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A case study of one instructor's lecture-based teaching of proof in abstract algebra: making sense of her pedagogical moves

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Abstract This paper is a case study of the teaching of an undergraduate abstract algebra course, in particular the way the instructor presented proofs. It describes a framework for proof writing based on Selden and Selden (2009) and the work of Alcock (2010) on modes of thought that support proof writing. The paper offers a case study of the teaching of a traditionally-taught abstract algebra course, including showing the range of practice as larger than previously described in research literature. This study describes the aspects of proof writing and modes of thought the instructor modeled for the students. The study finds that she frequently modeled the aspects of hierarchical structure and formal–rhetorical skills, and structural, critical, and instantiation modes of thought. This study also examines the instructor's attempts to involve the students in the proof writing process during class by asking questions and expecting responses. Finally, the study describes how those questions and responses were part of her proof presentation. The funneling pattern of Steinbring (1989) describes most of the question and answer discussions enacted in the class with most questions requiring a factual response. Yet, the instructional sequence can be also understood as modeling the way an expert in the discipline thinks and, as such, offering a different type of opportunity for student learning.

Keywords Undergraduate teaching · Proof presentation · Abstract algebra · Teacher questions

1 Introduction

Many researchers argue that lecture-based mathematics instruction is intimidating and misleading to students about the nature of mathematics, especially in proof-based courses (Cuoco, 2001; Thurston, 1994). Some contend that it hides much of the process used in mathematical thinking and makes it difficult for students to develop an appreciation for the

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discipline (Dreyfus, 1991). Others maintain that it ignores the important role that mathematicians ascribe to ideas such as elegance, intuition, and cooperation (Burton, 1998; Dreyfus, 1999; Fischbein, 1987). Still others claim that lecture-based instruction is not an effective way to promote student learning of mathematics content (Leron & Dubinsky, 1995). In fact, evidence suggests undergraduate mathematics teaching is one of the most important reasons that students change majors away from science, technology, engineering, and mathematics fields (Seymour & Hewitt, 1997).

One goal of ongoing efforts in mathematics education has been to develop interventions that change undergraduate teaching practices. Understanding the beliefs and practices of those who teach mathematics is important to the success of this reform effort (cf. Alcock, 2010; Speer, Smith & Horvath, 2010; Weber, 2004). Just as students come to class with preconceptions that must be addressed, so too do mathematics instructors. Without understanding current teaching practices, however, it is difficult to determine what those preconceptions are. This issue has partially been addressed via interviews (e.g., Alcock, 2010; Weber, 2011) in which instructors describe their teaching practices. Yet there is a need to study undergraduate teaching to make sense of the claims that mathematicians make in describing their teaching (Speer et al., 2010). This paper responds to the dearth of studies on undergraduate mathematics instruction by presenting a case study of how one mathematician, who self-identified as a traditional university teacher, used lecture-based practices in enacting her teaching of a proof-based abstract algebra class. In particular, this paper addresses the questions:

1. What aspects of the proof writing process and modes of thought supporting proof writing does one instructor model for her students?
2. How does she devolve responsibility for these actions to the students during the lessons, and what responsibility is devolved?

One way to define the difference between traditional, lecture-based courses and inquiry-oriented or reform-oriented classrooms is to compare how much of the responsibility for mastering cognitive processes is devolved to the students. Rasmussen and Marrongelle (2006) described a scale of teaching that ranges along a continuum from “pure telling” to “pure investigation.” They claimed that a well-designed inquiry-oriented course would fall close to the middle of this continuum, while a lecture-based course would be closer to “pure telling.” Similarly, McClain and Cobb (2001) characterized the spectrum of teaching as running from *non-interventionist* to *total responsibility*. In the class described in the present study, the instructor had almost total responsibility for daily classroom activities and mathematical content; the class was closer to pure telling than was an inquiry-oriented class. The instructor delivered instruction from the front of the room while the students sat in desks arranged to face the board.

Despite repeated suggestions for the study of varied teaching styles (Harel & Fuller, 2009; Harel & Sowder, 2007), Speer et al. (2010) lamented that “very little empirical research has yet described and analyzed the practices of teachers of mathematics” (p. 99). This is especially true at the undergraduate level. Mejia-Ramos and Inglis (2009) conducted a literature search and surveyed 102 educational research papers that studied undergraduate students’ experiences writing, reading and understanding proof. Of those 102 papers, Mejia-Ramos and Inglis found no papers describing how instructors presented proofs to students or how students understood these presentations. This lack of research implies that assertions about the differences between inquiry-based and traditional classrooms are essentially untested.

Speer et al. (2010) argued that "researchers' questions, methods, and analyses have not generally targeted what teachers say, do, and think about collegiate classrooms in an extensive or detailed way" (p. 105). Researchers have not studied the teaching of proof in undergraduate classrooms in depth. Consequently, researchers have little insight into why instructors choose a particular teaching style or learning goals for their students. Although it is important to better understand the reality of collegiate mathematics classes, it is also essential to develop research-based descriptions of traditional undergraduate classes in order to support and explain the results of studies of students' proof writing abilities. Furthermore, traditional lecture teaching at the university level is not well defined. It is possible that a traditional lecture includes a teaching style that inculcates independence and inquiry in students. A traditional university teacher might ask questions and expect answers, give challenging yet motivating problems, or engage with the material in such a way as to inspire similar behavior in students. Because researchers are not able to describe the range of instructional practices among those instructors who identify as traditional university teachers, some critics may paint with too broad a brush.

Burgan (2006), among others (e.g., Andrews, 1999; Wu, 1999), has defended the value of the traditional method of university instruction. Burgan described several possible values of an excellent lecture to students. For example, she introduced the idea of instructor as role model for students: "I believe students benefit from seeing education embodied in a master learner who teaches what she learned" (p. 32). She noted that "teachers are irreplaceable as models of knowledgeable adults grappling with first principles in order to open their students' understanding" (p. 33). In a proof-based undergraduate mathematics class, this describes an instructor who demonstrates the cognitive processes involved in doing the tasks of mathematics. Those tasks include writing proofs while speaking aloud and possibly writing out a representation of the reasoning involved. The principal means of instruction in this case might be exposing students to the cognitive processes of doing mathematics. Although the mathematical tasks done in class are not novel or challenging for instructors, they can still make explicit the mental processes that instructors engage in when tackling these tasks.

