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Differences between experts' and students' conceptual images of the mathematical structure of Taylor series convergence

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Abstract Taylor series convergence is a complicated mathematical structure which incorporates multiple concepts. Therefore, it can be very difficult for students to initially comprehend. How might students make sense of this structure? How might experts make sense of this structure? To answer these questions, an exploratory study was conducted using experts and students who responded to a variety of interview tasks related to Taylor series convergence. An initial analysis revealed that many patterns of their reasoning were based upon certain elements and actions performed on elements from the underlying mathematical structure of Taylor series. A corresponding framework was created to better identify these elements and how they were being used. Some of the elements included using particular values for the independent variable, working with terms, partial sums, sequences, and remainders. Experts and students both focused on particular elements of Taylor series, but the experts demonstrated the efficiency and effectiveness of their reasoning by evoking more conceptual images and more readily moving between images of different elements to best respond to the current task. Instead of moving between images as dictated by tasks, students might fixate on "surface level" features of Taylor series and fail to focus on more relevant features that would allow them to more appropriately engage the task. Furthermore, how experts used their images, supports the idea that they were guided by formal theory, whereas students were still attempting to construct their understanding.

Keywords Taylor series · Calculus · Advanced mathematical thinking · Grounded theory · Focal analysis · Qualitative methods · Expert and novice · Pseudostructural

In contrast to the extensive amount of education literature on the limit of functions and sequences (e.g., Alcock & Simpson, 2004, 2005; Mamona-Downs, 2001; Monaghan, 1991;

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Williams, 1991), there is a dearth of research specific to students' understanding of power series convergence. In most cases, a study addressing power series convergence does so while considering a broader topic, such as limit or function approximation techniques (Alcock & Simpson, 2004, 2005; Kidron, 2004; Oehrtman, 2009). Of the power series convergence studies, attention focuses on students' understanding of various convergence tests (Kung & Speer, 2010), the influence of dynamic graphical images on learning (Kidron & Zehavi, 2002; Kidron, 2002, 2004), and the effects of metaphorical reasoning (Martin & Oehrtman, 2010).

As a special case of power series, Taylor series brings together the concepts of function, limit, derivative, sequences, and series, and typical tasks force students to juggle topics such as error, upper bounds, interval of convergence, radius of convergence, and center. Taylor series are frequently used in physics and engineering to simplify complicated equations using approximations and they play a foundational role in the theory of complex analysis. Calculus students are typically introduced to function approximation techniques using Taylor series, and a single approximation task may require students to coordinate changes to an independent variable, the center, and index, while interpreting the role of the unspecified variable if using Lagrange's remainder formula. Therefore, for a calculus student first encountering Taylor series, it can be especially challenging to make sense of this complicated structure due to the interaction of all these concepts. Indeed, after observing students' struggles with power series convergence, Kung and Speer (2010) titled their paper, "Do they really get it?" Therefore, the study described herein explores the question, "What do students get?" and, "How does this compare to what experts get?" More specifically, when students are from traditional calculus classes (as opposed to Kidron's class), what are students' patterns of reasoning concerning Taylor series convergence when they are not guided to reason visually or forced to recall complicated series convergence tests? Finally, how do students' patterns of reasoning compare to experts' patterns?

1 Background

1.1 Operational verses structural understandings

The importance of the duality between conceiving of mathematical concepts as objects and as processes performed on and with objects has been well documented (Breidenbach et al., 1992; Gray & Tall, 1994; Sfard, 1991, 1992). Sfard described *structural conceptions* as conceiving mathematical notions "as if they referred to object-like entities" and *operational conceptions* as conceiving mathematical notions as computational processes acting on previously established objects (1992, p. 60). For Sfard, the transition from operational to structural conceptions occurs through *reification* when a process becomes viewed as an object, and hence, processes can be performed on the newly structured object, which can eventually be detached from the process that produced it. The notion of limit conceived as process is embodied in dynamic language and can be associated with ideas of unreachableness, such as $0.\overline{9}$ cannot equal 1 because the *process* of concatenating 9's to get 0.9, 0.99, 0.999, etc. can never really end (Davis & Vinner, 1986; Williams, 1991). Fortunately, the process of taking a limit can be reified into an object, and the limit of a sequence, like {0.9, 0.99,0.999,…}, can be obtained.

Sfard (1992) also observed that students' conceptions can be neither clearly operational nor clearly structural but both partial in operation and incomplete in structure, calling such conceptions *pseudostructural*. When knowledge is detached from a "previously developed system of concepts," student reasoning using such knowledge is pseudostructural (Sfard & Linchevski, 1994, p. 117). Symptoms of pseudostructural reasoning include regarding formulas and symbols as things that do not stand for anything else, making little to no connection between graphs and algebraic formula, and being inflexible in utilizing the appropriate structural interpretation at the proper time (Sfard, 1991, 1992; Sfard & Linchevski, 1994). Zandieh (2000) claimed that "a pseudostructural conception may be thought of as an object with no internal structure" (p. 107). Zandieh went on to purport that the thinking of a person reasoning using a pseudostructural conception should not be viewed as negative because a more complete conception may be available even though it was not currently evoked. Furthermore, not referring to the full structure may help an individual to reason more efficiently by not having to attend to so much detail (Martin & Oehrtman, 2010).

