this paper are:

- *Aptness to herself or himself*, i.e., extent to which a poser attempts to pose a problem that would satisfy her or his personal criteria, as a problem solver, of an interesting or meaningful problem
- *Aptness to the potential evaluators*, i.e., extent to which one's problem posing is driven by the wish to get a positive evaluation by a person who assigned the problem-posing task
- *Aptness to the potential solvers of a posed problem*, i.e., extent to which the poser's performance is driven by her or his wish to create a problem that would be suitable for the potential solvers

3 Two success stories

The cases below are abridged reconstructions of events that occurred with two prospective mathematics teachers, Ian and Alla (pseudonyms), when working on the Billiard Task. Every case is a "success story" in its own sense. Ian was a second year undergraduate mathematics education student who had no prior experience in problem posing. Alla was in transition from her B.Sc. to M.Sc. degree in mathematics education and met problem-posing tasks in her past studies.

Ian and Alla took part in the course "Advanced workshop in teaching mathematics" jointly taught by the authors of this paper in 2010. Problem posing was one of the major themes of the course. The course included various problem-solving and problem-posing activities, systematic discussion of research papers on problem posing and three guest lectures by highly reputed problem-posing experts. The data presented in this paper are based on a homework assignment given for 2 weeks in the middle of the course. The assignment included, in addition to the task presented in Fig. 1, a request to solve some of the posed problems and to answer the following questions in writing:

> How did you start? How did the idea of the problem emerge? Which ideas did you abandon during the posing and why? How did you decide that the problem is complete?

Ian and Alla granted us permission to use their submitted homework and draft materials in our study after the end of the course. Face-to-face conversations with the students about the details of their written works served as a complementary data source. In addition, a draft of the following two subsections was sent to the students, and they acknowledged that our interpretations of their thinking had been accurate. In Ian's words, “You really captured what I did.” In Alla's words, “You understood me perfectly. The description [of my work] reflects very well what was happening in my mind.”

3.1 Success of Ian: the posed problem incorporates several “challenges” for middle school students

Ian read the task and quickly wrote down two problems:

1. Calculate the area of the quadrilateral part of the billiard ball's path in Table 1 [see Fig. 1], given that AB=4, and AD=6.
2. Calculate the length of the billiard ball's path in Table 1.

The problems' structures and questions (“calculate the area” and “calculate the length”) are in line with many geometry problems in Israeli textbooks. Consequently, it can be assumed that Ian considered geometry textbook problems as prototypes.

Ian proved that the quadrilateral path of the ball (see Table 1 on Fig. 1) was a square and calculated its side by the Pythagorean theorem. The side was $\sqrt{8}$ and the answer to the problem was a “nice” number 8. He then used the calculated side $\sqrt{8}$ in his solution to the second problem and found that its answer was not a nice number $2\sqrt{32}+2\sqrt{8}$. (Noticeably, Ian did not consider solving the problem in a simpler way or even simplifying the answer.) The problems were not especially interesting to Ian, and he decided to formulate the next problem so that it would be more mathematically complex, but still have a nice numerical answer. This decision indicated the presence of considerations of aptness to himself in Ian's problem-posing reasoning. His third problem was as follows:

3. In the table shown below [Fig. 2a], a billiard ball is shot from corner A to point G at a 60° angle to BC. The length of the ball's path from A to B is 42 cm. Calculate the length of border AB.

Ian explained that he was thinking “from the end to the beginning” when formulating the problem. Namely, he chose the angle of incidence and the length of the path so that a nice number would be the answer.

> I understood that if I make the angle of incidence 60°, there appear many isosceles and regular triangles on the table [Fig. 2b]. I paid attention to the phenomenon that all segments inside the rectangle are equal, and that there are 6 segments. As a result, I chose as the length of the path 42 so that it would be divisible by 6.

