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A problem-solving conceptual framework and its implications in designing problem-posing tasks

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Abstract The links between the mathematical and cognitive models that interact during problem solving are explored with the purpose of developing a reference framework for designing problem-posing tasks. When the process of solving is a successful one, a solver successively changes his/her cognitive stances related to the problem via transformations that allow different levels of description of the initial wording. Within these transformations, the passage between successive phases of the problem-solving process determines four operational categories: decoding (transposing the text into more explicit relations among the data and the operating schemes, induced by the constraints of the problem), representing (transposing the problem via a generated mental model), processing (identifying an associated mathematical model based on the mental configurations suggested by the problem and own mathematical competence), and implementing (applying identified mathematical techniques to the particular situation of the problem, with the purpose of drafting a conventional solution). The study of this framework in action offers insights for more effective teaching and can be used in problem posing and problem analysis in order to devise questions more relevant for deep learning.

Keywords Problem posing · Problem solving · Decoding · Representing · Processing · Implementing · Multiple choice questions

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1 Rationale

In everyday life, people naturally solve problems in order to satisfy various needs. Problem solving (PS) is also practiced in school; in fact, one of the major goals of school education is to train students for solving problems pertaining to a variety of domains. Unlike the problems in the school context, everyday life issues are much less structured. Thus, while a “school problem” usually contains explicit assumptions and restrictions, everyday life problems require identification and reformulation to be tackled with the resources at hand. Therefore, in everyday life it is not only useful to solve problems but the capacity to synthesize the complexity of situations seems to be important as well, in order to anticipate possible problems. Actually, identifying such problem situations, assessing the possible implications, and finding solutions to avoid various consequences or at least mitigate their impact on the environment or the person can sometimes be of vital importance.

How can school contribute to the development of students' abilities in this regard? Conversely, can peoples' natural ability for identifying and reformulating real-life problems be valued in school? A working hypothesis of an extensive research project (see, e.g., Singer, Pelczer & Voica, 2011; Voica & Singer, 2012) is that we can meet these desiderata by organizing problem-posing (PP) activities in school.

There are different terms that are used in reference to problem posing (PP), such as problem finding, problem sensing, problem formulating, creative problem-discovering, problematizing, problem creating, and problem envisaging (Dillon, 1982; Jay & Perkins, 1997). In all these meanings of the term, PP involves extracting/identifying a (new) problem from the multitude of data or information available to the proposer. More recent studies (e.g., Brown & Walter, 2005) show that problem posing is deeply embedded in problem solving. Is PP ability determined by PS ability? Trying to see how these abilities are correlated, we surprisingly found that relatively many teachers—some of whom are experienced problem solvers—do not seem to have developed skills in problem posing (Pelczer et al., 2011; Singer, Ellerton, Cai & Leung, 2011).

Further, is there any need to have teachers who are able to pose problems? Rowland, Huckstep, and Thwaites (2003) consider that in the teaching practice, teachers should display PP competence, at least reformulation of a given wording, in order to adapt it to an educational purpose. We also assume that teachers should have the capacity to pose or modify problems in order to get relevant wordings for the students' learning. Given the fact that teachers are in the position of organizing learning activities, they inevitably pose tasks for students and, moreover, this capacity seems natural. For example, in a study with 29 prospective teachers, Silver, Mamona-Downs, Leung, and Kenney (1996) revealed teachers' spontaneous personal capacity for mathematical problem posing. Although this natural predisposition for PP exists in teachers, it should be adequately trained in order to achieve techniques for effectively using it in the teaching practice.

Therefore, there are many reasons to assume that in order to ensure the success of PP tasks in the classroom, pre-service and in-service teacher training programs should contain problem-posing sessions. This does not usually happen, as the TIMSS studies have shown (Noveanu & Singer, 2008). Consequently, we have started a research program focused on the identification of ways to train teachers better for developing PP activities. Thus, we looked for a general framework for the problem-solving process that could generate a mental tool able to be rapidly mobilized in PP contexts. The present study focuses on the conceptual resource that such a program should be provided with. Consequently, the main research question of this study is: What cognitive characteristics of the problem-solving approach might help teachers to better design learning tasks for students in mathematics school contexts? And, even more specifically: Is there any PS conceptual framework as an effective tool for teacher training programs to address this need?

There are many papers, some classical (e.g., Pólya, 1945; Bloom & Broder, 1950; Duncker, 1945; Silver, 1985), concerning mathematical problem solving. There are also various models describing PS. Why would it not be sufficient to use one of them? Pólya's model (Pólya, 1945) on the stages of solving (understanding the problem, drawing a plan, plan implementation, retrospective look) is focused on describing how teachers can help students develop skills in PS. From Pólya's view, the teaching (didactical) component is essential. For this reason, the explanations differ depending on the target domain (algebra, geometry, etc.) and on the level of difficulty of the problem. However, we look for a model describing stages the solver him/herself crosses while solving a problem, with a focus on cognitive processes and not on the teaching ones.

Another classical model is Newman's error analysis hierarchy used for detecting students' errors in PS (Newman, 1977). In this model, PS is described as the result of the following series of sub-activities: read the problem; comprehend what is read; carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy; apply the process skills demanded by the selected strategy; encode the answer in an acceptable written form. This model has a more developed cognitive component, but its construction is limited by the target, that of the description of solving standard one-step word problems. The model was successfully applied in primary education (Newman, 1983), as well as in lower secondary classes (Ellerton & Clements, 1992). We seek a more general model, applicable to a variety of populations and contexts, which incorporates the cognitive processes activated when solving a problem. While Pólya's PS framework is described in behavioral terms, Newman's model merges cognitive and behavioral levels. For example, *read, comprehend, apply the process skills demanded by the selected strategy* refer to specific behaviors, while *carry out a mental transformation* addresses cognition.

More recently, Christou, Mousoulides, Pittalis, Pitta-Pantazi, and Sriraman (2005) proposed a theoretical model consisting of the following cognitive processes that interact in PP and PS: editing quantitative information, selecting quantitative information, comprehending and organizing quantitative information, and translating quantitative information from one form to another. While this model is designed to separate students into categories of abilities, we are looking for a framework that can be used at various levels of complexity by a wide variety of users, irrespective of their level in mathematical processing.