1.2 Power series

Studies suggest that depending on the problem situation, students will attend to and operate with different elements of the underlying mathematical structure of Taylor series convergence, such as terms, polynomials, the center, and errors. Kidron (2002, 2004) and Kidron and Zehavi (2002) describe a teaching experiment explicitly designed to support students in making connections between algebraic representations and dynamic visual images of Taylor polynomials created using a CAS (Computer Algebra System). Students noticed things such as "the higher degree of the approximating polynomial, the bigger is the interval in which $f(x)$ and the polynomial coincided" (Kidron, 2004, p. 318), and that "the error decreases" as approximations get "better" (Kidron & Zehavi, 2002, p. 220). Moreover, Kidron's approach was designed to help students reify the limit process in this context instead of concentrating on the "divergent process of adding terms" (Kidron, 2002, p. 210). Taylor polynomials were operated in a process that for some students reified into an infinite series as they coordinated both algebraic and dynamic graphical representations of convergence. Similar attempts to support reification of the limit of remainder were made as well. Although other process/objects conceptions can be seen in excerpts, process/object conceptions related to other components of Taylor series convergence were not well explored.

Kung and Speer (2010) indicated that students can have great difficulty when reasoning about Taylor series to the extent that they can reach a state of "cognitive overload" on fairly standard tasks:

> Throughout the interviews, students took a very mechanical view of testing series, at times working through the steps of a particular test without being able to clearly articulate the underlying ideas, what they were testing for, or at times, even what the conclusion of the test was. (p. 9)

On the few occasions where Alcock and Simpson (2004, 2005) specifically referenced student responses to their Taylor series task, students struggled. Alcock confirmed that they "reported very little about students' responses to this question, in part because many students were simply confused about what the question was asking" (as cited in Kung & Speer, 2010, p. 3).

Students may try to reduce cognitive strain by using metaphor (Oehrtman 2009), but in so doing, neglect key features of Taylor series convergence (Martin & Oehrtman, 2010). By using tasks assessable to individuals from diverse mathematical backgrounds, this paper attempts to add to this body of literature by further detailing expert and student conceptions based on different elements of the underlying structure of Taylor series convergence. In so doing, differences between how experts and students identify, relate, and use such conceptions may shed light on the source of the confusion observed by Alcock.

1.3 Expertise

An expert in mathematics has been described as having a more robust mental imagery, more numerous images, ability to switch efficiently and effectively between different images, ability to focus attention on appropriate features of problems, and more cognizance of their thought and of how others may think (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994). According to Hiebert and Carpenter (1992), individuals with a coherent understanding of a particular mathematical topic have a complex system of internal and external representations that are joined together by numerous strong connections to form a network of knowledge. In contrast to experts, a student's system of representations of a mathematical topic may be deficient in number and deficient in connections to form an adequate network of knowledge (Hiebert & Carpenter, 1992; Lester 1994). Bezuidenhout (2001) observed that calculus students' understanding of limit, continuity, and differentiability "rests largely upon isolated facts and procedures, and that their conceptual understanding of the relationships between these concepts is deficient" (p. 498). Ferrini-Mundy & Graham (1994) report that "graphical contexts and algebraic contexts may function for students as separate worlds," where algorithms are seen as unrelated to corresponding graphical representations (p. 42). Both the deficient conceptual understandings and the disconnect between graphical and algebraic contexts can stem from pseudostructural reasoning (Sfard, 1991, 1992; Sfard & Linchevski, 1994; Zandieh, 2000).

In other scientific domains, experts are consistently identified as having a lot of knowledge that is well connected and influenced by underlying scientific theory whereas students may focus on surface level features explicitly stated in a problem that tend to not be well connected to underlying theory (Chi et al., 1981; Kozma & Russel, 1997; Larkin, McDermott, Simon, & Simon, 1980). Even when knowledge is connected to some scientific theory, it can still be incomplete or unconnected to other pieces of knowledge (diSessa 1988; Kozma & Russel, 1997).

2 Framework & methods

2.1 The emergent framework

To allow themes about Taylor series convergence to emerge from the data, open coding (Strauss & Corbin, 1990) was employed (See Layer 1 in Fig. 1). Elements of individuals' concept images (Tall & Vinner, 1981) emerged, and the most prominent images were directly related to components of the underlying mathematical structure and operations that could be performed on those components (Layer 2 in Fig. 1). Therefore, this study adopted Sfard's operational and structural notions of duality as a theoretical perspective. By emphasizing this dichotomous view instead of a trichotomous view (e.g. Briendenbach et al., 1992), further distinction was made between operational and structural conceptions.