In a lecture, instructors can, as Burgan (2006) argued, "raise questions that inspire students to seek answers together. In doing so, they also can provide a shortcut for the student through the thicket of detail and argument that the presenters already know by heart" (p. 34). Wu (1999) argued that it is critical to introduce students to topics while acknowledging that most of the content cannot be learned at the pace he is required to cover in class. He conceded, for example, that learning the Euclidean division algorithm is difficult, and described a "torturous" 2-h tutoring session with a student. Wu admitted that "it is likely that for most students this is the only way to learn [the topic]" (p. 3).

Although there are significant critiques of a lecturing style of teaching, there is also some debate in the education community about whether a lecturing style of instruction might inculcate independence and inquiry. This is related to the second goal of instruction: devolving responsibility for performing the cognitive tasks of learning mathematics to the student. This paper examines the practice of one instructor who claims to teach in a traditional style and explores the responsibility for cognitive tasks that she gives students during class.

2 Literature

The following section describes various aspects of proof that Selden and Selden (2009) claim students need to learn to handle mentally and technically in order to be successful provers. The section then presents four modes of thinking that some mathematicians claim support for successful proof writing (Alcock, 2010). There is a short discussion of previous studies of undergraduate mathematics instructors' teaching of proof. Because this study also focuses on the classroom dialog between the instructor and students, the final section describes the lens of Steinbring (1989) for analyzing classroom discourse.

2.1 Aspects of proof writing

To describe what it means to write proofs, this study draws on the work of Selden and Selden (2009) and Alcock (2010). The list of aspects of proof and ways of thinking described below should not be thought of as exhaustive, but rather as a preliminary attempt to determine what provers should pay attention to in creating a proof, and how they might do so. Selden and Selden define five aspects of a proof that they claim must be attended to when constructing proofs with meaning:

- The Hierarchical Structure, which includes knowing what the proof has to accomplish and coordinating any subproofs or constructions, including lemmas.
- The Construction Path, which is the means for actually creating the proof (as distinct from the way the proof is written for publication). The description of the construction path relies on the concept of an idealized prover who, as Selden and Selden (2009) described, "never erred or followed false leads" (p. 340).
- The Proof Framework, which encompasses the conventions of proving things in mathematics but does not require understanding the meaning of any of the terms. For example, it includes the logical structure of different types of proofs (direct, contradiction, and contrapositive), as well as the concepts of hypotheses and conclusions.
- The Formal–Rhetorical part of a proof, which requires primarily *behavioral knowledge* to complete; as Selden and Selden (2009) argued, "it is not important that a student be able to articulate such behavioral knowledge, it is important that he/she can act on it" (p. 344). It includes the ability to do algebraic and technical symbolic manipulations within the structure of the proof system such as those required to deal with quantification or logical implication. This behavioral knowledge includes knowing that if a theorem says, "For all real numbers …" then the proof should start by introducing an arbitrary real number, "Let x be a real number…" (Selden & Selden, 2009, p. 343).
- The Problem-Centered part of the proof, which includes determining the key idea(s) and coordinating aspects of the proof (especially if they include nonstandard argument structures), and requires what Selden and Selden (2009) call "conceptual knowledge, mathematical intuition, and the ability to bring to mind the 'right' resources at the 'right time'" (p. 344).

The Problem-Centered and the Construction Path aspects can be understood as the problem-solving portion of proof writing. Consequently, if a student reads a proof as a finished product, he or she may not understand the complex problem-solving processes used in creating the argument. This set of constructs will be used to present the data in Section 4.2 and in the analysis Section 4.3.1.

2.2 Modes of thought that support proving

The aspects of proof writing by Selden and Selden (2009) are supported by certain modes of thinking about proof and proving. Alcock (2010) identified four modes of thinking that working mathematicians expect of someone who engages in proving, and four modes of thinking that the interviewed mathematicians claimed were required to successfully write proofs. Each of the modes can be classified as either semantic, which involves thinking about the mathematical objects to which the statement refers, or syntactic, which involves thinking about and manipulating a statement based on its form (based on the work of Weber & Alcock, 2004). Syntactic work is that in which the proof author builds arguments based on formal reasoning via theorems and known results without developing any links to informal representations of mathematical ideas—that is, without instantiating. Alcock identified *structural thinking* as the mode of thought that occurs when drawing on the formal structure of the definition, related/associated definitions, and known results. Structural thinking can be understood as giving provers the ability to apprehend what Selden and Selden (2009) call the Proof Framework and Hierarchical Structure.

Semantic thinking involves what Alcock (2010) describes as "thinking about the mathematical objects to which a statement refers" (p. 78). Three modes in Alcock's analysis are semantic in nature. The first of these is *instantiation*, where a person writing a proof will "meaningfully understand a mathematical statement by thinking about the objects to which it applies" (Alcock, 2010, p. 78). The second is *creative thinking*, which requires examining particular instantiations in order to determine the "property or set of manipulations that can form the crux of the proof" (Alcock, 2010, p. 78). Finally, *critical thinking* is the process by which a person writing a proof ensures that a claim is correct through means such as searching for counter-examples, or asking what the claim implies.

The aspects of proof writing that Selden and Selden (2009) described are about a proof, not the type of thinking needed to support proof writing. As a result, these aspects do not include proof validation (looking back over a proof to ensure that it does what is needed). But, Selden and Selden have written that "validating a proof for oneself appears to be an essential part of constructing it" (2000, para. 4). As the construct of critical thinking by Alcock (2010) supports checks on validity of claims, this study takes the position that critical thinking is a mode of thinking that also supports proof validation.

Several of the mathematicians whom Alcock (2010) interviewed claimed that what Selden and Selden (2009) would call the Problem-Centered part of proof writing was most likely to be accomplished via semantic thinking, especially through the use of instantiation. Hence, Alcock's modes of thinking could be understood to support this aspect of proof writing, although the problem-centered part could also be completed via syntactic thinking. Table 1 summarizes the different aspects of proof that Selden and Selden (2009) describe as well as the modes of thinking that Alcock (2010) identified that support students in becoming proficient at producing those aspects of proof.

Students must be given responsibility for practicing each of these aspects of proof writing, as well as regular assessment of their progress, in order to develop their facilities. Different instructional models could devolve this responsibility in different ways. In the section that follows, there is a summary of the work of Weber (2004) on the teaching of one instructor who chose *not* to devolve responsibility to students for these modes of thought and aspects of proof writing in the classroom. In the case study presented in this paper, I show an instructor who devolved some responsibility to students within the context of classroom instruction. This set of constructs will be used to present the data in Section 4.2 and in the analysis Section 4.3.1.