Because we assumed a specific target, the PS framework we are looking for is meant to complement rather than to challenge previous descriptions of problem-solving processes.

Another category of assumptions refers to *whose* behaviors to look at in order to analyze the PS process: of students? of teachers? of high achievers? of low achievers?, etc. A large body of literature on cognition favors the hypothesis of domain specificity: this is the idea that all concepts are not equal, and that the structure of knowledge is different in important ways across distinct content areas. In other words, a series of cognitive abilities are specialized for manipulating specific types of information (e.g., Carey & Spelke, 1994; Singer, 2007). Moreover, there are innate predispositions of the human mind for some privileged domains (e.g., Spelke, 2003). Assuming the perspective of domain specific learning, the proposed framework should reflect, on the one hand, natural predispositions, and, on the other hand (and most important) the strategies of the most "advanced thinker" in that domain, i.e. in our case, the expert in mathematics. In this paper, we use the term "expert" in PS in a broad sense, referring to three types of persons: the mathematician who runs research activity, the teacher who trains students for math competitions, and the high achiever student validated by the results obtained in math competitions.

Therefore, when we point to a certain framework for problem solving that we want to value efficiently in the educational practice, two questions become important: *Does the framework describe the experts' problem-solving strategies?* (Schoenfeld, 1980) and one that has been even more tackled: *What are the strategies that would help individuals to make sense of the critical data in a problem that triggers effective solving?* The proposed framework should be able to face these challenges. More specifically, in this paper, we intend to develop a framework for problem-solving processes and to use this proposed model of PS as a means to develop problem posing, in particular to construct multiple choice questions.

2 Methods

2.1 Participants

For this study, we have used data obtained from two different samples.

The first sample consists of 150 experienced in-service teachers who participated in a short teacher training program. The involvement in the program was selective, by invitation: only teachers with good results in training students for mathematics competitions were invited to participate.

The second sample consists of 120 students with high achievement in mathematics from grades 3 to 6, who voluntarily answered to a call for problems. These respondents represent 55 % of the students who participated in a mathematical summer camp. The participants in this summer camp had been selected via a two-round national competition. We assume that this group of students is highly competitive because, in their quality of winners, they represent less than 0.2 % of the total number of participants in the two-round mathematics competition.

We consider that these two samples are relevant for our study for at least two reasons. First, both the experienced teachers and the high scoring students (although very young) have a certain level of expertise in problem solving. Second, both groups were involved in mathematics competitions (as trainers, organizers, or direct participants); consequently, they have at hand types of problems that could be recalled from their memory when they are in the situation of devising new problems. We assume that a completely random selected sample (of teachers or students) would be unlikely to be able to pose interesting problems.

2.2 Procedure

The present study reports on the conclusions we got by processing data from three categories of explorations.

First, we introspected into our own ways of solving problems. The authors' introspection during personal problem solving offered valuable insights into the way of organizing mental stances when approaching problems of various levels of difficulty. On the other hand, our intuitions where confronted with and validated by empirical studies that have analyzed, in different contexts, both the students' and the teachers' approaches to problem solving and posing.

Second, we observed habits and attitudes of the teachers involved in our study (the first sample) when they designed problem-solving tasks for students. At the beginning of a four day training program, teachers received a 2-h instruction in PP based on a variety of examples and concrete discussions. After that, they were asked to work in small groups (two to five in a group) in order to pose problems. Each group was asked to conceive about

20 problems, which were to include complete solutions of the posed problems, as well as relevant hints to help the student in solving. As a result, about 1,000 problems have been proposed. As part of the training, during the group discussions, the participants were invited to express their opinions about a selection of the posed problems, concerning: clear/unclear wording, interesting/uninteresting data, easy/difficult questions, usual/unusual theme or context of the problem, well-defined problems, ill-defined problems, etc. These discussions brought interesting information about teachers' views on PP and PS.

Third, we explored students' approaches (the second sample) to problem solving and posing. During the camp activities, the students were asked to create two problems—an easy one and a difficult one—and to deliver their posed problems (including the solutions) after a few days. The sample consisted of the 120 students who responded to this task. Subsequently, we have chosen and interviewed 40 respondents. The choice was made depending on the nature of the problems the students posed and their solutions. For the interviews, we selected children who: posed a completely non-standard problem; gave an algebraic solution above her/his class level; inappropriately used a specific technique; made interesting mistakes (some suggesting interesting developments); gave unusual solutions; provided an interesting generalization; made surprising classifications for easy or difficult problems; or, in general, made comments with potential for further discussions.

We structured the interview protocol around questions such as: Are there redundant/insufficient data in your proposal? Can you define a more general situation? Is there any interesting particular case? Can the given data of your problem be changed? (And, if yes, what will happen? Can you devise new data that match your problem?) What happens if you change a smaller/larger part of the problem? etc.

Although this study is based on relatively large samples, the research reported in this paper is qualitative because we were interested in the details of individuals' ways of thinking when dealing with various types of problems, and not in the statistical analysis of the data.

3 Results: steps to the conceptual framework

3.1 A starting point: introspection

As problem solver participants in math competitions and, later on, as trainers of gifted students in mathematics, the authors proceeded to make explicit their approach in solving various problems. In this process of re-describing PS activities, they tried to extrapolate what is general beyond a specific problem to be solved.

Introspection is a good way of emphasizing mental processes that are relevant for phenomena under study. Moreover, by introspection, one can better discern and explain individual perceptions of experiences and personal reactions to them (Duffin & Simpson, 2000; Stehlikova, 2003). In addition, one can infer about mental processes of other people with more “sensitivity” (Mason, 1998). The authors used introspection to configure stages in problem solving. This led to a four-step framework that appeared to cover the solving of a large gamut of problems.

In order to validate/invalidate this framework, a confrontation with various PS contexts was needed. We found that the most productive situation is the one in which one poses a problem, solves it, and eventually modifies some components in order to obtain a result/solution that satisfies certain criteria.