These images consisting of both operational and structural elements will be referenced as *structural images* of Taylor series convergence. The calculation of an approximation may be viewed as a process in which more terms are iteratively added to a Taylor polynomial to produce a new polynomial that better approximates the function. In this case, the terms themselves are used as objects concatenated onto the current Taylor polynomial. In another situation, Taylor polynomials may be viewed as objects that are a part of a sequence $\{T_n(x)\}$ used to verify that the Taylor series converges to a given function. Therefore, a structural image can refer to multiple components of Taylor series convergence in which each component may be conceived operationally or structurally. These situations also suggest that individuals might make references to terms, partial sums, and sequences of partial sums

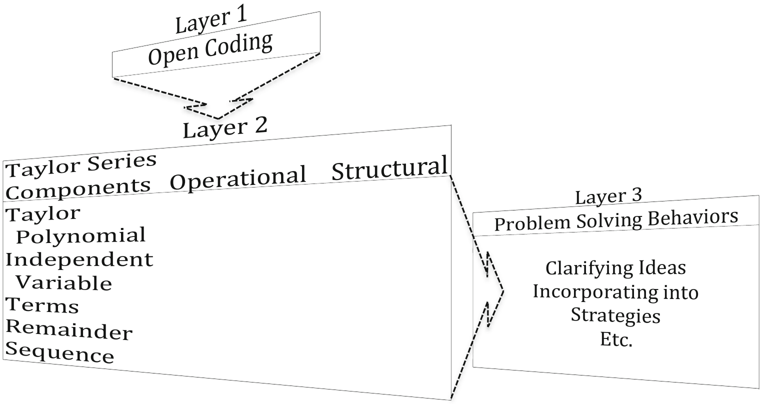

Fig. 1 Framework for analyzing understandings of Taylor series convergence based on identifying structural images (*Layer 1*), operational and structural elements of those images (*Layer 2*), and roles in problem solving behaviors (*Layer 3*)

without attending to the details of other components of the underlying mathematical structure, such as the remainder, and in so doing, demonstrate a pseudostructural image.

To reveal how participants were using their structural images, Carlson and Bloom's (2005) *multidimensional problem solving framework* (Layer 3 in Fig. 1) was adopted. They describe "formal and informal knowledge about the content domain, including facts, definitions, algorithmic procedures, routine problems, and relevant competencies about rules of discourse" (p. 48) as resources that individuals use when engaging in difficult problems. Therefore, elements of an individual's concept image constitute resources for that individual. They also listed problem solving behaviors that included accessing resources, strategizing, and clarifying ideas as one moves though phases of planning, executing, and checking. Thus, the effectiveness of one's resources are dependent upon a degree of control one has to access appropriate resources at appropriate times. Therefore, an individual's ability to move between different structural images to address tasks meaningfully would demonstrate a type of "well connectedness" of understanding frequently displayed by experts.

The frameworks discussed in this section constitute multiple layers of a Taylor series framework comprised of structural images of convergences and how images are used. This framework is used to reveal what an individual understands and how he/she uses such understandings about Taylor series convergence on a component-by-component basis. Using this framework, the research questions are refined to the following:

1. What are some different ways that spontaneously emerge in which experts and students conceptualize Taylor series convergence relative to the underlying mathematical structure and operations performed on that structure?
2. In what ways do experts and students use these conceptions of convergence and how might these conceptions aid or hinder individuals in basic proof and estimation tasks involving Taylor series convergence?

2.2 Method

At a mid-size, four-year university and a regional community college, data were initially collected from five faculty and two graduate students who had either taught or used series in their research. Subsequently, data were collected from 131 undergraduate students in calculus and introduction to numerical or real analysis. Data consisted of questionnaires and audio/video recordings and written work during individual interviews. Pseudonyms are used in the reporting of data.

Faculty and graduate students participated in no more than three 60-min, tasked-based, individual interviews (Goldin, 2000), and most completed all tasks in two interviews. Both faculty and graduate students are identified as experts since there was no notable difference in their reasoning. Experts were chosen to go first because conceptual resources at their disposal should elucidate concept images in ways not previously considered (Carlson & Bloom, 2005), and their perception of student thinking could help with designing the next phase.

After having previously studied Taylor series using books by Stewart (2008) or Hass, Weir, and Thomas (2007) to solve typical approximation tasks, 103 calculus students completed a questionnaire. They were introduced to proofs using Taylor's inequality (Stewart, 2008, p. 773) or the Lagrange remainder formula (Hass et al., 2007, p. 560). Also, 16 students from introduction to real analysis completed the questionnaire after they had covered sequences and series (Bartle & Sherbet, 2000, pp. 52–95), and 11 students from introduction to numerical analysis completed the questionnaire after they had revisited approximation tasks typically seen by calculus students (Burden & Faires, 2005). Both real and numerical analysis students are reported together as "analysis" students to emphasize the distinction between calculus students and students from more advanced mathematical backgrounds.

The questionnaire consisted of 33 questions of various types. Some of the most informative questions concerned graphing Taylor polynomials for sin x given sine's graph and questions related to structural images where students could select multiple images that they perceived themselves as employing (see Appendix). Five calculus, one real analysis, and two numerical analysis students additionally participated in no more than two 60-min interviews. This paper will focus on the analysis and discussion of interview tasks consistent between the expert and student groups (Table 1) and will refer to other results as needed.