2.3 Styles of proof presentation

Weber (2004) observed and identified three basic styles of teaching proof in a real analysis class "Dr. T" taught: *logico-structural*, *procedural*, and *semantic*. Weber characterized the

Table 1 A summary of aspects of proof writing and supporting modes of thought

Aspect of proof writing	Mode(s) of thought that support proof writing
Hierarchical Structure: knowing what must be proved; use of lemmas; coordination of sub-proofs	Syntactic–structural Semantic–creative thinking
Construction Path: the means for creating the proof	Semantic–instantiation Semantic–creative thinking Syntactic–structural
Proof Framework: conventions of proving things in mathematics	Syntactic–structural Semantic–creative thinking
Formal–Rhetorical: behavioral knowledge, including algebraic and technical skill for dealing with quantified statements and logical implications.	Syntactic–structural
Problem-Centered: determining key ideas and coordinating aspects of the proof; requires conceptual knowledge and mathematical intuition	Semantic–instantiation Semantic–creative thinking
Proof validation: (included here, although not an aspect of proof writing) checking to ensure that the proof is complete, correct and demonstrates what is needed	Semantic–critical thinking

logico-structural approach by its reliance on formal mathematical statement, its conspicuous lack of diagrams, its lack of any semantic meaning of the concepts or the proof, and its emphasis on careful use of definitions to both start and conclude a proof. It can be understood to model the Hierarchical Structure aspect and the Formal–Rhetorical part of proof writing, as well as structural modes of thought.

Weber (2004) characterized the *procedural* style of teaching proof by its lack of semantic meaning—students were expected to learn the structure of the argument and creative heuristics for proof writing, but not much about the mathematical constructs themselves. Again, this style of proof presentation modeled the Hierarchical Structure and Formal–Rhetorical aspects of proof writing and demonstrated structural modes of thought. This style can also be understood to model the Problem-Centered Part in a way that gives a procedure for handling algebraic manipulations and avoids deep engagement with the material.

The *semantic* teaching style is characterized by the instructor's use of intuitive descriptions of concepts and relationships. Thus, while this style of proof presentation may model the Problem-Centered part and the Construction Path, Weber's text only showed one example of Dr. T introducing a definition. Moreover, this style of teaching only occurred at the end of the course.

While Dr. T wanted the students to take responsibility for proof writing, it was only in homework assignments that he devolved any responsibility for this task to the students (Weber, 2004). During class, Dr. T essentially delivered a monolog, presenting the material without giving the students any explicit responsibilities. He did ask questions during the procedural proof presentation, but he answered them without student input. Weber (2004) described the students' level of engagement by noting that "Students asked questions only infrequently and rarely participated in class discussions" (p. 118). Thus, there is little evidence that the students were given any responsibility for any aspects of proof writing or modes of thinking during the class meetings. This is not to criticize Weber's Dr. T's instructional decision, but merely to establish that there are instructors who call themselves traditional university teachers who essentially give monologs. The case study below will offer a contrasting pedagogical practice by showing a university teacher who claimed to use traditional practices, but asked questions of the students, and also expected and got answers.

2.4 Instructors' claims about how they support students' learning

There is little data about what happens in undergraduate mathematics classrooms based on observation. Alcock (2010) and Weber (2011) carried out separate interview studies with university mathematics faculty about their teaching practices in courses designed to introduce students to mathematical proof. Alcock interviewed five mathematicians and identified the modes of thinking described above from their responses. She formed a series of claims about how the mathematicians engaged in teaching practices that supported each of the four modes of thinking. Similarly, Weber interviewed nine mathematicians about their experiences teaching an introduction to proof class and had them describe their teaching. He found that the mathematicians more frequently presented proofs to teach students ideas and techniques than to convince them that the results were true.

The two groups of mathematicians interviewed described many similar practices with respect to teaching proof writing. For example, both groups emphasized the importance of teaching students to instantiate by assigning students problems that required example generation (Alcock, 2010). Additionally, eight of the nine mathematicians in the study of Weber (2011) said that they normally accompanied a proof with an example or drew a diagram that was intended to support students' understanding of the proof.

The interviewed faculty offered few concrete suggestions for teaching the Problem-Centered part of proof writing. One suggested that students label parts of their paper as "analysis" and "proof," where the analysis section was meant to be exploratory and allow the students to think on paper (Alcock, 2010). Several of the instructors stated that they would indicate features of the proof being presented that were new to the students and had not been part of their prior proof experience, perhaps as a way to suggest new ways to think about proving (Weber, 2011). In terms of teaching critical thinking, the mathematicians suggested asking students to be alert for implied properties, presenting false statements and asking for counter examples, and even designated a student to look for errors in the presented work (Alcock, 2010). Another possible way to teach the Problem-Centered part was shown in the discussion in the study of Weber (2004).

Mathematicians in both studies had a number of strategies for teaching the Hierarchical Structure and Formal–Rhetorical aspects of proving and modes of structural thinking. They recalled all relevant facts prior to presenting a proof (Weber, 2011) or asked students to do so (Alcock, 2010). They might spend more time on steps they believed the students would find difficult to understand (Weber, 2011). But the most common tactic was to provide a series of heuristics that students could follow to create a proof, such as algorithms for operationalizing definitions, templates for certain proof types, and templates for introducing objects to be used in the proof (Alcock, 2010).

Weber (2011) found one mathematician who claimed to use classroom dialog to include students in the proof writing process. This claim, especially in the context of the calls for deeper understanding of undergraduate teaching (Harel & Fuller, 2009; Harel & Sowder, 2007; Mejia-Ramos & Inglis, 2009; Speer et al., 2010), is worth further investigation. The following analysis is an attempt at the kind of research called for, and details a case study of a teacher who self-identified as a traditional type of university teacher but used a significant amount of dialog with students in her proof presentation. It examines the qualities of that dialog with the help of a lens for studying classroom interactions.

2.5 Studying classroom interaction

The class used, as a case study, a question–answer dialog pattern. At the advanced undergraduate level, there is relatively little research on classroom interactions, and thus, the analysis in Section 4.3.2 draws from theory developed in lower-level classes. Research has focused on instructor questions across the K-16 spectrum and examined the nature and distribution of teachers' questions (cf. Nickerson & Bowers, 2008; Pollio, 1989), and the value of questions to student learning (Silver, 1996; Stein & Lane, 1996). Pollio (1989) examined teacher questioning at the undergraduate level across 10 fields and found that almost 70 % of questions went unanswered by students, and more than one third only required yes or no answers. Silver and colleagues argued that based on their work in K-12 settings, the questions most likely to lead to student learning required the students to engage in high-level thinking and reasoning (Silver, 1996; Stein & Lane, 1996).