The two samples that we described above were requested to come up with posed problems accompanied by explicit solutions. By putting the participants in the situation to confront their own posed problem with their proposed solution, we intended to highlight general traits of the PS strategies they used when they accomplished the PP tasks. The PP tasks in this case were meant to spontaneously reveal solving techniques and approaches better than a simple PS task, where the problem comes from somebody else. For example, we noticed that the more the student advances in the abstract dimension of his/her posed problem and its context, the more mathematically relevant are the newly obtained versions (Singer et al., 2011). Our interactions with the two samples, via group discussions, or interviews, offered us the opportunity to correlate PS abilities with PP abilities for the same person.

3.2 Teachers as problem posers: their difficulties

The analysis of the teachers' posed problems showed us that although the teachers involved in the study have had more than 5 years of teaching experience and extended involvement in math competitions for students, they had difficulties in posing problems that are relevant for students' learning. We used the problems posed by teachers from the first sample as a basis to analyze three elements that bring about a successful PP process, namely clarity, coherence, and novelty/originality. In addition, the mathematical correctness of the problem was considered an implicit criterion. We found that a substantial number of problems were incomplete (17 %), incorrect (8 %), and the vast majority of them (70 %) was assessed by a team of experts as not interesting, being just scholastic. (For example, the following problem: “Compute 1.5+0.6.” was included in this last category.) These outcomes are even more surprising if we think that the participant teachers were experts in PS.

3.3 Students as problem posers: their understandings

To better understand PP individual approaches, we selected from the posed problems of the second sample a large gamut of students' proposals: correct problems, ill-defined problems, problems with interesting/correct/wrong/incomplete solutions, easy and difficult problems, etc. Each time we looked for more hints into the students' insights.

During the interview sessions, the students could re-read their initial proposals, and then they responded to various specific questions within the interview protocol. During these interviews, we paid attention to students' explanations concerning the content and the solutions of their posed problems. The questions asked for the purpose of seeing how students understand the consequences of modifying some parameters of their posed problems highlighted distinct phases of students' thinking in approaching their own proposals.

To focus the target of this article, we present here only a few posed problems, each of them revealing important aspects of a certain phase of the PS process.

Example 1: Andrei's proposal (grade 5)

> Ten players compete in a chess contest. The players are numbered from 1 to 10. Each round is eliminatory. The winner of one round gets 1 leu (leu, plural lei, is the Romanian currency—authors’ note), and the winner of the big final gets 5 lei. Number 2 won the contest. How much money did he earn, in total?

Figure 1 presents Andrei's solution to his problem. Because his text is not clear enough (not all the relations among the data of the problem are made explicit), he decodes the problem in a personal way, adding constraints to his solving which are not mentioned in his text. Thus, although his solving is correct, it is only a part of the complete solution because he analyzes only one possible situation. Therefore, his decoding of his own text is incorrect; Andrei already had a mental representation of the described situation, but this was not entirely satisfactory for the problem text.

Example 2: Octavia's proposal (grade 6)

> The tap A pours 20 l of water per minute in the first pool. The tap B pours 35 l of water per minute from the first pool to the second. At the moment when both taps were opened, there was 600 l water in the first pool, while the second was empty. After how much time will the quantities of water in the two pools be equal?

Initially, Octavia gave an algebraic solution to her posed problem, which was based on the variation of the two quantities of water per minute. The interviewer expressed doubts that the pools would ever hold the same amount of water. Trying to explain the reason for which her problem has a solution, Octavia described the phenomenon by gesture (moving her hands vertically, in opposite directions), thus showing that while in one pool the level of water decreases, in the other one the level increases. Her conclusion, expressed by gesture, was that inevitably the pools will at some point have the same level. This shows that Octavia developed for her problem a mental model based on the idea of balance, which allowed her to represent the data of the problem. Asked why she considered this problem to be a difficult one, she said that she had to carefully choose the numbers (of the problem text) in order to ensure that the water level variation of the two pools allows

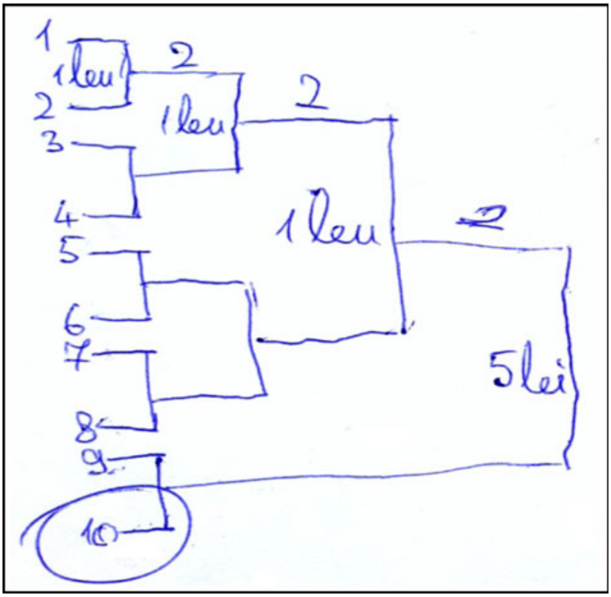

Fig. 1 The scheme drawn by Andrei to explain his solution to the problem from example 1

increase–decrease. Therefore, she coded–decoded the relationships among the data of the problem and put them into connection with a mental model she had of the phenomenon described within this problem.

Example 3: Maram's proposal (grade 6)

> A tiger, a duck, and a monkey started the mufflers crochet contest. The tiger crochets 1 m in 2 hours, the duck—1.5 m in one hour, and the monkey crochets 8 m in 4 hours. Who has made the longest muffler if the contest took 10 hours?

Initially, Maram's solution to her problem consisted in computing the length of each character's muffler. During the interview, she was asked if this is the only way to solve the problem. She proposed an approach in which she referred not to the length of the mufflers, but to the working speed. Therefore, Maram can activate a mathematical model of the problem (comparison of rates/products) and she can implement this model in various ways in order to get to the solution.