Many of the interview tasks were designed to be accessible to all the participants (Goldin, 2000) by not constraining participants to using difficult to recall procedures, such as the ratio test or the Lagrange Remainder Theorem (Kung & Speer, 2010). Open form questions allowed for students to be more spontaneous to the extent that they were less constrained to

Table 1 Common interview tasks between expert and student groups

Number	Task
1.	What are Taylor series?
2.	Why are Taylor series studied in calculus?
3.	What is meant by the "=" in "cos $x=1-x^2/2!+x^4/4!-x^6/6!+-\cdots$ when x is any real number?"
4.	What is meant by the "(−1,1)" in, "$1/(1-x)=1+x+x^2+x^3+\cdots$ when x is in the interval (−1,1)?"
5.	What is meant by the word "prove" if you were asked to, "Prove that sine is equal to its Taylor series?"
6.	What are the steps in proving that sin $x=x-x^3/3!+x^5/5!-x^7/7!+-\cdots$?
7.	How can we estimate sine by using its Taylor series?
8.	What is meant by the "approximation" symbol in "sin $x\approx x-x^3/3!$ a Taylor polynomial for sine when x is near 0?"
9.	What is meant by the "near" in "sin $x\approx x-x^3/3!$ a Taylor polynomial for sine when x is near 0?"
10.	How can we get a better approximation for sine than using $x-x^3/3!$?

evoke particular structural images based on instructor or visual cues as was the case in Kidron's work. This allowed for more discussion about Taylor series convergence than previously observed by Alcock. While addressing the first four tasks, participants would typically make reference to "limit" or "convergence," at which point the interviewer would probe for more detail of their understandings. Participants were asked to respond to simple proof and approximation problems in Tasks 6, 7, and 10 to capture how they were using their structural images in basic problem solving situations. Since in calculus, the approximation and proof questions are more commonly asked of Taylor series than power series, using Taylor series terminology was more consistent with instruction.

2.3 Analysis

Data were analyzed using the layers in Fig. 1. Once concept images emerged though the first layer of analysis, mathematical structures that were the focus of an individual's attention were identified using key expressions revealed in utterances, written word and symbols, or in the form of gestures while engaged in discourse. Using this notion of expression, Sfard (2001) defined the pronounced focus as the "expression used by [the participant] to identify the object of her or his attention," and the attended focus as what a participant is "attending to—looking at, listening to, etc." (p. 34). During interviews, participants were attending to the task in front of them and to certain structural components of Taylor series as indicated by their pronounced focus. To illuminate the operational and structural elements of these images (Layer 2), efforts were made to distinguish between what participants were operating *with* (structural component) versus what they were operating *on* (operational component). To help explain how structural components were being used by individuals (Layer 3), as participants engaged each task, components that were attended to were recorded. To explain reasons behind these conceptual movements, the data were analyzed for indicators of images being coordinated with problem solving behaviors, such as using images to clarify ideas, incorporate into strategies, etc.

3 Results

The emergent components of the structural images included: terms, the independent variable, Taylor polynomials, the expanded Taylor series, and the tail of the expanded series. Descriptions of the images associated with these structural components and what participates were operationally constructing can be found in Table 2. Other components, such as the center of the series, were mostly not attended to by the participant groups.

The dynamic partial sum image was one of the most often used images of both groups and was common amongst all individually interviewed participants in response to different types of approximation questions. Of analysis and calculus students, 61% and 42%, respectively, selected a dynamic partial sum image for convergence on the questionnaire. For approximation tasks, Taylor polynomials were conceived through a process that involved operating with terms to construct a desired polynomial by iteratively adding terms until some condition was achieved. Frequently, the dynamic partial sum image was embodied in chopping motion gestures where every "chop" suggested another term being added to a Taylor polynomial.

> Expert Wallace: Estimating sine by using its Taylor series can be uh, with the particular parts [*holds both hands up as if holding something between*] that we take from Taylor series expansion. For example [*now moves right hand away from left*

Table 2 Common structural images observed between groups

Layer 1		Layer 2	
Image	Description	Structural component	Operational component
Dynamic partial sum	Focus on a single polynomial. Convergence conceived through the action of successively adding terms to a Taylor polynomial.	Terms	Taylor polynomials, and/or limit of Taylor polynomials (Taylor series)
Particular x	Focus on a single value for the independent variable. Convergence conceived through the action of substituting specific values for x, and not on convergence over an interval.	Independent variable	Taylor polynomials, and/or limit of Taylor polynomials (Taylor series)
Sequence of partial sums	Focus on the sequence of Taylor polynomials. Convergence conceived through a limit process of polynomials approaching a function.	Taylor polynomials	Limit of a sequence of polynomials
Remainder	Focus on the difference between Taylor polynomials and approximated function. Convergence conceived through a limit process of remainders approaching zero.	Taylor polynomials and generating functions	Limit of a sequence of remainders
Remainder as Tail[a]	Focus on the "tail" of the expanded Taylor series. Convergence conceived through a limit process of tail approaching zero.	Expanded Taylor series, Taylor polynomials, and tail	Limit of the tail
Termwise[a]	Focus on Taylor series terms. Convergence justified by terms approaching zero.	Terms	Limit of the terms

[a] Only appeared in the student group

> *hand by making chopping motions*], first two terms, first three terms, first four terms, and we can look at the estimation by going that way.