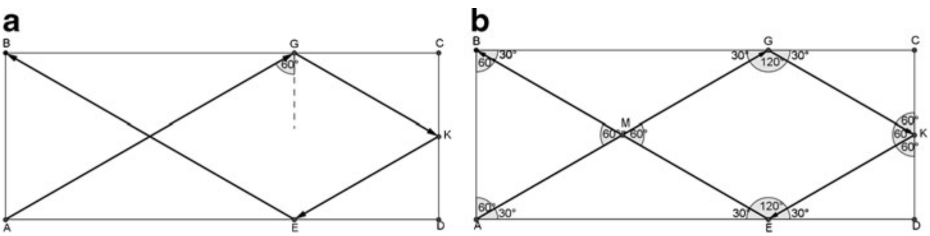


Fig. 2 **a**, **b** Drawings intended by Ian for his third problem

Ian's choices imply that he used a modification of the chaining strategy. Indeed, his third problem took into account not only a type of the problem's question, which he adapted from his second problem, but also what he had not achieved so far (i.e., making a problem with a nice number). The adherence to nice answers strengthens the suggestion that Ian's reasoning was shaped by recalling some geometry textbook problems as prototypes.

Ian liked his third problem more than the previous ones because of the informed choices of the parameters he had made when formulating it. However, he was not fully pleased by the solution to the problem as it was, he believed, too simple. His last problem was as follows:

4. A ball is shot from corner A, hits border BC at point G, and then, it hits borders CD and DA and enters pocket B, as in the drawing below [see Fig. 3]. The borders of the table are made of a special material so that the angle of reflection is 20 % less than the angle of incidence. Find the angle of incidence AGN, given that angle ABE is 40°.

He explained:

> I wanted this problem to be of a higher level than the previous ones, which led me to the idea to look at the angle of incidence as a variable, and the last angle as a given. In addition to the challenge of reversing the problem, I decided to incorporate in it an additional challenge. I considered a billiard table made from an unusual material that changed the angles. I chose the condition about 20% randomly, and started to solve the problem for $\alpha[ABE = \alpha]$, from the end to the beginning. When I found a general expression for the angle, I substituted α with 40° and found the answer that the solvers of the problem are supposed to find, 50°, which was a nice answer.

By "reversing the problem," Ian meant that one of the conditions of the given task became a goal of a new problem, i.e., he had utilized a symmetry problem-posing strategy. The process of finding the general expression for α is presented in his draft (see Fig. 4).

Ian was pleased that the calculations involving α were quite complex. It seems that he was not aware of the fact that his problem had a much simpler solution for the chosen—

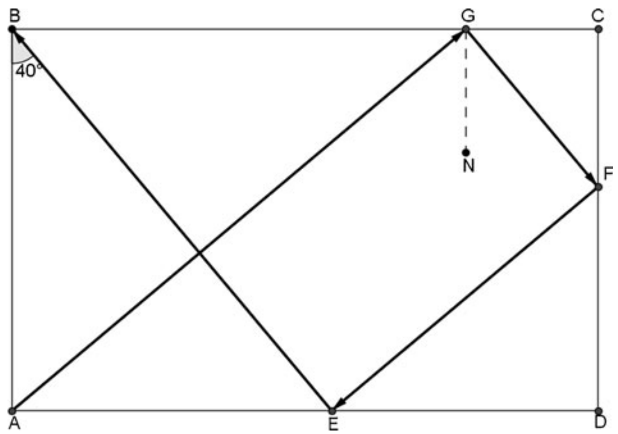


Fig. 3 A drawing intended by Ian for his fourth problem

apparently, for the sake of a nice answer—angle 40°. He noted that middle school students "can learn something" from going through the same processes as he did when developing the expression for angle AGN. The last assertion implies that consideration of aptness to potential solvers were also present in Ian's reasoning.

Ian told us that he had offered the problem to his friend, a mathematics education student, to solve, even though it was not a part of the homework. The friend solved the problem very quickly and evaluated it as "very transparent, almost obvious." Ian explained us that this is because the problem formulation does not necessarily require developing a general formula for α, as he meant it for the solvers. The friend's opinion lowered Ian's own opinion about the problem, but he still felt that the problem was quite interesting. We, the authors of this paper, also liked this problem, in particular, because it did not immediately occur to us that the condition "the angle of reflection is 20 % less than the angle of incidence" preserves the angle of incidence in a sequence of hits for the chosen ending angle. Thus, the problem somewhat surprised us.

In sum, the case of Ian is a success story in the following meaning: Ian succeeded to pose a problem which, by his considerations of aptness to himself and to the potential solvers, was mathematically and pedagogically interesting.