Example 4: Denisa's proposal (grade 3)

> Tudor and Vlad are of the same age, i.e. 10 years old. Their father is 40 years old. After how many years will the sum of Tudor and Vlad's ages will equal their father's actual age?

Denisa's solution is given in Fig. 2. Unlike Maram, Denisa does not have a mathematical model of the problem and this is why she solves it by trials. She can, however, implement steps of solving because she activates a mental model of the problem. More explicitly, she knows how age varies year by year, but she cannot conceptualize an equation (mathematical model) to express the problem. We searched for new evidence of this fact and asked her to solve a similar variant of this problem. Even in this case, she applied the same type of trial and error strategy although she could have shortened the solving process by using the results of her previous solution; but she did not take this option. This confirms the fact that she had not structured yet a mathematical model to rely on for this problem.

Example 5: Claudia's proposal (grade 6)

> I am 13 years old, and my sister, Carla, is 3 years younger than me. What difference will be there between us 5 years later?

The problem can be considered a mathematical charade (the difference between the ages remains constant!). However, the solution given by Claudia (see Fig. 3) shows that she was not aware of this fact. It seems that in proposing this problem, Claudia has started from a predefined mathematical model: she wanted to use an equation to solve a problem. In other words, in posing this problem, she wanted to encrypt an algebraic formulation into a text. The persistence of the mathematical model (which actually is not related to the problem) prevented her from realizing the meaning of her problem and its immediate solution. Therefore, the preexistent mathematical model impeded her from implementing an optimal solution.

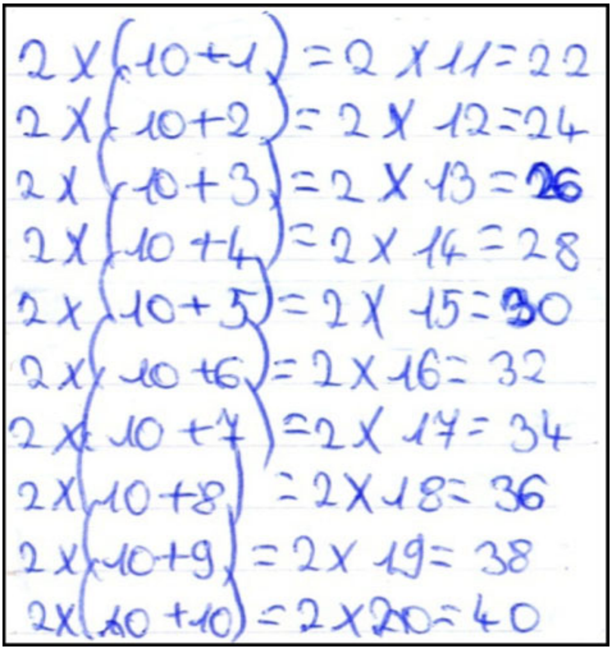

Fig. 2 The step-by-step solving by Denisa of the problem from example 4

4 A conceptual framework for problem solving and its connections to problem posing

In the steps described above, we looked for a simple architecture of the cognitive involvement in the solving approach. We present below the phases of the resulting conceptual framework (see also Singer & Voica, 2008). We introduce each phase by a brief description.

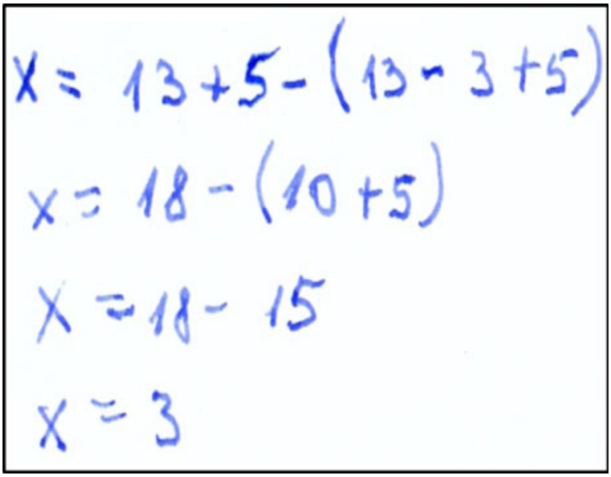

Fig. 3 The algebraic proof given by Claudia to the problem from example 5

Decoding In general, the text of a problem contains: a background theme, (numerical) data, operators (or operating schemes), constraints over the data and the operators, and constraints that involve at least one unknown value of a parameter.

The *background theme* represents the general context in which the problem happens or is described. The problem context is important in both word problems and questions given in a formalized or figural language. In general, the background theme expresses “what the problem is about”. For example, it might be about: triangles, lines, trees, chess contests, ages, trigonometric functions, etc. The background theme of a problem is characterized by one or more parameters. *The data* are (numerical or literal) values associated to these parameters. *The operating schemes* are actions suggested by the text of the problem. Among the operating schemes there might be mathematical operations, but they can also refer to activities such as “plot”, “draw”, “trace”, “intersect”, “cut”, etc. Some operating schemes may act implicitly within the problem, while others should be explicit. *The constraints imposed on the data and the operators* are restrictions that state the relations of the background theme with the data and the operators (operating schemes). *The constraints that imply at least an unknown value of a parameter* are those restrictions that state the relations among the data, operating schemes and the problem question. The constraints individualize a specific problem within the set of possible problems having the same background theme, (numerical) data, and operators.

In order to understand the wording of a problem, a solver tries to relate the two types of constraints. This search resolves into a primary decoding process. This decoding process (that usually means understanding the text) leads to making explicit the relations induced by the constraints among the data and the operating schemes. For simple (one-step) problems, the relations among the data and the operating schemes frequently coincide with the constraints. In this case, as well as in an expert case, decoding might be mindless, automated, while reading/hearing the problem.

Summarizing, in order to understand the problem, the solver moves his/her focus from the wording to the relations among the data and the operating schemes that are induced by the given constraints. We called this phase *decoding*.