In contrast to the dynamic partial sum image, for the particular x image, terms and polynomials were not conceived merely as formulas, but as formulas evaluated at a certain point, indicated by 38% of calculus and 36% of analysis students. In explaining the meaning of the equality, Expert Dylan initially attended to the action of substituting 37 rad:

> If you want the cosine of uh, 37, or whatever, measured in radians, all you have to do is plug that number into the formula on the right and take it out sufficiently far and you will get an actual decimal approximation for that number.

By being able to operate with this particular value, Dylan indicated that he had conceived of the independent variable as an object. His focus was on what happened to the values of the Taylor polynomials evaluated at a particular value during the process of "taking" the

Taylor polynomial "out." Others might coordinate particular values for x with the entirety of the series.

The least selected image on the student questionnaire, with only 34% of all students indicating, was the sequence of partial sums image. In contrast, experts frequently demonstrated this image in conjunction with the formal definition of series convergence as the limit of partial sums. While responding to what it meant for Taylor series to converge, Expert Dean's focus moved to the sequence structure of convergence, and stated that "for a series to converge, it means for a sequence of partial sums to converge." To reiterate, he then noted, "that's the definition of convergence of series."

Expert Marshal noted that one of the steps for proving that sine's Maclaurin series is equal to sine involved "looking at the difference between what you think it converges to and the n^{th} partial sum and then showing that that error goes to zero." Indicating this remainder image were 40% of calculus and 50% of analysis students. Not seen in the expert group, one interviewed student focused heavily on the remainder as the "trailing terms" in the expanded series. This image materialized when Jordan incorrectly viewed the limit of the "tail" equaling zero as a sufficient condition for Taylor series convergence to a given function. He viewed the Taylor series as an "infinite polynomial" that he could "cut" at any place and produce a polynomial that he could "pick up" and use and a "tail" that "had value" that became closer to zero for larger polynomials. His actions on the expanded series and the Taylor polynomial, and his conception of the tail as having value suggests that he had conceived of these as objects. On the questionnaire, 56% of all students indicated "tail" convergence to zero sufficed to justify Taylor series convergence to a given function.

While all experts quickly recalled the harmonic series to disprove a termwise conception of convergence, few students ever alluded to the harmonic series as a counterexample. Perhaps the most noticeable difference between the expert and student groups was in the high affirmation of the termwise image by the students. As the most selected image on the questionnaire, 58% of calculus and 61% of analysis students indicated that termwise convergence to zero was a valid sufficient condition for determining series convergence. Even though, none of the interviewed students indicated a termwise conception on their questionnaire, during interviews one student stated, "So each figure [term] gets smaller and smaller and smaller and is adding less and less and less until it's basically adding almost nothing… That's what I think convergence is."

3.1 Graphical images

At multiple instances throughout the interviews, experts sketched Taylor polynomial/series graphs to help better explain their reasoning. There was never any indication of experts struggling to produce accurate graphs. In the questionnaire, students were asked to graph Taylor polynomials for sine, given the graph of sin x (Question 5 in the Appendix). Of those students who attempted this task, approximately 30% (21 students) produced graphs of polynomials whose approximations to sin x became progressively more accurate toward an implied center and 60 students (over 45%) did not even attempt this problem. When asked to graph the Taylor series given the graph of sin x (Question 6 in the Appendix), of those students attempting the task, only 36% (20 students) produced graphs that resembled tracing over sine and 75 students (over 57%) did not even attempt this problem. Common incorrect solutions included polynomials that approximated sin x well over a finite interval, or they simply repeated a version of their incorrect response to Question 5.

For Question 5, Student Philip produced two very accurate graphs of cubic and quintic Taylor polynomials. In response to being asked to graph the Taylor series for sine, Philip graphed four approximating polynomials (Fig. 2). From the graph it was unclear if he viewed the Taylor series as the *completion* of all the Taylor polynomials, or if he viewed the series as the *set* of all polynomials. The interviewer drew three individual Taylor polynomials of different orders on separate axes. Philip indicated the series as the collection of all polynomials that were built up in the progression of the graphs. "If you're calling it a series, then it's all [*sweeps his right hand from left to right over the graphs*] of the different ones." His conception is consistent with the sequence of partial sums image with polynomials viewed as objects, but the result of convergence was the *sequence* of Taylor polynomials and not a *function*.

3.2 Differences in problem solving behaviors: Layer 3

Table 3 reveals that the remainder and partial sum structures were heavily relied on by the expert group, while the sequence of partial sums was the least relied upon image. Using Carlson and Bloom's framework, these structural images can be viewed as resources available to the individual. In this section, the focus is on when images were accessed and why they may have been accessed by identifying indicators of problem solving behaviors, most notably appearing were the behaviors of clarifying ideas and incorporating into strategies.

For a closer analysis, consider the case of Expert Dean who relied upon a particular x to clarify his meaning of convergence. He used particular x as a means to relate Taylor series convergence to series convergence, and then he eventually related series convergence to sequential convergence.