3.2 Success of Alla: the posed problem is based on an exploration of "what is going on in the game"

Alla had begun from making up a list of questions, which eventually did not appear in the submitted homework. In her words:

> When I looked at the drawings [included in the formulation of the Billiard Task; see Fig. 1], I thought about the principle of reflection in geometry. This is because of the problem I remembered from the course 'Selected problems in mathematics' [taught by one of the authors of this paper]. When I read the task carefully, I was surprised that there was much more in there, and quite easily wrote down 20 questions. Most of the questions were based on one of two

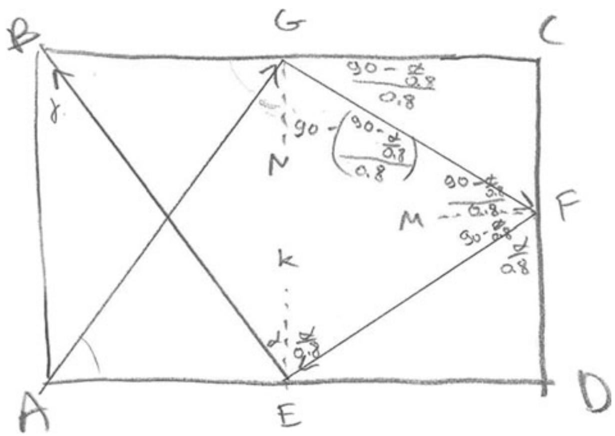


Fig. 4 A draft for Ian's fourth problem

> ideas: changing the conditions of the task and searching for generalization or a rule related to some characteristics of the game… Naturally, not all the questions were of the same level of difficulty and interest to me.

The questions on the list concerned the relationships among the dimensions of the tables, the ball's path, and the final pocket the ball enters. Many questions were remarkably close to those posed by the participants in the study of Silver et al. (1996). Namely, there were questions in which the given angle of incidence was changed from 45° to other angles, a series of questions calling to explore what happens when the angle of incidence would be not equal to the angle of reflection (cf. the story of Ian), and questions about possible meetings of two balls shot under different conditions. For this reason, and because of the above explanation of Alla, we suggest that she generated the list by using chaining, constraint manipulation, goal manipulation, and generalization strategies, as the participants in the study of Silver et al. (1996) did. Alla generated the list in about 15 min and then decided to stop and choose one of the questions to pursue.

> I looked over the list of questions, and decided to focus on what really interested me: to understand what is going on in this game. I mean that I did not change the angle of incidence or reflection, and not the places of the pockets on the table etc., but only the table sizes.

Specifically, she chose to explore the final pockets the ball enters on the tables of all integer sizes. To us, her choice was quite remarkable. Alla decided to make up an interesting problem from an exploration of one the most natural questions embedded in the Billiard Task (Silver et al., 1996), the solution to which was unknown to her. To recall, Ian made up an interesting problem by incorporating more or less unusual conditions, but he never experienced uncertainty about how to solve the problems he had posed. Thus, it can be assumed that considerations of aptness to herself were the most important ones for Alla at this stage.

She started the exploration from looking at the 8×2, 8×4, 8×6, and 8×10 tables and then at the 10×2, 10×4, 10×6, 10×8, and 10×12 tables. She evidently tried to be systematic and drew only the tables with even–even dimensions, in which one side was kept constant (Fig. 5). Note that, mathematically speaking, this decision was unfortunate. Indeed, if Alla had started from exploring the tables of odd–odd or odd–even dimensions, she would have seen the pattern more easily (cf. Cifarelli & Cai, 2005, for the case of Sarah). When having nine tables with even–even dimensions in front of her, Alla stopped drawing and started to look for regularities. This course of exploration of the Billiard Task was called by Cifarelli and Cai (2005) data-driven reasoning.

Alla hypothesized that there was a connection between the differences between the table dimensions and the final pocket, but could not see any pattern. Then she acted as follows:

> In every table that I've explored I drew the path of the ball. For this reason, it was ‘easy’ for me to see that the paths on [bigger] tables are some sequences of reflections or translations of the paths at some [smaller] tables. So I tried to divide the [bigger] tables into symmetric parts [see Fig. 5] that would be the same at the different tables. This division of the tables created mini-tables with odd dimensions, so I drew the following tables: 5×3, 5×7, 5×9 and 5×11. I was trying to think about all the tables in front of me [i.e., even-even and odd-odd], and simultaneously about odd-even tables, though I did not draw them.