In example 1, Andrei does not pay attention to the data and does not make explicit the various constraints he envisaged when he developed the problem. Thus, the number of players (10) supposes that some rounds of the tournament are “incomplete” (that means some players do not have partners, so they don't play), and the supposition that the winner of the round 9 vs.10 qualifies for the big final is a constraint not mentioned in the text. In other words, Andrei's proposal does not allow *decoding*. Yet, this example allows us in turn to highlight the relevance of this phase.

Representing Reading/hearing the problem text with understanding leads to a *mental model*. The mental model is a structured ensemble of mental representations induced by the wording, which is oriented by the purpose of solving the problem. The mental model can be made of: images; configurations; graphic organizers; drawings; schemes; graphs, etc., with or without (mental) visual support. The mental model might also consist of the problem transposition into a language that is more accessible to the solver. Therefore, as a result of reading/hearing and understanding the wording, the solver builds a mental model that is expressed via images, movements, physical objects, schemes, or sentences in a more familiar (internal) language. We call this phase *representing*.

In example 2, we can observe that Octavia built a mental model for her problem. Her gestures showing why the pools should get the same water level mean that she owns a *representation* for the variation of the water levels in the two pools.

Processing When solving a problem, the mental configurations suggested by it and the solver's mathematical knowledge and understanding lead him/her to the identification of a *mathematical model* associated with that problem. The mathematical model is a conceptual structure the transformation of which is clearly defined by the existence of specific mathematical concepts and techniques. Such a mathematical model can be: an equation; a system; the steps of a graphical representation; various computing algorithms, etc.

Briefly, the mental configurations suggested by the problem put the mathematical competence in action with the purpose of developing a *mathematical model* associated to the problem. We called this phase *processing.* Examples of processing might be: exploring the possible situations through a drawing; the decision of eliminating some variables; the inference of new relations of congruence, similarity, equality, inequality; the transposition of the text into an equation, etc.

In example 3, Maram manipulates a mathematical model. She is aware that she can compare the results of the products by comparing their factors. In order to do this, she needs to possess a certain mastery of the mathematics involved.

Implementing The passage from the mathematical model to the actual solution of the problem supposes applying specific techniques that are learned by studying mathematics. These techniques might be, for example: identifying the adequate ways of computing or proving; eliminating the values that do not satisfy the constraints of the problem; using auxiliary constructions, substitutions; using a known algorithm. They might also suppose comparison or optimization.

Because this phase is based on the application of already mastered techniques to a certain determined situation in certain known conditions, we called it *implementing*. Even in the most basic mathematical problems (such as, for example, "Ann had three apples and gets two more. How many does she have now?"), there is a phase of implementation, consisting of becoming aware of the mathematical model: addition—developed from the proto-quantitative operation of "putting together" (Singer, 2009), and of applying it.

In example 5, Claudia actually starts from a mathematical model that is established *a priori* as part of her problem-posing activity (using an equation). She developed her problem with the obvious intention to *implement* that mathematical model.

The human mind uses representations to make sense of environmental stimuli. Therefore, the generation of mental models is natural (e.g., Singer, 2009; Spelke, 2003), as is the search for reconciling a mental model with an evoked mathematical model (Piaget viewed this as part of the assimilation–accommodation process). From its evolutionary history, the human mind tends to arrive at a solution when a problem appears, that is to put in action a functional mental schema. Consistent with these ideas, the identified framework seems to have the characteristics of a natural construal.

The above presentation illustrated by students' work highlights the general phases of the PS process. Figure 4 gives a graphical image of these phases. Briefly, in order to find a solution to a problem, a solver starts from the wording, searches for relations among the data and the operators that lead to mental representations, deduces relationships that call for a known mathematical model adequate to the problem, and finally uses techniques adequate to the identified mathematical model to get to the final result(s)/solution of the problem.

When the process of solving is a successful one, from the text of the problem to its solution, a solver successively changes his/her cognitive stances related to the problem via transformations that are equivalent and reversible, allowing different levels of description for the initial text. Within these transformations, the passage between successive stages of the model determines four operational categories: decoding (transposing the text into more

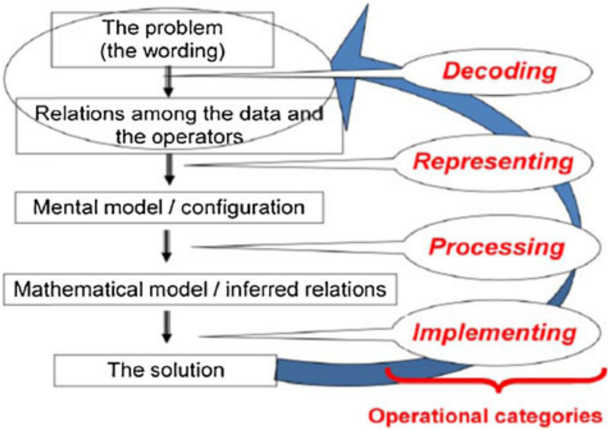

Fig. 4 The phases of the conceptual framework for the PS process

explicit relations among the data and the operators induced by the constraints); representing (understanding the problem via a generated mental model); processing (using mental configurations suggested by the problem and personal mathematical competence for the identification of a mathematical model that can be associated with the problem) and implementing (applying techniques that are specific to the found mathematical model and adaptable to the given particular situation, with the purpose to obtain final results for the problem). When this cycle of transformations is mastered, the solver is able to connect the formal solution of the problem with the initial data. The arrow in Fig. 4 shows the closing of the solving cycle.

5 Using the problem-solving conceptual framework in designing problem-posing tasks

The effectiveness of this conceptual framework might consist in its capacity for implementation in designing PP tasks. In order to check this capacity, we have applied it in developing multiple choice questions. To better explain this potentiality, we will compare the manner of posing distracters to the same problem in two circumstances: one, in which teachers set up the distracters by ad hoc reasoning, and the other in which the framework was applied. We detail below two examples for these situations, from two distinct mathematical domains: geometry and algebra.

During the teacher training program mentioned above, we asked teachers to propose multiple choice problems. In this case, we requested that the alternative answers (distracters) should reflect, as much as possible, typical errors, including incorrect strategies, misconceptions, computing errors, etc.