Building on this work, Steinbring (1989) described discourse patterns that *funneled* students' responses in such a way that the teacher does the bulk of the intellectual work and the students' role is to respond to the primarily factual question that the teacher asks. Such a pattern might begin with a challenging question and pass through a series of questions that funnel the students' attention to particular features of the problem. Wood (1998) argued that this method is problematic because "the student needs to know only how to respond to the surface linguistic patterns to derive the correct answer" (p. 172). To illustrate this pattern, Herbel-Eisenmann and Breyfogle (2005) showed an interaction where a teacher asked a class what the slope of a particular line was. When there were no responses, the teacher asked a set of factual questions focused on a particular computation that funneled the students to the correct slope. Other questions in this discourse pattern might ask students to supply the next step in a procedure or a particular fact or definition. Wood contrasted this with discourse patterns that *focused* student attention on the important aspects of the mathematics while also giving them primary responsibility for the work.

3 Methods

3.1 The teacher and institution

Abstract algebra was chosen as the mathematical content of this study because of its importance in the undergraduate mathematics curriculum (Committee on the Undergraduate Program in Mathematics, 1971; Mathematical Association of America, 1990). When this study began, Dr. Tripp (a pseudonym) was an assistant professor working toward tenure at a mid-sized doctoral granting institution in the USA that claimed a strong focus on undergraduate teaching. She was selected for study for a number of reasons. The first was because she was interested in educational issues, having co-authored a teaching manual as a graduate student. She had also participated in Project NeXT, a professional development program for new or recent Ph.D.s in the mathematical sciences, with the goal of changing her instructional practices in introductory mathematics courses.

An initial interview with Dr. Tripp revealed that she had earned a doctorate in algebra and taught abstract algebra a number of times prior to the study. This meant that she was an expert in the mathematics field and had relatively settled pedagogical practices. Her class placed significant emphasis on developing students' proof-writing abilities. The final criterion for her selection was that she self-identified as a traditional teacher of abstract algebra.

To her, this meant that lecturing in front of the class would be the predominant method of instruction and that proofs would "form the backbone of this course." On the continuum of teaching from pure telling to pure discovery, Dr. Tripp claimed that she would be closer to pure telling. In short, she was selected because in her teaching of abstract algebra she claimed to use traditional teaching practices, she was a subject matter expert, and because her pedagogical expertise suggested she would be a good teacher.

3.2 The class

Dr. Tripp's organization and presentation of material closely followed that of the text *Abstract algebra: An introduction* (Hungerford, 1997). The major topics of the course were rings, fields, and finally group theory. The students in the class were typically in their third year of undergraduate study who had completed a two-semester calculus sequence and an introductory course on mathematical proof; 13 such students were enrolled in the course section being studied, as well as one student in her second year, and one student in his final year of undergraduate study. The class met four times each week for 50-min sessions during the spring semester. Besides the course meetings, there were no other regular or organized class activities (e.g., discussion sections). The students were expected to regularly produce proofs on their homework and examinations.

3.3 Methodology for analysis

I observed 18 of the class meetings, 15 of which were video recorded. The video camera was positioned at the back of the room and faced the front of the room to record the board and overhead projector because all proof presentations were done in the front of the room. The camera primarily recorded the instructor, as she was the principal focus of all class activities and the second goal of the research was to understand how she devolved responsibility to the students. Students were only video recorded when they presented at the board or engaged in dialog. In each class meeting, I took detailed field notes of all classroom activity including all text written on the board or written on a transparency and displayed via the overhead projector. I transcribed everything that Dr. Tripp and those students who had signed consent forms said in order to further the description of the modeled proof writing and devolvement of responsibility.

I then reviewed all classroom video recordings, field notes, and transcripts to log all episodes that included proof writing or presentation. An incident was noted as a proof production or presentation when any member of the class community described, wrote, or presented a formal mathematical proof that drew on symbolic notation and logical reasoning. Working with particular examples of structures was treated as proof, as long as the presenter used algebraic techniques (demonstrating that the integers under the operations of addition and multiplication form a ring would be treated as a proof).

During the logging process, I first distinguished between who wrote or presented the proof, classifying it as student or teacher authored. For each of the proofs that Dr. Tripp presented, I engaged in two rounds of coding. The first round of coding was used to respond to the first research question, asking what aspects of proof writing the instructor modeled for her students. As a result, I coded each utterance and all mathematical text that she wrote on the board or displayed via overhead projector. The utterances and written text were coded to reflect the aspects of proofs of Selden and Selden (2009) by examining their role in the proof. Statements were coded as modeling thinking about the Hierarchical Structure if they described what a proof had to accomplish, the coordination of subproofs, or showed that the proof was complete. Statements were coded as modeling thought about Proof Framework if they described a choice of proof route, the conventions of proof, or the structure of the proof. Statements were coded as modeling thought about the Formal–Rhetorical part of the proof if they related to symbolic manipulation or moves. Statements were coded as modeling thought about the Problem-Centered part of a proof if they discussed key ideas, the coordination of nonstandard argument structures, or conceptual understanding of the constructs under study. Statements were coded as modeling thought about the Construction Path if they described a difference between proof writing and the final or presented form of the proof.

Dr. Tripp's statements and mathematical writings were further analyzed to note the modes of thinking of Alcock (2010), which support proof writing in semantic or syntactic ways. Any statements that represented only algebraic manipulations were described as syntactic. References to mathematical objects or previously worked examples were described as semantic; this characterization was further refined as instantiation when a known example was referenced, as critical thinking when example(s) were referenced to support or structure the proof, and again as critical thinking if the statement served to check that the claim was correct or the proof was complete.

The second research question asks how Dr. Tripp devolved responsibility to the students and what responsibility she devolved. In responding to that question, I performed a second round of coding of all dialog during Dr. Tripp's proof presentations. I described the types of sentences used during proof presentation as declaring knowledge, process or procedure. For the questions that Dr. Tripp asked, I similarly classified the response that Dr. Tripp attempted to solicit as seeking a statement of a factual response, or a contribution to the proof writing (that required more than a factual statement), based on the student response. For those that required more than a factual statement, I described the contribution's effect using the previous coding via the framework of Selden and Selden (2009) or the modes of thinking of Alcock (2010). In the present article, I give a representative example of her modeling of proof writing and devolvement of responsibility chosen from week 11 of the semester.

4 Results

This section begins by giving an overview of the proof writing and presentation practices in Dr. Tripp's class. Next, it shows an example of one proof presentation that represents her pedagogical practices and organization of results. After this case study, there is a summary and analysis of the instructor's actions. The analysis is presented first in terms of the proof-writing framework and then in terms of the responsibility for proof writing that Dr. Tripp devolved to the students during class meetings.