This course of exploration is closer to what Cifarelli and Cai (2005) called hypothesis-driven reasoning. Still, Alla could not see any regularity and came back to browsing her list of questions. At this point, the correct hypothesis came to Alla as an insight. In her words:

> I don't know how to exactly explain what happened in my mind in the next several moments, but I'll try. I think that several ideas came along: one of the questions in my list, which was about multiplication of the sides of the table by some integer factor + the tables that I drew and divided into mini-tables = the idea to reduce all the dimensions by a constant, instead of multiplying… From here I was just thinking about splitting all the tables into equivalence classes, where the representatives of each class were ordered pairs of relative primes. I am sure, almost for 100%, that I I've seen this idea at my first year group theory course, but at that time I could not see any relation of the equivalence classes to situations like a billiard… I just checked what happens on the odd-odd, odd-even and even-even tables with relative primes, and saw the pattern.

Eventually, Alla formulated the following problem:

1. In continuation of the given situation, try to formulate a rule that would match an ordered [horizontal side/vertical side] pair of table's dimensions to the final pocket the ball enters.

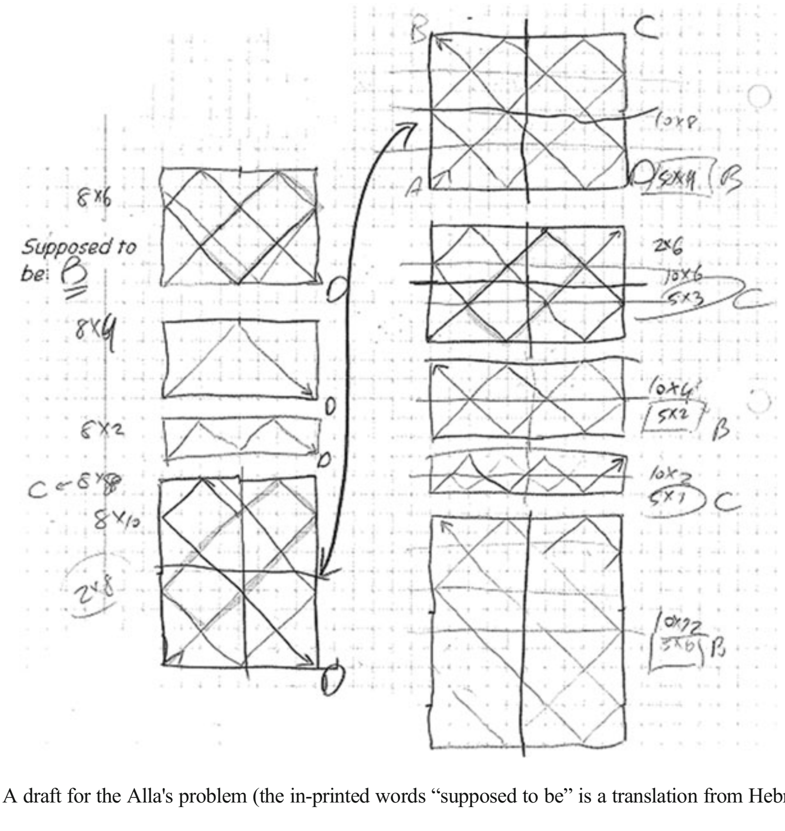


Fig. 5 A draft for the Alla's problem (the in-printed words "supposed to be" is a translation from Hebrew)

In addition, Alla wrote, as a solution, how the tables of all sizes should be divided into three equivalence classes, and indicated which final pocket the ball entered for every class (with reference to Table 1 on Fig. 1, the answer is pocket C for odd–odd tables, pocket D for even–odd tables, and pocket B for odd–even tables). Note that Alla formulated her problem so that it did not require from the solver proving the discovered relationship. Alla knew that she had not found a general proof and explained:

> I did not feel the necessity to prove the rule I had found, because I was quite sure that it would be analogous, if not the same, to the proof that I've seen at the group theory course.

This explanation is instrumental for understanding why Alla was quite delighted by the problem she posed: she assigned herself a challenging goal and succeeded in pursuing it. Simultaneously, she asserted that she was not interested in the "technical" verification of the problem, which already fitted her considerations of aptness to herself. It is interesting to note that Alla's submitted homework included three additional problems and their solutions. Alla generated them after discovering the above rule and formulated so that she could quickly and fully solve them; the problems were somewhat similar to the third and fourth problems by Ian as they all included "unusual" conditions. Alla explained that the problems were not particularly interesting to her, but she included them in her work in order to be sure that it would satisfy the expectations of the evaluators. Thus, some tension between considerations of aptness to the evaluators and considerations of aptness to herself can be observed. In sum, the case of Alla is a success story in the following meaning: she succeeded in posing a problem which, by her considerations of aptness to herself, was challenging and surprising and, thus, interesting.