There is a rich literature that addresses the issue of writing distracters for multiple choice questions, in order to diagnose the solver's mathematical level (e.g., Ashlock, 2002; Coben, 2003; Ma, 1999; Rowland, Heal, Barber & Martyn, 1998; Ryan & Williams, 2000). However, we were interested in this aspect not from a research perspective, but from that of the teacher's preparation for posing multiple choice questions to be used in the classroom. We would expect teachers to be able to write better distracters as they gain experience in teaching, because they learned in the meanwhile about the students' misconceptions. However, analyzing the proposals of the in-service teachers from our sample, we found that the ability to pose relevant distracters seems not to come as a consequence of the teaching experience.

The problem further examined was proposed by Daniela (15 years of teaching experience):

> In the triangle *ABC*, points *D*, *E*, and *F* are the midpoints of the sides, points *M*, *N*, *P* are the midpoints of the sides of the triangle *DEF* and points *X*, *Y*, *Z* are the midpoints of the sides of the triangle *MNP*. If the area of the triangle *XYZ* is 16 (cm^2), what is the area of the triangle ABC?

For this problem, Daniela proposed the following distracters:
(a) 1,024; (b) 64; (c) 128; (d) 256; (e) 80.
Daniela's comments for this selection of the distracters are given in Table 1.

Of course, it is possible for a student, in solving this problem, to make one of the mistakes that Daniela expected to and, therefore, to choose the appropriate distracter. But we think that the analysis of this problem in terms of the operational categories of the conceptual framework proposed in this paper can better indicate the source of students' mistakes. This fact allows effective feedback on written papers, in the absence of direct interaction with the respondent. Below, we explain how the framework can be used in designing distracters for Daniela's problem.

Decoding In order to obtain relevant distracters concerning the students' comprehension difficulties, it is important to emphasize the essential relationship between the data and the operating schemes during the decoding phase. In the case of this problem, we consider essential the process of successive crossings from a bigger triangle to a smaller one. Since the numerical data of the problem refer to the (final) triangle *XYZ*, and the question concerns the (initial) triangle *ABC*, a primary decoding requires the notification of a transition from smaller to bigger.

Incomplete decoding would show that the student has a perception of the process described in the text, but he/she cannot see the need for reversibility. As a result, the student applies halving three times for the only numerical data that appears in the text of the problem. In this way, one can reach the distracter 2. The option for this distracter clearly shows us the decoding failure: otherwise, this distracter could be eliminated immediately because the area of the triangle *ABC* (="the whole") cannot be less than the area of the triangle *XYZ* (="a part").

Representing A mental image of this problem is supported by drawing a representation of the problem statement. Drawing involves iterating (three times) a single procedure of a

Table 1 Distracters and comments proposed by Daniela for her problem

	Distracter	Commentary
(a)	1,024	The student knows the relationship between the areas of the two triangles and correctly applies it to solve the problem.
(b)	64	The student multiplies the area of the triangle *XYZ* by 4, thinking that the area of the triangle *ABC* is 4 times bigger.
(c)	128	The student thinks that area *MNP*=2 × area *XYZ*, area *DEF*=2 × area *MNP*, and area *ABC*= 2 × area *DEF*.
(d)	256	The student correctly finds the area of the triangle *MNP*, but further successively multiplies by 2.
(e)	80	The student gives an answer at random, without actually solving the problem.

fractal type, namely drawing the midpoints of the sides of a triangle. What would suggest that this image is "distorted" in the mind of a solver? For example, that the student, rather than considering triangles with areas becoming smaller and smaller, believes that all parts of the drawing have equal areas. This distorted image can be highlighted by the answer 160 (which can be reached by considering—erroneously—that *ABC* was divided into ten disjoint triangles of equal areas). The conscious choice of this variant clearly indicates dysfunctions in the mental representation of the problem.

Processing Solving the problem involves the use of the following mathematical tools: the median properties; congruence of triangles; and similarity of triangles. What is mathematically essential in this phase? The problem focus is not on the median, but beyond that, on the expression of the ratios of the areas of similar figures. What distracter could indicate that the mathematical model is not well constructed in the student's mind? Bad processing can lead to the result 128, where the student transfers the ratios of the sides of the similar triangles to the ratios of their areas.

Implementing Once the processing is done and the mathematical model clarified, it remains to transpose the identified mathematical model to the specific context of the problem. Where do errors occur in this phase? Solving involves a sequence of three steps, in which the previous result is multiplied every time by 4. Compressing these steps could lead to a computation in which the factor 4^3 (each time, multiply the area by 4) is replaced by the factor 4×3 (multiply by 4, 3 times). So, inadequate implementation can lead to the distracter $192=16\times4\times3$.

In conclusion, we propose the following distracters:

(a) 2; (b) 160; (c) 128; (d) 192; (e) 1024.

Conversely, we further analyze all proposed distracters in order to find out to what extent the choice of one of them shows a relevant error for a certain category of the conceptual framework.

(a) 2: the choice of 2 indicates an aberrant option, clearly showing that the student did not understand the relationship (such as part–whole, smaller–bigger) between the problem data. So, decoding does not work.
(b) $160=10$ times 16. The choice of 160 indicates a focus on the number of parts at the expense of their size, so the problem representation is distorted.
(c) $128=16\times2\times2\times2$. The choice of 128 indicates that the student has reversed the process of successive division, therefore he/she has a correct representation of the problem, but the mathematical model is blinkered because he/she does not understand that the mathematical quantities referred to are of different types (length versus area).
(d) $192=16\times4\times3$. The choice of 192 indicates that the student has built a mathematical model, but its customization in the problem situation fails in a confusion of operations.

Of course, the proposed distracters for a multiple choice problem cannot be unique. Usually, in order to be relevant, the distracters are determined based on a pre-test, but this actually remains limited to that specific problem. A more efficient strategy would be to devise problems that allow distracters that are meaningful for the students' learning with understanding.

We further include a second example. The following problem was posed by a team of two teachers (Aurelian and Stefan):

Find the last two digits of the number 2^{2009}.