4.1 Proof writing with dialog

During the 18 class meetings that I observed, the students and teacher combined wrote 29 analytical verifications of properties or other results. Seven of these proofs were given entirely by students; in all seven cases, these proofs were property–verification arguments (such as demonstrating that the associative property held in a given algebraic structure). Dr. Tripp wrote one proof without any dialog with students, asking rhetorical questions without waiting for a response. The remaining 21 proofs had a consistent structure: Dr. Tripp asked a large number of questions during her lecture that solicited student feedback while writing a proof, and did not give a monolog lecture. I call this type of interaction a proof presentation with dialog. In all such proofs, the dialog began with a teacher-initiated question, most often directed at the whole class. This question generally solicited a student response that was relatively short, which was followed by a comment by Dr. Tripp and possibly another question.

4.2 An example of the proof presentation with dialog scheme

I have chosen to show a proof that the kernel of a ring homomorphism is an ideal of the domain. To better understand Dr. Tripp's actions, it is important to consider what the students had previously done and what Dr. Tripp said and did immediately before presenting the proof. First, Dr. Tripp introduced the claim while discussing ideals in rings and noted, "So far, we've only looked at a couple of examples of things of the form *R* mod *I*, namely polynomials, quotients of polynomial rings like this, and *Z* modulo *n*. Where else do these things show up?" Then, she reminded the students that they had previously done homework determining the kernel of a specific function, treated many specific cases of homomorphisms of polynomials (in the construction of roots of irreducible polynomials), and that on the previous examination the students were asked to show that the kernel was a subring of the domain. Finally, before beginning the proof, Dr. Tripp worked through a proof that the kernel of the standard homomorphism between the integers and the integers modulo three is an ideal. It was only after this discussion and example that Dr. Tripp began the proof that the kernel of any ring homomorphism is an ideal of the domain.

Let us now examine the proof presentation that the kernel of a ring homomorphism is an ideal of the domain. The written proof is included near the end of this section.

> Dr. Tripp: Let's see why the thing called *K*, which has a name, it's the kernel, is an ideal all the time. So, we need to get back to this ring homomorphism. If we have any ring hom *f* from *R* to *S*, let's show *K* is an ideal. What do we have to do to show it's an ideal? [pause] You have to show it's closed under addition, closed under multiplication, it's non-empty, every element has an additive inverse. What do those four things tell us?
> S: Subring.
> Dr. Tripp: Subring.

In this section of the proof presentation, Dr. Tripp introduced the goal of the proof and then asked a rhetorical question: "What do we have to do to show it's an ideal?" This question structured the proof and can be seen as modeling thinking related to the Hierarchical Structure of the proof, as Dr. Tripp explicitly stated what must be proved. She finished the comment with a factual question that did not require any understanding of the context of the proof. Note that the student's response contributed to the proof by showing that the proof structure would satisfy the definition of ideal, which states, "A subring *I* of a ring *R* is an ideal…" (Hungerford, 1997, p. 135). Thus, little responsibility was devolved to students via this question, since Dr. Tripp had already structured the proof and the students could, therefore, assume that it was complete.

> Dr. Tripp: [continuing] And then, I need what? I multiply an element in *R*, I take anything in *K* and I multiply by any element in *R*, and that product comes back into *K*. That's the ideal condition. That last condition actually includes, like we said, that multiplication is closed. So, then we just need to check addition, that it's non-empty, that additive inverses work out, and the ideal condition. So, let's do that.

In this comment, Dr. Tripp completed modeling the creation of the Hierarchical Structure by stating that the proof must check the ideal condition. Then she noted a possible change to the proof, verifying that the ideal condition also shows multiplicative closure. This can be understood as potentially modeling a Construction Path. However, the proof could be completed by verifying only the ideal condition if it were noted that the ideal condition implies multiplicative closure. Thus, this can also be understood as proposing two different forms of the proof, both correct, rather than proposing a Construction Path distinct from the final form of the proof.

Dr. Tripp began the subsequent subproof by verifying the non-empty property of subgroups. Although she did not explain why this was the first property that she demonstrated, this is traditionally the first property verified because it is often the easiest and usually verifies the identity property as well.

Dr. Tripp: How could I show it's non-empty? How do I show that there's something that goes to zero?
S: [Inaudible]
Dr. Tripp: Yeah. So, we know that $f(0_R)$ equals 0_S, so there's something there. So, 0_R is in K. So, it's got something in it.

Dr. Tripp modeled completion of an appropriate Hierarchical Structure aspect of the proof by stating that the subproof was complete. In this section, Dr. Tripp did not give the students any responsibility for the proof writing. She then stated the next subring properties to verify and began the second subproof:

Dr. Tripp: [continuing] That may be all that's in it, and if so, that tells us something very special about that map. Okay, let's take two things, not r and s, how about a and b. If a and b are in K, I want to show their sum is in K. How do I show their sum is in K? You have to use the definition of big K. The only thing you know about big K is, well, it consists of stuff that gets mapped to zero. So, what do I have to show about $a+b$ to show it's in K?
S: It gets mapped to zero.

In this set of comments, Dr. Tripp modeled the behavioral knowledge needed for the Formal–Rhetorical part of the proof by demonstrating that a set is closed under an operation by first choosing two arbitrary elements. She also modeled thinking related to the Hierarchical Structure aspect by stating the definition of closure, which told students which subproof to complete. Finally, it is important to note that when Dr. Tripp called into being two arbitrary elements, she was careful in her choice of representatives ("not r and s") to avoid possible confusion with previous variables. This seems to be an explicit pedagogical choice and a possible demonstration of avoiding confusion of variables when writing proofs.

Dr. Tripp devolved responsibility to the students for explicitly stating what must be proved about $a+b$. This is an example of demonstrating the Hierarchical Structure aspect of proof writing and may have required structural thinking. However, this devolvement was a reformulation of Dr. Tripp's previous statement.

Dr. Tripp continued by modeling the remainder of the closure subproof.

Dr. Tripp: Ok, so, let's look at what f does to $a+b$. So, [Student], what can I say about $f(a+b)$?
S: It equals $f(a)$ plus $f(b)$.
Dr. Tripp: Is there anything I know about $f(a)$ now?
S: It equals zero.
Dr. Tripp: Great, and $f(b)$? And what do I know about zero plus zero? That's zero. Great. So, $a+b$ meets the condition it needs to be in K. [pause] So, that's the property

of f preserving addition that we just used, and that gives us that the kernel is closed under addition.