4 Discussion

The presented success stories occurred with individuals of different mathematical knowledge bases and experience in problem solving and problem posing. Consequently, it is not surprising that their self-attributed criteria of success and the resulting problems appeared to be so different. It is probably more surprising that, despite the differences, certain common traits can be identified in the presented stories. In this section, we attempt to pinpoint the participants' success and draw some implications.

4.1 The role of *uninteresting* problems in posing the interesting ones

Let us begin from considering Ian and Alla's work in terms of the following problem-posing stages: (a) *warming-up*, (b) *searching for an interesting mathematical phenomenon*, (c) *hiding the problem-posing process in the problem formulation*, and (d) *reviewing*. These stages have to be considered as interlacing ones rather than as a linear progression from a problem-posing stimulus to a resulting problem, in line with what is done in some contemporary problem-solving frameworks (e.g., Carlson & Bloom, 2005) and problem-posing models (Ramirez, 2006; Pelczer & Gamboa, 2009).

At the warming-up stage, the problem posers spontaneously associated the given task with particular types of prototypical problems. In both cases, the recalled or quickly formulated initial problems served only as departure points. Indeed, two initial geometry problems posed by Ian and most of the questions on Alla's list were considered and discarded by them. Moreover, Alla chose not to present the entire list of questions in her submitted homework.

This observation may have a methodological implication. Namely, in many studies on problem posing, the overall number of problems posed by an individual or a group in response to a given problem-posing stimulus is frequently reported as an informative datum (e.g., Cifarelli & Cai, 2005; Harel et al. 2006; Silver et al., 1996; Toluk-Uçar, 2009). This datum, may, however, be of minor importance in situations where posing interesting problems is put forward (see also Kontorovich, Koichu, Leikin, & Berman, 2011).

At the searching for an interesting mathematical phenomenon stage, Ian and Alla restricted their attention to selected aspects of the given task and looked for the mathematical foundation of the forthcoming, interesting, problem. At this stage, the initial (prototypical) problems served for the students as a material to be critically considered and modified.

Ian's search seems to be driven by a modification of chaining strategy. To recall, Silver et al. (1996) defined chaining as expanding an existing problem in a way that a solution to a new one would require to solve an existing one first. In contrast, Ian's aspiration was not to (straightforwardly) use the solutions of his early problems, but rather to achieve in the next problems what he had not achieved so far.

Alla's search was essentially shaped by the decision to explore the most interesting to her question without being sure that she would succeed. It is worthwhile noting that such a course of action required from the poser both self-confidence and intellectual courage. According to Movshovitz-Hadar and Kleiner (2009), such intellectual courage is an important component of mathematical creativity.

At the hiding the problem-posing process in the problem formulation stage, Ian and Alla wrote down the problem formulations so that the processes they went through when posing/solving the problems became not transparent to the potential solvers. Indeed, the formulation of Ian's last problem did not reflect the fact that he had developed the general formula for α in order to choose an initial angle so that the final angle would be nice. The formulation of Alla's problem contained a hint about the importance of the order in which dimensions of the tables are given, but it was not transparent about ways of looking for the pattern she went through. To our knowledge, the phenomenon of hiding the process of problem posing in a problem formulation has not been pointed out in past research on problem posing by students or teachers.

At the reviewing stage, Ian and Alla shaped their opinions of the posed problems. Ian lowered his opinion about the problem after testing it with a peer. Still, he considered the problem instructive for middle school students. Alla liked her problem very much, but assumed that the evaluators may not be pleased by her problem. As a result, she compensated for the absence of the general proof to the most interesting (to her) problem by formulating and fully solving three less interesting to her problems. In other words, Ian and Alla's courses of actions imply that different types of considerations of aptness do not always come alone (e.g., aptness to the potential solvers and aptness to potential evaluators).