They proposed for this problem the distracters listed below:
(a) 12; (b) 18; (c) 56; (d) 35; (e) Impossible to determine.
They also added the following "complete" solution:

$$2^{2009} = 2 \cdot 16^{502}$$

$$u_2\left(16^{502}\right) = 56$$

The powers of 16 repeat the last two digits with a period of 5.

We begin by referring to the solution proposed by the two teachers. This solution neglects important steps in solving the problem. Actually, the two proposers reduced the problem to an already known algorithm. Just as Claudia wanted to implement an equation for which she posed a problem (see example 5), Aurelian and Stefan wanted to implement a known property (*The powers of 16 repeat the last two digits with a period of* 5) within their proposal. This is a common behavior we met among the teachers-problem posers. In this way, they actually miss the difficulty load of the problem, a fact that can be seen also in the way they comment on their proposed distracters (Table 2).

A brief analysis of the distracters proposed by the two teachers shows the following:

At least two of the distracters are not likely to be chosen by a student: 56 results from applying a property that is unlikely to be known by a sixth or seventh grader. The teachers had in mind a solver who is trained in knowing by heart various mathematical wordings. Even worse, the distracter 35 seems to be put at random, being completely irrelevant. Moreover, the comment for distracter (e) deserves special attention. It seems that PS is perceived like a prey-and-predator race in which the teacher (or the trainer) arranges traps from which the student tries to escape, with more or less success.

Although we believe that this problem is very difficult for most students, we chose to analyze it for two reasons: on the one hand, the two teachers were convinced that their proposal was a good option to be included in a multiple choice test for students in grades 7–8. On the other hand, we consider that their proposed distracters are irrelevant to their problem. Under these conditions, we wanted to see to what extent an analysis of the problem made using the PS conceptual framework can lead to distracters that can reveal levels of understanding of the item. We resume the problem:

Find the last two digits of the number 2^{2009}.

Table 2 Distracters and comments proposed by Aurelian and Stefan

	Distracter	Comment
(a)	12	"Correct answer"
(b)	18	"Computing mistake"
(c)	56	"Partial computing correct, but wrong final"
(d)	35	"Absurd answer"
(e)	Impossible to determine	"You wanted to easily escape!"

The question is, in this case, what distracters could highlight the coverage of the framework phases for this problem? To answer, we analyze the stages of its solving from an expert view and search for possible relevant errors. The data of this problem consist in the numbers 2 and 2009, and the operating scheme is computing a power.

Decoding supposes establishing relationships between the data and the operating scheme, facilitated by the constraints of the wording. A primary decoding means to be aware of the specific notation (2^{2009} is “something else” than 22009). The notation 2^{2009} has an implicit nature: decoding supposes making explicit the mathematical “condensed” notation and its replacement with $2\times2\times\ldots\times2$. In the absence of this decoding, a possible answer of the solver might be 09 (these are the last two digits of the “number” from the wording).

Representing Once the decoding has been made, we can mentally represent the computation by the following recurrent sequences of operations:

$$2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} \ldots$$

The recursive representation can fail by compressing the exponentiation (which supposes a computation “step by step”) and replacing it with the product 2×2009. This is why the distracter 18 can indicate difficulties in the phase of representing.

Processing supposes understanding that, among all the recurrent computations, we are interested at each step in only the two final digits of each result. In mathematical terms, this means that we use the remainder class ring modulo 100. If such a mental construction is missing, the recurrent computations become more and more difficult because the terms of the sequence are bigger and bigger. This is why an appropriate distracter might be: “It is impossible to compute”.

Implementation consists in finding the period and finishing the computations. Except for the first term (2^1), the last two digits of subsequent terms repeat with a period 20. In other words, the numbers 2^2 and 2^{22} both finish in 04. A thoughtless implementation can lead to the conclusion that the recurrent sequence is periodical, of period 21. For this reason, a distracter that highlights this implementing error can be 84.

Finally, the correct answer is 12.

Therefore, to highlight the errors generated by the blockage on a certain layer of the solving framework, the distracters for this problem might be:

(a) 09; (b) 18; (c) It is impossible to compute; (d) 84; (e) 12.

Using the above examples, we have argued that by choosing appropriate distracters it is possible to emphasize the difficulties the solver faces in a certain phase of the PS process. Having such a diagnosis, the teacher can adapt her/his teaching strategies more specifically to the students' needs.

6 Conclusion

Starting from observations of an empirical nature, we have identified a conceptual framework for problem solving that highlights four operational categories. Briefly, the solver starts from the text of the problem and through a *decoding* process, self-*represents* a mental model. Based on this representational model, he/she *processes* relationships that call for a mathematical model. Further, during the phase of *implementing,* he/she uses adequate techniques to get to the solution.

These phases (*decode–represent–process–implement*) succeed one another almost linearly in the solving, that is, when solving a problem, one goes sequentially through them. This assertion includes the fact that depending on the solver's expertise, some phases might be implicit. The recursiveness of the framework has an important consequence: students' mistakes in one of the phases indicate a failure at a certain stage of the solving. Consequently, teachers can thus obtain patterns of students' problem-solving competence.

Being aware of this four-phase framework for problem solving, the teacher can pose specific problems that distinctively focus on one of the phases, or more precisely, problems that allow some phases to be implicit, while only one or two need attention and more detailed analysis and description. Thus, the framework can be used by teachers in order to orient their PP activity towards certain values of the parameters involved in a problem in order to focus relevant educational purposes.

In particular, the PS conceptual framework can be used in devising relevant distracters for multiple choice problems. In the absence of a paper and pencil test in which participants write down detailed solutions, or of face-to-face interviews, the above analysis identifies from the multitude of possible distracters, the one that potentially reveals a fundamental mistake at each phase of the PS process. Therefore, this framework has potential for posing multiple choice questions—a category of PP that is of great interest. More specific, the design of questions according to this four-phase framework allows teachers to offer consistent feedback on multiple choice written examinations.