Dr. Tripp suggested a way to begin the subproof and asked students to make use of the behavioral knowledge that is required in the Formal–Rhetorical part of a proof to complete the next step. When she claimed that the subproof had been completed, she again modeled thinking about the Hierarchical Structure aspect. In this interaction, Dr. Tripp devolved the responsibility for stating a known fact about $f(a+b)$. This was the next algebraic step, and again, may have required structural thinking.

Dr. Tripp then essentially repeated the closure subproof creation process for multiplication. She modeled the same aspects of proof writing in demonstrating the ideal property:

Dr. Tripp: And, let me change something… Also, how about we make this for any a in K, well, we already said a was in K, and how about for any r in R? Well, if I change b to be r, actually we could have just left this as b, but then we'd get zero. And that includes a and b being in K, as we were just talking about. Likewise, $f(ra)$, what is that going to equal?
S: $f(r)$ times $f(a)$.
Dr. Tripp: Great. And what is that going to equal?
S: $f(r)$ times 0, zero.
Dr. Tripp: So in other words, it doesn't matter, they're both the same. So, ra and ar are in K. So, what do I need left to check that this is an ideal?
S: Inverses.

In this last set of exchanges between Dr. Tripp and the students, she appeared to do two things: model the importance of choosing good notation again and model proving the ideal condition in the same way she modeled the closure arguments. Dr. Tripp again demonstrated thinking about the organization and structure of the proof here by clearly stating that one subproof was complete ("ra and ar are in K") before asking the students what subproof still needed to be done. That is, she modeled creation of some aspects of Hierarchical Structure, but devolved some of the responsibility for creating the Hierarchical Structure to the students. Similarly, she gave the students responsibility for structural thinking by asking them to supply subsequent algebraic steps.

Dr. Tripp finished writing the proof by writing a subproof that the kernel of a homomorphism includes additive inverses. At the same time, she cycled through thinking related to the Hierarchical Structure aspects and called on the behavioral knowledge required for the Formal–Rhetorical parts of proof. She began by giving a statement describing the Hierarchical Structure of the proof:

Dr. Tripp: Finally, let's look at what f does to negative a. For all a in K, $f(-a)$, what can I say about that? What does f do to negative a?
S: $-f(a)$.
Dr. Tripp: Yeah, this is another one of our properties of homomorphisms. f carries an additive inverse to the additive inverse of the image, to negative $f(a)$, and that equals negative zero! Yeah! And what's the additive inverse of zero? Zero, so we get zero. So, K is an ideal. [Underlines: K is an ideal.] So as a consequence, any time you have a homomorphism of rings, you get an ideal.

Here, she modeled one last type of thinking related to the Hierarchical Structure of the proof by claiming that the proof was complete by claiming that demonstrating each required property was sufficient to guarantee that the kernel is an ideal. Dr. Tripp devolved responsibility for structural thinking to the students by eliciting the statement that $f(-a)$ is equal to $-f(a)$; (Fig. 1).

In all of the above exchanges, Dr. Tripp expected students to respond to questions that she asked. Yet, the questions were all product questions, and each time she evaluated the student's response by saying, "Yeah," "Great," or asking a follow-up question. While the students were active in the sense that they were expected to answer questions about the proof, their contributions exclusively involved straightforward syntactic reasoning. None of their responses required semantic reasoning or reasoning about other aspects of the proof, such as the Proof Framework, the Construction Path, or the Problem-Centered part. Instead, their responses stated a fact or a subsequent algebraic step that had become routinized after multiple similar proofs. While the questions did not prompt the students to be meaningfully engaged in the proof-writing process, the subsequent section will describe how Dr. Tripp's questions and statements did model thinking related to aspects of proof writing.

4.3 Analysis of various aspects of proof writing with dialog

4.3.1 Aspects of proof writing

Dr. Tripp was consistent in modeling the aspects of proof writing described in Section 2.2 and modes of thought described in Section 2.1. For example, in all observed cases of proof, Dr. Tripp instantiated the ideas via relevant examples; in the proof presented above, the students had already worked with a number of examples, and Dr. Tripp presented another example immediately before presenting the proof. This suggests that she indicated to the students that instantiation is a critical action for understanding. However, she did not explicitly link instantiation with any of the subsequent proof-writing activities.

Table 2 takes as the unit of analysis the 22 individual proofs that Dr. Tripp wrote and presented during the observed class meetings. It summarizes how often (in terms of the proportion of proofs) Dr. Tripp modeled the thinking related to each of the aspects of proof writing and modes of thinking that support proof writing. For example, if Dr. Tripp modeled the Hierarchical Structure at any point during the entire proof, this thinking would be counted here. The third line describes how frequently Dr. Tripp involved students in that

Fig. 1 Dr. Tripp's written proof

Let $f: R \to S$ be a ring homomorphism and let $K = \ker f$.
Then, $f(0_R) = 0_S$.
Let $a, b \in K$.
Then, $f(a+b) = f(a) + f(b) = 0_S + 0_S = 0_S$.
$f(ab) = f(a)f(b) = 0_S \,.\, 0_S = 0_S$

Added to line 3:
Let $r \in R$.

Revised the last line to read:
$f(ar) = f(a)f(r) = 0_S \,.\, f(r) = 0_S$

Added:
$f(ra) = f(r)f(a) = f(r).\, 0_S = 0_S$
$f(-a) = -f(a) = -0_S = 0_S$

Thus, K is an ideal. [Later underlined]

Table 2 A summary of the rate at which Dr. Tripp taught aspects of proof writing

	Aspect of proof writing				
	Proof Framework	Hierarchical Structure	Formal–Rhetorical aspect	Problem-Centered	Construction Route
Modeled	More than 50 %	Always	Always	Never	Never
Students involved	Never	More than 50 %	More than 50 %	Never	Never

teaching via previous homework or questions during the proof, without judgment about the quality of the involvement.

In every proof that she presented, Dr. Tripp said aloud and wrote statements that modeled the thinking related to the Proof Framework, Hierarchical Structure aspects, and Formal–Rhetorical part of proof writing. In most proofs, she made multiple statements of these forms. Her teaching could therefore be understood as helping students to develop these proof-writing skills for themselves. Yet most of her statements modeling Proof Framework, Hierarchical Structure, and Formal–Rhetorical thinking did little more than describe what needed to be proved and then verify that it had been proven.