4.2 Interesting problems—in response to an interesting task given in an appropriate context

A rationale for using problem-posing activities in mathematics teachers' education relies, in part, on the following chain of considerations: (a) problem posing is a powerful means of learning and teaching mathematics, as it is an authentic mathematical activity (e.g., Kilpatrick, 1987) related to enhancing mathematical creativity (Silver, 1994), developing problem-solving skills, mathematical aptitude, and learning autonomy (Silver & Cai, 1996); (b) successful use of authentic mathematical activities in teaching requires that the teachers would gain experience with such activities for themselves as learners of mathematics (e.g., Crespo & Sinclair, 2008; Putnam & Borko, 2000; Wilson & Berne, 1999). Consequently, in-service and prospective mathematics teachers should be offered opportunities to pose their own problems.

This logical chain, as compelling as it is, cannot be a warrant against poor problem-posing classroom practices if and when the intended tasks fail to pass the criteria of an authentic mathematical activity. This happens, for instance, when the teachers are engaged in problem-posing situations, which are devoid of connection to what mathematics practitioners actually do when posing problems (cf. Brown, Collins, & Duguid, 1989) or when the situations do not evoke in the teachers establishing any personal relationship with the posed problems (Crespo & Sinclair, 2008; Putnam & Borko, 2000).

From this perspective, the success of Ian and Alla is stipulated by the fact that students succeeded in establishing personal relationships with the task and to make their work progress beyond what was needed in order to formally fulfill it. In terms of the proposed theoretical framework, Ian and Alla's problem-posing reasoning involved considerations of aptness to themselves and to the potential solvers and not only to the evaluators. This phenomenon—posing problems that would be interesting to solve also to the poser—was rarely observed in the past studies. (The study of Perrin, 2007, in which the students posed problems for follow-up teacher-guided investigation, is an exception.) For instance, Crespo and Sinclair (2008) reported that mathematics teachers in their study posed many tasty problems, but only in one case were the problem posers were curious about how to solve their own problem.

We also deem important that the task was not isolated from the context, in which it was proposed. To recall, it was offered in the framework of a course, in which problem posing was one of the central themes. We deem particularly important that the participants got familiar with experience of three problem-posing experts and had a chance to learn that for the experts the aspiration to pose interesting problems for real use (in contrast to posing many problems for no further use) was an established norm.

5 Concluding remarks

Our first remark concerns the adequacy of the chosen conceptual framework and the second one is on a possible pedagogical implication of our findings.

5.1 A methodological remark

As has been hopefully evident from the above exposition, narrative in terms of the problem posers' knowledge bases and considerations of aptness was instrumental for (partially) capturing the complexity of the processes involved in posing interesting mathematical problems. At the same time, we found that the analysis in terms of problem-posing strategies was unequally applicable to the cases under consideration. Problem-posing strategies, as considered in this paper, were apparent in the entire work of Ian and only at the beginning of the work of Alla. A similar observation can be made regarding the distinction between data-driven exploration and hypothesis-driven exploration made by Cifarelli and Cai (2005), when applied to the presented data. Namely, these two categories were applicable only to the case of Alla and only partially captured the processes preceding her insight.

These observations are reminiscent of those of many scholars (e.g., Cifarelli & Cai, 2005; Mamona-Downs & Downs, 2005; Singer et al., 2009) pointed out that further work should be done towards the development of a system of categories that would be sensitive to the subtlety of the problem-posing processes and simultaneously applicable to a broad range of problem-posing situations and tasks. For instance, problem-posing strategies identified in past research in the situations where the subjects were asked to generate many questions under strict time constraints, as in the study by Silver et al. (1996), may be unequally applicable in the situations requiring from the subjects to pose a small number of interesting problems under less strict time constraints, as in our study.

5.2 A pedagogical remark

The undertaken dissection of two success stories may have some pedagogical implications. We observed that posing interesting, first of all to the posers, problems involved searching for an interesting mathematical phenomenon and hiding the problem-posing process in the problem formulation stages. It is reasonable to assume that problem-posing tasks formulated so that students or teachers would be given a chance to experience these stages may result in posing better problems. To this end, our findings support Crespo and Sinclair's (2008) and Perrin's (2007) suggestion that problem-posing tasks should not be separated from mathematical explorations. We also suggest that the request to pose a small number of problems that would be interesting to solve either for the problem poser or his or her peers should be reinforced in future problem-posing tasks. More generally speaking, our small-scope study suggests, in line with some past studies, that attention to the authenticity of situations and contexts, in which problem-posing tasks are offered, may encourage learners to become better problem posers.

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