Of course, not all problems have unique ways of approaching them and many do not allow anticipation of the solving trajectory. However, part of the art of teaching is to choose learning tasks that match the learning goals. Therefore, the problems posed as tasks should be relevant for effective assessment, in order to detect the students' range of abilities and orchestrate appropriate training.

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References

Ashlock, R. B. (2002). *Error patterns in computation: Using error patterns to improve instruction* (8th ed.). Upper Saddle River, NJ: Merrill Prentice Hall.

Bloom, B. S., & Broder, L. J. (1950). *Problem-solving processes of college students*. Chicago: University of Chicago Press.

Brown, S., & Walter, M. (2005). *The art of problem posing* (3rd ed.). Mahwah, NJ: Lawrence Erlbaum Associates.

Carey, S., & Spelke, E. (1994). Domain-specific knowledge and conceptual change. In L. Hirschfeld & S. Gelman (Eds.), *Mapping the mind: Domain specificity in cognition and culture* (pp. 169–200). Cambridge, UK: Cambridge University Press.

Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B. (2005). An empirical taxonomy of problem posing processes. *ZDM, 37*(3), 149–158.

Coben, D. (2003). Adult Numeracy: Review of research and related literature. Research review. *National Research and Development Centre for Adult Literacy and Numeracy*. http://www.nrdc.org.uk/publications_details.asp?ID=35#. Accessed 9 January 2011.

Dillon, J. T. (1982). Problem finding and solving. *The Journal of Creative Behavior, 16*(2), 97–111.

Duffin, J., & Simpson, A. (2000). When does a way of working become a methodology? *The Journal of Mathematical Behavior, 19*, 175–188.

Duncker, K. (1945). On problem solving. *Psychological Monographs, 58*(5). (Whole # 270.) Washington, DC: American Psychological Association.

Ellerton, N. F., & Clements, M. A. (1992). Implications of Newman research for the issue of "What is basic in school mathematics?". In B. Southwell, B. Perry, & K. Owens (Eds.), *Space: The first and final frontier* (pp. 276–284). Sydney: MERGA. Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia.

Jay, E. S., & Perkins, D. N. (1997). Problem finding: The search for mechanism. In M. A. Runco (Ed.), *The creativity research handbook* (pp. 257–293). Cresskill, New Jersey: Hampton Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mason, J. (1998). Researching from the Inside in mathematics education. In A. Sierpinska & J. Kilpatric (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 357–377). Norwell, MA: Kluwer.

Newman, M. A. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. *Victorian Institute for Educational Research Bulletin, 39*, 31–43.

Newman, M. A. (1983). *Strategies for diagnosis and remediation*. Sydney: Harcourt, Brace Jovanovich.

Noveanu, G. N., & Singer, F.M. (2008). Romania. In I.V.S. Mullis & M. O. Martin (Eds.) *TIMSS 2007 Encyclopedia a guide to mathematics and scienece education around the world*, vol 2. (481–490). IEA, TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

Pólya, G. (1945). *How to solve it*. Princeton University Press.

Pelczer, I., Singer, F. M., & Voica, C. (2011). An analysis of relevant hints in problem solving. In B. Ubuz (Ed.), *Developing mathematical thinking, vol 1* (p. 370). Ankara, Turkey: PME. Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education.

Rowland, T., Heal, C., Barber, P., & Martyn, S. (1998). Mind the “gaps”: Primary teacher trainees' mathematics subject knowledge. In E. Bills (Ed.), *Proceedings of the British Society for Research in Learning Mathematics day conference at Birmingham* (pp. 91–96). Coventry, UK: University of Warwick.

Rowland, T., Huckstep, P., & Thwaites, A. (2003). The knowledge quartet. In J. Williams (Ed.), *Proceedings of the British Society for Research into Learning Mathematics 23*(3), 97–103.

Ryan, J., & Williams, J. (2000). *Mathematical discussions with children: Exploring methods and misconceptions as a teaching strategy*. Manchester, UK: Centre for Mathematics Education, University of Manchester.

Schoenfeld, A. H. (1980). Teaching problem solving skills. *The American Mathematical Monthly, 87*(10), 794–805.

Silver, E. A. (Ed.). (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale, NJ: Erlbaum.

Silver, E., Mamona-Downs, J., Leung, S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education, 27*(3), 293–309.

Singer, F. M. (2007). Beyond conceptual change: Using representations to integrate domain-specific structural models in learning mathematics. *Mind, Brain, and Education, 1*(2), 84–97.

Singer, F. M. (2009). The dynamic infrastructure of mind—A hypothesis and some of its applications. *New Ideas in Psychology, 27*, 48–74.

Singer, F. M., & Voica, C. (2008). Operational categories for assessing problem solving (in Romanian). *ROMAI Educational Journal, 3*, 36–41.

Singer, F. M., Ellerton, N., Cai, J., & Leung, E. (2011). Problem posing in mathematics learning and teaching: a research agenda. In B. Ubuz (Ed.), *Developing mathematical thinking, vol 1* (pp. 137–166). Ankara, Turkey: PME. Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education.

Singer, F. M., Pelczer, I., & Voica, C. (2011). Problem posing and modification as a criterion of mathematical creativity, In In T. Rowland & E. Swoboda (Eds.) *Proceedings of the 7th Conference of the European Society for Research in Mathematics Education (CERME 7*) University of Rzeszów, Poland, 9–13 February, 2011.

Spelke, E. (2003). What makes us smart? Core knowledge and natural language. In D. Gentner & S. Goldin-Meadow (Eds.), *Language in mind* (pp. 277–311). Cambridge, MA: MIT Press.

Stehlikova, N. (2003). Emergence of mathematical knowledge structures. Introspection. In N. Pateman, B. Dougherty & T. Zilliox (Eds.), *Proceedings of the 27th International Group for the Psychology of Mathematics Education*, Honolulu, Hawaii, Volume 4, 251–258.

Voica, C., & Singer, M. (2012). Creative contexts as ways to strengthen mathematics learning. In M. Aniței, M. Chraif & C.Vasile (Guest Eds.), *Procedia—Social and Behavioral Sciences*, *PSIWORLD 2011*, Volume 33, 538–542.