> If you just write the series for any x whatsoever it just, it's a function, but for a particular x it becomes a series which has entries real numbers and then the question can become whether the sum actually makes sense in the sense of whether it's a real number or not. If it's a real number then it's, the series converges. Um, but of course convergent, convergence of a sequence is another matter, but we're not talking about that here. I guess I could say, for a series to converge it means for a sequence of partial sums to converge.

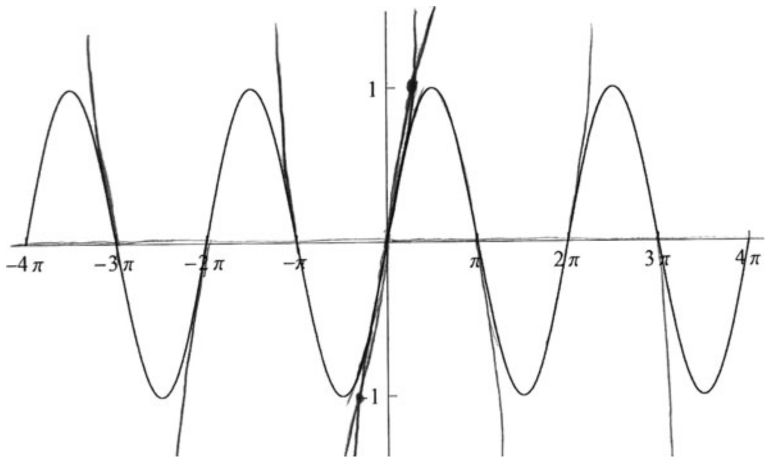

Fig. 2 Student Philip's graph of Taylor series

Table 3 Tallies of experts attending to structures related to structural images

Structure	Interview tasks					
	1 & 2	3 & 4	5 & 6	7	8 & 9	10
Particular *x*	1	6	1	5	3	2
Partial sum	3	7	7	6	4	7
Sequence	0	6	2	0	0	1
Remainder	2	2	7	7	7	5

Expert Dean's response illustrates how experts connected Taylor series convergence to other conceptions, such as sequential convergence, and that they were able to draw upon these connections when talking about Taylor series convergence. The two experts that did not connect Taylor series convergence to convergence of sequences in this context, evoked images of remainder.

Dean did not continue to incorporate his particular x image in response to Tasks 5 and 6. While addressing these tasks about proving Taylor series convergence to sin x, as if by default, all experts responded with a remainder image. In these cases it was a part of a strategy that typically involved 1) using a formula to find the terms of the Taylor polynomials and 2) showing the difference between the Taylor polynomials and the function converges to zero. Dean indicated that he understood that Taylor's inequality yielded a "sequence of upper bounds" for remainders. He also understood, together with the sandwich theorem, that demonstrating that the upper bounds converged to zero sufficed to show that the polynomials converged to the generating function.

References to the remainder or bounding the remainder persisted in expert images as they moved through Tasks 7–9. In response to these tasks concerning the approximation properties of Taylor series, all but one expert alluded to a given error bound as an initial part of a strategy for approximation. For Dean this bound on the error appeared as "level of precision" in the following: "So if you're given a level of precision, from Taylor inequality you can estimate which power of the Taylor polynomial you need and then you can calculate that Taylor polynomial at the value you need."

Finally in Task 10, adding more terms to a Taylor polynomial typified all expert responses to, "How can we get a better approximation for sine than using $x–x^3/3!$?"

Table 4 reveals that students tended to access partial sum and particular x images as resources during interview tasks. Like the experts, all students utilized a partial sum image in response to Task 10, but unlike the experts, only four of eight students accessed a remainder

Table 4 Tallies of students attending to structures related to structural images

Structure	Interview tasks					
	1 & 2	3 & 4	5 & 6	7	8 & 9	10
Particular *x*	1	4	2	5	4	1
Partial sum	1	4	4	7	8	8
Sequence	0	1	0	0	0	0
Remainder	0	2	4	5	2	0
Terms to 0	0	1	0	0	0	0

image in response to Tasks 5 and 6. In fact, a closer look at the data revealed that three students never referenced a remainder image while responding to any task. Noticeably missing from all but one student was a sequence of partial sums image even though three students indicated a sequence of partial sums image on their questionnaire.

For a closer analysis, consider the cases of Student Andy and Student Dave. Like Expert Dean, Andy used his particular *x* image to clarify his understanding and as a part of a strategy. In response to Task 3, Andy immediately referred to substituting in values for *x*, but unlike the experts, Andy did not link Taylor series convergence to sequential convergence. Instead he concluded that the function is "just basically the limit of the series." When asked to elaborate on what he meant by his reference to "limit," he reverted to talking about convergence in the context of a function approaching a horizontal asymptote. Likewise, Dave did not allude to sequential convergence in Task 3, but instead framed his response in infinitesimal language of the series being within "$10^{-\infty}$" of cos *x*.