In terms of Proof Framework, Dr. Tripp was never observed to discuss conventions of proving nor why she chose a particular proof route (direct, indirect), although this may be because the students had already taken a transition-to-proof course. Similarly, Dr. Tripp was only observed to model certain types of thinking related to the Hierarchical Structure aspects and Formal–Rhetorical parts. At the beginning of a proof, she consistently stated what the proof needed to show. She began the proof that a kernel of a homomorphism is an ideal: “What do we have to do to show it’s an ideal? [pause] You have to show it’s closed under addition, closed under multiplication, it’s non-empty, every element has an additive inverse.” These statements modeled for the students the creation of a list of properties to be shown and determined the appropriate subproofs. Furthermore, when Dr. Tripp modeled thinking about the Hierarchical Structure at the end of proofs, she essentially stated that the proof did what was required. As an example, consider, “So in other words, it doesn’t matter, they’re both the same. So, ra and ar are in K.” Although these statements are important in comprehending the Hierarchical Structure aspect of a proof, they do not, by themselves, give the students any tools for proof writing beyond “write out all the properties that must be proved and then verify that you prove them.” Yet these statements about the Hierarchical Structure show that Dr. Tripp taught one mode of thinking that supports proof writing, that which Alcock (2010) called critical thinking, or the habit of looking back and ensuring that the proof accomplished what it needed to. Thus, Dr. Tripp’s statements about the Hierarchical Structure aspects of a proof were a useful teaching move.

Moreover, although Dr. Tripp always stated what proof route she would use, she never explained why she chose a particular proof route. She would simply begin and indicate via her proof the structure that she had chosen: direct, contradiction, or contraposition. Thus, while Dr. Tripp modeled thinking about significant aspects of the Proof Framework, she was never observed to help the students choose the Proof Framework themselves.

Finally, as in the proof shown above, Dr. Tripp was never observed to write an aside or in other ways indicate that the Construction Path was different from the route for writing the proof in any way other than a very trivial modification (changing the order of properties that need to be verified or noting that the ideal property includes multiplicative closure). This should not be seen as a criticism of her instruction, but rather a disciplinary issue, since it is possible that there is little need for different Construction Paths in algebraic proofs.

All of Dr. Tripp's proofs were completely and correctly written from start to finish; thus Dr. Tripp's proof presentation could be understood to model concept of an ideal prover of Selden and Selden (2009). At no time was she ever observed to make a major mistake in logic or other aspects of proof, nor was she observed to pursue an incorrect or ineffective avenue. An analysis of all proof episodes further shows that Dr. Tripp was consistent in the aspects of proof that she did (or did not) engage in and what aspects she expected the students to engage in.

Dr. Tripp was observed to model the types of thinking Alcock (2010) identified as supporting proof writing in two other important ways. First, she always instantiated the concepts and involved students in doing so more than half of the time. But she was never observed to draw on the instantiation in writing the proof, nor did she ever explicitly suggest that the students do so. Thus, while she instantiated the concepts to promote understanding, which is important to proof writing, she was never observed to fully demonstrate instantiation in terms of using examples to support the proving process. Second, at the end of each observed proof, Dr. Tripp stated that the proof had shown all that was necessary, and, as a result, she modeled a type of critical thinking. The students were only infrequently involved in this check, and always in the form of a question to the class: "Did we show that…?" to which they always responded, "Yes." Dr. Tripp would then reiterate why the proof was complete. In other words, although Dr. Tripp modeled this type of critical thinking, she only did so in a cursory way; her method may not have offered the students the opportunity to learn to read and validate proofs.

4.3.2 Asking questions, devolving responsibility

The following section describes how Dr. Tripp's questions can be understood as modeling mathematical thinking, but, at the same time, did not give the students a meaningful chance to practice such thinking. Dr. Tripp asked 294 questions during observed proof writing or presentation episodes, and the students responded to 171. Many of the questions were an immediate rephrasing of a previous question without any pause, as in "For all a in K, $f(-a)$, what can I say about that? What does f do to negative a?" When she asked these linked questions, the subsequent questions almost always reduced the cognitive load of answering. This pattern can be understood as *funneling* the students toward a certain end while Dr. Tripp did the serious intellectual work (Steinbring, 1989). Similarly, the manner in which Dr. Tripp asked questions resulted in students giving factual responses based on linguistic and contextual clues. Even the questions that could be thought of as proof-strategy questions were basically factual because they were part of *funneling* patterns (Steinbring, 1989). For example, Dr. Tripp used very similar phrasing and gave other verbal cues when asking about proofs of different properties exactly as Wood (1998) described. Yet the discourse sequences Dr. Tripp used in class can be understood as the way that a mathematician might approach the task of proof writing. For example, in the pair above, the first question asks "What can I say about that?" which leaves a number of possibilities open; the second question asks students to recall a specific fact about homomorphisms: that $f(-a)=-f(a)$.

Similarly, earlier in the proof presentation Dr. Tripp asked:

> How do I show their sum is in K? You have to use the definition of big K. The only thing you know about big K is, well, it consists of stuff that gets mapped to zero. So, what do I have to show about $a+b$ to show it's in K?

Responding to the first question would require the students to recall the definition of K (the kernel) and translate that definition into an operable form for the proof. Dr. Tripp answered this question and reduced the cognitive load of the question to a level where all that was required was rephrasing her previous statement, "$a+b$ is mapped to zero."

Although repeatedly rephrasing the question reduced the cognitive demand for the students and the level of responsibility they were given for the proof, Dr. Tripp modeled a mathematically meaningful proof-writing strategy that allowed her to write the next step of the proof. If she had asked the questions in the reverse order (or only asked the final question), she would not have modeled an appropriate proof-writing strategy. She would not have given the students meaningful responsibility for the proof, or the opportunity to think about an important question for proof writing. This set of questions balanced the modeling of proof-writing strategies with devolving responsibility for proof writing to the students. Consider a final example, where she defined a homomorphism:

> And, we asked ourselves before, what's a natural way to go from R to R mod I? [pause] What should I do? Take an element and where can I send it? What is R mod I? [pause] The set of all cosets mod I. And, what form do we write those in?

The first question is high level and requires significant understanding on the part of the students, but the final question in the string asks, "What form do we write [cosets mod I] in?" This last question required students to state a fact that should be familiar from their repeated use of coset notation.

This claim is strengthened by an analysis of the language Dr. Tripp used just before asking a question. For example, before asking a question about the meaning of the term homomorphism, she said, "You have to know your terms down cold, but you also have to be able to translate them into symbols." Then she asked, "So, what do I need to do to show this is a homomorphism?" The student who responded appeared to understand that he was being asked to restate the definition in symbolic form. In the proof presented above, Dr. Tripp asked the students 12 questions, all of which required a factual response. Generally, the students responded as expected, stating an appropriate term or result of an algebraic manipulation.