Both Student Andy and Student Dave, included particular *x* as a part of a strategy to prove Taylor series convergence, whereas no expert included this image in their strategy. Andy initially stated that "this function, the answer if you plug in a certain number, show that that equals the series." Although he alluded to Taylor polynomials becoming closer and differences between polynomials as decreasing as the index increased, he never directly referenced the sufficiency of remainder convergence to zero for proving series convergence. In contrast, Dave did not allude to any remainder formula but only to a vague notion of finding ways to show that things are "very close." In total, three students included a particular *x* image as a part of a strategy to prove convergence, four students alluded to remainder, two of those four alluded to the sufficiency of the remainder converging to zero for series convergence, and only one had a strategy resembling that of the experts.

In response to Tasks 7–9, Andy persisted in substituting in values for *x* to talk about Taylor series in the context of approximations. Even so, Andy was not evoking any formal pointwise conception of convergence because approximations were not dependent upon a given error bound but were simply made accurate by adding more terms to partial sums. This description also characterized three other students, including Dave. The remaining four students, three of which were the three analysis students, expressed a need for a given bound on the error.

Excluding the termwise image, this analysis showed that, to various degrees, both the interviewed expert and the student groups accessed all of the structural images as resources and used images to clarify ideas or incorporate into strategies. Even so, questions that prompted images and how particular images were used were very different between the groups. Experts tended to use their various images of Taylor series convergence in meaningful ways that allowed them to clarify their understanding and present appropriate strategies for addressing tasks. This appeared most prevalent when they explained their understanding of convergence and when they engaged in an explanation for proving Taylor series convergence. For the former, experts tended to connect their understanding to Taylor series convergence, to series convergence, and then to sequence convergence. For the later, all experts produced a strategy that involved demonstrating remainder convergence to zero as a sufficient condition for series convergence. In contrast, most students lacked the control that experts had when reasoning about Taylor series convergence. For example, Student Andy used an image and continued to use that image in response to almost all tasks even though that image was not directly appropriate for all tasks.

Per participant, experts averaged 13.3 and students averaged 10.4 instances of focusing on these structural images during interview questions, including follow-up questions. For some tasks, students "drew a blank" and were not be able to make progress, and other times, student responses were based on informal "approaching" type language without details specific to Taylor series. Furthermore, when experts evoked an image while responding to a question, 52.7% of the time the expert would also evoke an additional image in response to the same question. In contrast, students used multiple images in only 30.1% of the instances where they had already evoked an image in response to the same question.

4 Conclusions and discussion

This paper presented an exploratory study designed to reveal different ways in which experts and students conceptualized Taylor series convergence by identifying and categorizing particular reasoning patterns. In particular, two main research questions were posed:

1. What are some different ways that spontaneously emerge in which experts and students conceptualize Taylor series convergence relative to the underlying mathematical structure and operations performed on that structure?
2. In what ways do experts and students use these conceptions of convergence and how might these conceptions aid or hinder individuals in basic proof and estimation tasks concerning Taylor series convergence?

This study adapted Sfard's (1991, 1992) operational and structural notions of duality to introduce the idea of describing concept images of Taylor series convergence in terms of focus on underlying structural components of Taylor series and operations performed on those components. Together, the structural components and the operations constituted what I called structural images. In addition, to unveil how experts and students were *using* these structural images, this study introduced Carlson and Bloom's (2005) problem solving framework coordinated with the observed structural images. These frameworks, merged as one (see Fig. 1), revealed differences between experts' and students' conceptions and how these conceptions were used.

4.1 Expert and student reasoning using structural images

Structural images that emerged were identified as particular x, dynamic partial sum, sequence of partial sum, remainder, remainder as tail, and termwise (see Table 2). Indicators of most of these images have appeared in other literature (e.g., Kidron, 2002, 2004), but the operational and structural components have not been detailed for all of these images. Depending on the structural image, different structural components were conceived as objects with which to be operated. Movements between images (indicated by Layer 3) at one moment, had participants conceiving of a particular component of a structure being formed by a process and in the next moment, conceived as an object. For example, Taylor polynomials created through a process in the dynamic partial sum image were conceived as objects in a sequence of partial sums image. Furthermore, many of these images appeared formulaically driven, especially for students. In the dynamic partial sum image where terms were progressively added to a Taylor polynomial *formula*, the terms as formulas were operated on like "object-like entities" (Sfard, 1992, p. 60). Likewise, for Student Jordan, he performed the operation of "cutting" on an expanded Taylor series *formula* and then "picked up" the Taylor polynomial *formula*.

Student inability to draw graphs further supports the conclusion that student conceptions of terms and Taylor polynomials may be mostly formulaic. Hence, an image like the dynamic partial sum image may be much more akin to a frequently used algorithm than an indicator of some deep understanding of convergence. This would be consistent with Ferrini-Mundy & Graham's (1994) findings of calculus students failing to connect graphical and algebraic representations in other limit contexts. Additionally, the dynamic partial sum image also suggests a process view of limit that can contribute to an unreachable conception (Davis & Vinner, 1986; Kidron, 2002; Williams, 1991) which could explain why students who had not experienced Kidron's dynamic graphs were unable to draw Taylor *series* graphs even when given the graph of the generating function.

The third layer of analysis demonstrated experts as systematic in their use of sequence of partial sum and remainder structural images in response to clarifying convergence and incorporating into strategies for proof tasks. Their ability to focus their attention on appropriate problem features is further supported when considering how experts switched to using a remainder image when addressing proof tasks, no matter what image they had been previously using. In contrast, most students were not reasoning with the systematicity seen in the expert group. Individual students fixated on certain structural images across multiple tasks and used that image in ways no expert did. For example, students used particular x as a part of a proof strategy. Idiosyncratic variations of images, such as series as "all" partial sums, also demonstrated lack of systematicity within the student group. Furthermore, the observed disconnect between the harmonic series and its implications on convergence indicated a lack of connection to theory which may have contributed to the students' termwise conception as one of the most selected images in the questionnaire. All of these findings are consistent with prior expert/novice literature (e.g., Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994).

Students' high usage of the dynamic partial sum, particular x (during interviews), and termwise (on the questionnaire) structural images suggests students' comfort with using these structures as one would use a "surface level" feature (Chi et al., 1981; Kozma & Russel, 1997). These images of Taylor series convergence can be viewed as "surface level" in the sense that the objects they reference, the independent variable and term, are readily available in any expanded Taylor series formula. Even Student Jordan's remainder as "tail" image was oriented toward an expanded Taylor polynomial *formula* that does not necessarily need to involve the coordination with the approximated function. The high percentage of students' selections of the remainder as "tail" supports the likelihood of this image as a surface level feature. In contrast, the remainder and sequence of partial sums structural images can be viewed as "deeper" since these images involve the coordination of polynomials with the approximated function, or a *set* of Taylor polynomials. The low percentage of students' selections of these images supports the likelihood of these images as "deeper" images that are less accessible for students.

The students' use of "surface level" structures, their fixations on images, their inflexibility to move between images at appropriate times, and their use of formulas that are disconnected from graphs are all symptoms of pseudostructural reasoning (Sfard, 1991, 1992; Sfard & Linchevski, 1994; Zandieh, 2000). For many students their concept image may not be much more than what was evoked as some students never evoked certain images during any task. Therefore, student conceptions of convergence may be truly pseudostructural and thus, be incomplete and ill-connected (diSessa, 1988; Kozma & Russel, 1997). These types of pseudostructural reasoning by the students could hinder their ability to reason rigorously about Taylor series convergence and contribute to the confusion observed by Alcock and Kung and Speer (2010). Although pseudostructural reasoning can contribute to confusion, it should not exclusively be viewed in the negative (Zandieh, 2000). Just by using structural images, by focusing on particular components and not attending to all the details of Taylor series convergence, experts demonstrated a type of pseudostructural reasoning that allowed them to manage tasks without having to continually coordinate all aspects of convergence. Pseudostructural reasoning can allow someone like Student Philip to produce Taylor polynomial graphs very quickly based more on correct notions of proximity and shape than on meticulously coordinating the roles of x and y for each polynomial.

4.2 Limitations and questions

Because of the relative ease of the tasks, participants, especially experts, did not move much beyond the planning or executing phases as described in Carlson and Bloom (2005). Yet these tasks still revealed notable differences between experts and students, and student struggles suggest that for some students these tasks were not so easy. Additionally, this study focused on those evoked images that were related to the underlying mathematical structure of Taylor series convergence. This does not mean that there cannot be other elements of concept images related to Taylor series convergence that are not directly related to the underlying mathematical structure, such as images based on metaphorical reasoning (Martin & Oehrtman, 2010). Furthermore, some structures specific to Taylor series, such as the role of the center, were not accounted for by the current set of participants. This does not mean that a different group would not demonstrate a different focus. Fortunately, the first layer of the framework would account for different structures that emerged.

The results and limitations from this exploratory study suggest several questions. How are these structural images used when engaging in more complicated problem solving activities and to what extent do these images aid or hinder progress? Kidron (2002, 2004) and Kidron and Zehavi (2002) have already demonstrated how a CAS may be used to develop students' understandings and demonstrated the importance of connecting algebraic representation to graphical interpretations during the learning process. How else might these structural images develop and in what ways are notions of Taylor series convergence assimilated into previous knowledge schemas, such as limit or function schemas? These questions can be addressed with well designed studies that have students engaged in more problem solving activities, that closely follow students through the learning process of calculus topics leading up to and including Taylor series, and that have students consistently use and interpret power series graphs.

4.3 Conclusion

Taylor and power series are topics that have only recently come into the scrutiny of mathematics education research. This type of framework can be used to take note of what a student understands and how he/she uses such understandings about power series convergence on a component-by-component basis. It is also conceivable that adaptations to this framework could be beneficial for identifying relevant structural images and how these images are used in other mathematical notions involving multiple concept interactions.

This study indicates that the gap between expert and novice reasoning in the case of Taylor series convergence exists primarily in graphical depictions and proper selection and use of structural images, in particular non-surface level images. By better comprehending what constitutes the gap between expert and students, we can design curriculum to help make explicit those things that are a hallmark of expertise but students tend to miss.

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