During proof presentations, Dr. Tripp also frequently made statements about the goals of the proof that changed her questions about proof structure into factual questions. For instance, she occasionally described the outline of the proof and made statements that organized the verification of individual properties, as in, "OK, so, let's look at what f does to $a+b$." This told the students the next part of the proof to work on. As a result, although Dr. Tripp did ask a number of proof-strategy questions that required high-level thinking, she responded to all of them without pause, just as the teacher of Herbel-Eisenmann and Breyfogle (2005) did. Subsequently, using both statements and questions *funneled* the students' attention to syntactic aspects of the proof-writing process.

In the episodes of proof writing with dialog, Dr. Tripp balanced modeling aspects of proof writing and the thinking that supports proof writing with devolving responsibility to the students. She gave students frequent responsibility for working on examples related to proofs, the Hierarchical Structure, and Formal–Rhetorical aspects of proof writing (Selden & Selden, 2009). She gave this responsibility by asking factual questions to which the students would respond. Their answers stated the next part of a proof or the next algebraic step. Asking for the next algebraic step may be understood as giving the students responsibility for the structural mode of thought (Alcock, 2010). Dr. Tripp also gave the students responsibility for some components of proof writing during the class meetings; there were seven proofs entirely written or presented by the students where the students had full responsibility for the work.

4.3.3 A summary of Dr. Tripp's proof presentation

Dr. Tripp consistently taught the types of thinking required to write proofs as well as the modes of thought that support proof writing. She also appeared to cede some responsibility to her students for proof writing and presentation through consistent questioning. She also consistently began by describing what must be proved and ended with, "Have we shown X?" But, as argued above, the students did not have a significant role in proof writing during class because of the nature of the questions.

Dr. Tripp consistently modeled thinking about aspects of Proof Framework, Hierarchical Structure, and the Formal–Rhetorical parts of a proof, as well as instantiation, structural, and critical modes of thinking. Dr. Tripp was only observed to model a nontrivial construction path once, when she responded to a question about a homework problem on which the students were stuck by spontaneously constructing a proof during class. The students saw little evidence of struggle, and there was no evidence that any of the proofs demonstrated might require a different route to create from their final form. Unlike Dr. T in the paper of Weber (2004), Dr. Tripp never divided the board into different sections for thinking and writing the proof. All of her proofs were written as the one above, proceeding from the beginning to the end, without revision, or even erasure of anything but notation errors. In every proof, she stated the claims to be proved, asked for a definition of appropriate terms, and then proceeded from beginning to end without ever making a mis-step or mistake other than notational errors. She never described or demonstrated that the construction path might be distinct from the final form of the proof, nor did she demonstrate any creative thinking that might model the Problem-Centered skills needed for proof writing. Thus, at no point during Dr. Tripp's proof presentation were students shown that proving is difficult or might require anything other than algorithmic stating of definitions; her proof required only translation via the behavioral knowledge that is required to complete the Formal–Rhetorical part of a proof and symbolic manipulation.

Yet, two cautions are in order. First, in all of the proofs that Dr. Tripp presented, there was no need to construct the proof in a way that differed from the way that the proof would be presented. Thus, this should not be understood as a critique of Dr. Tripp's pedagogical choices, but rather an observation about the opportunities that students had to learn.

Second, the manner in which Dr. Tripp asked the students questions generally reduced the cognitive demand to a level of factual recall and sometimes mere restatement. This happened because these factual questions often came after a string of questions that began with a high-level question following a funneling pattern. For example, in the proof presented above, Dr. Tripp repeatedly asked questions like "What can we say next?" or "What does f do to…?" These factual questions have a factual response that Dr. Tripp recorded as part of her proof—the next step. When asked in the context of the proof, initial questions required the students to understand the mathematical content of the theorem to be proven, but the final questions did not, meaning that the students' responses had been funneled to the correct answer. Yet, the sequence of questions modeled appropriate mathematical thinking by a master of the discipline. The question sequences can be understood as the internal dialog that a mathematician might engage in when approaching a proving task. As a result, claims about possible learning in Dr. Tripp's class based on discourse patterns should be tempered with acknowledgement of other possible avenues for learning.

5 Significance and directions for future study

The first significant aspect of this study is that it describes and analyzes the teaching of an undergraduate abstract algebra class. Before this, there were relatively few studies of lecture-based undergraduate instruction and even fewer from abstract algebra classes. Since this is a single case study, it is inappropriate to draw generalizations from it. Yet, without a body of empirical evidence, there is no basis for more theoretical work. Because of the exploratory nature of this study, the conclusions I draw below are tentative.

The first conclusion is that Dr. Tripp consistently modeled certain aspects of proof and modes of thinking that support proof writing, but not others, as summarized in Section 4.3.3.

The second conclusion I draw from Dr. Tripp's teaching is that the students were likely to participate in proof writing or presentation activities in this "traditional" classroom. In 29 observed proofs, the students presented seven by themselves, and Dr. Tripp engaged in proof writing with dialog schemes 21 times. She only once delivered the caricatured "lecture" where students sat quietly and took notes without questions or comments. Dr. Tripp asked questions and expected responses from students; students were expected to regularly participate. She modeled the types of questions that a mathematician should ask while writing proofs, such as, "What does that mean?", "What comes next?", and "What do I still need to do?" In short, this instructor, who claimed to give traditional lectures, actually gave lectures that were more student-centered than some might expect. However, the questions Dr. Tripp asked were generally factual questions. Further, these questions usually were limited to a very specific aspect of a proof—recalling a definition or performing a straightforward manipulation—and the students' responses were typically funneled to a particular set of responses. Furthermore, Dr. Tripp answered many of the questions that required significant understanding without waiting to see if a student would respond. In other words, Dr. Tripp asked a large number of questions throughout the class, and these questions can be understood as teaching skills needed in writing proofs. But she answered the most difficult of the questions herself, devolving only limited responsibility to the students for the proof writing. The students were expected to engage in all aspects of proof writing on their homework and examinations so the claim about devolving responsibility should understood as limited to Dr. Tripp's in-class proof presentation.

These observations illustrate how well-intentioned instruction, even instruction that includes a lot of student participation, might provide students with less responsibility than is reflected by the number of questions asked and answered. Dr. Tripp's questioning of students is admirable; it engaged the class and required active thought about the material being presented. However, these questions were also limited, because they did not provide the students with opportunities to practice some important skills for proof writing. Rather, Dr. Tripp executed the types of reasoning supporting proof writing herself (though it is important to note that she did this in a way that could be understood as an expert modeling appropriate mathematical behavior). Examining whether teachers in other student-centered classrooms in advanced mathematics also reserve this responsibility for themselves would be a useful topic for future research.

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