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Levels of students' "conception" of fractions

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Abstract In this paper, we examine sixth grade students' degree of conceptualization of fractions. A specially developed test aimed to measure students' understanding of fractions along the three stages proposed by Sfard (1991) was administered to 321 sixth grade students. The Rasch model was applied to specify the reliability of the test across the sample and cluster analysis to locate groups by facility level. The analysis revealed six such levels. The characteristics of each level were specified according to Sfard's framework and the results of the fraction test. Based on our findings, we draw implications for the learning and teaching of fractions and provide suggestions for future research.

Keywords Procedural and conceptual understanding · Fraction · Part–whole subconstruct · Measurement · Equivalence · Comparison · Rasch model

1 Introduction

In order to understand how students learn new mathematical ideas, two distinct, though closely interrelated approaches have been identified: procedural and conceptual. Several researchers (e.g. Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991) have suggested that in concept formation procedural understanding precedes conceptual. The concept construction occurs through concrete situations, procedures and processes and moves on towards the abstraction of mathematical concepts, to understanding of symbols and mental concepts. This development has been described as: action–process–object (Dubinsky, 1991), procedure–process–procept (Gray & Tall, 1994), and interiorization–condensation–reification (Sfard, 1991).

Sfard proposed a scheme of three stages of conceptualization or "structuralization", describing the characteristics that students exhibit at each stage in terms of behaviors and skills. Some researchers used this scheme to investigate students' construction of knowledge in algebra (Goodson-Espy, 1998) or the fraction duality as process and object (Herman, Ilucova, Kremsova et al., 2004).

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Based on theoretical and empirical evidence (Sfard, 1991; Gray & Tall, 1994; Kerslake, 1986), the present study accepts that procedural understanding precedes conceptual understanding. It also assumes that full development of a concept is only achieved when a student possesses both conceptual and procedural understanding of the concept. To investigate these views, the study examined students' understanding of fractions using the three stages described by Sfard (1991). The choice of fractions was motivated by the fact that many students fail to understand them conceptually, remaining at the procedural, superficial knowledge level (English & Halford, 1995).

An integration of the research concerning student's conceptualizing of fractions and Sfard's theoretical framework will contribute to deeper understanding of students' way of developing fraction knowledge. The results of the study will inform teachers and facilitate them in sequencing related topics of instruction (Noelting, 1983).

The rest of this paper is organized in four sections. In the next section, we summarize the theoretical background and define the research questions. In the third section, we describe the methods and in the fourth one, we present the findings of the study. Last, in the concluding section, we discuss the findings and draw implications with regard to theory and practice.

2 Theoretical framework

2.1 Concept formation

Researchers asserted that learners acquire new mathematical ideas in variable ways, mainly procedurally/operationally or conceptually/structurally (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). Learners proceed through these two ways of learning to enhance their mathematical knowledge. Mathematical knowledge is defined in this study as "an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organizing, in her or his mind, mathematical processes and objects with which to deal with the situation" (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996, p. 171). Procedural knowledge or operational conception refers to processes, algorithms and actions, or to processes of actions (Sfard, 1991; Hiebert & Lefevre, 1986). Conceptual knowledge is defined as the knowledge loaded in relationships (Hiebert & Lefevre, 1986; Hallett, Nunes, & Bryant, 2010). Sfard described conceptual knowledge as structural conception that treats a mathematical notion as an abstract object, as a static structure.

These two distinct types of knowledge lead to different results in terms of mathematical performance. Even though both the procedural and conceptual approaches can be efficient in problem solving, and regardless of which of the two types of learning comes first in the acquisition of a mathematical idea, researchers (Charles & Nason, 2001; Sfard, 1991; Gray & Tall, 1994) suggest that learners who rely on conceptual knowledge may have an advantage over those who develop only the procedural knowledge. Learners relying on conceptual knowledge develop sophisticated mathematical thinking, while those who rely on procedural knowledge face difficulties in handling complicated conceptual structures. Procedural learning, although indispensable in mathematics, can only be stored in "unstructured, sequential cognitive schemata" which cannot be easily processed and therefore may lead to insufficient understanding and mere completion of routine mathematical tasks. Conversely, conceptual understanding includes "static, object-like representations", which compress the operational information into a whole, and develops cognitive schemata into more convenient structures (Sfard, 1991, p. 26). These compact abstract entities encompass mathematical ideas procedurally and conceptually, providing the learner with higher-order thinking and proficient knowledge (Gray & Tall, 1994).

While research is more consistent as to the importance of conceptual understanding, a discussion is still going on, concerning which of the two approaches precedes the other in students' learning of a mathematical notion (Byrnes & Wasik, 1991; Rittle-Johnson et al., 2001). Hallett et al. (2010) have argued that children may possess some mix of conceptual and procedural knowledge and that there exist individual differences in the way of combining these two approaches. However, research seems to be inclined that students perform best in mathematics when they acquire both conceptual and procedural understanding of a mathematical notion (Sfard, 1991; Hallett et al., 2010).

Researchers viewing that in the concept formation procedural/operational understanding precedes the conceptual/structural one, developed frameworks describing the process of concept formation (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). The main core in these frameworks is essentially similar and can be related to Piaget's notion of "reflective abstraction", in which actions on known objects become interiorized as processes and then encapsulated as mental objects of thought (Pegg & Tall, 2005).

The framework of Sfard (1991) differs from the others in a major way: even though she recognized the ontological gap between operational and structural conceptualization, she underlined the complementary nature of the two faces of learning, supporting the dual nature of mathematical constructs. In this framework, it seems that the ability of a learner to develop a mathematical concept is a gradual progress leading to reified objects whose structure gives a conceptual growth focusing on the properties of the objects. Sfard (1991) distinguished three hierarchical stages, which correspond to three degrees of structuralization. At the interiorization stage, the learner becomes accustomed to the processes performed on lower-level mathematical objects and gradually develops competence to perform these processes. This competence allows the learner to interiorize the process, that is to think of a process without actually carrying it out.

At the condensation stage, the learner becomes capable of thinking about a complicated process as a condensed whole, without feeling an urge to go into details. During this stage, the learner gradually becomes capable of compressing lengthy sequences of operations into more manageable elements in a form that it is easier for him/her to use and think; he or she deals with alternative forms and representations of the concept, combines processes, and makes comparisons and generalizations.

Finally, at the reification stage, the learner becomes able to conceive of a concept as a "fully-fledged object" (Sfard, 1991, p.19); the various representations of the concept are integrated in the learner's reified construct so that the construct no longer depends upon any process. The learner perceives the construct by understanding the conceptual category in which it belongs and its main properties and he/she notices relations between its representatives. The reified concept can be used as an input in higher-order processes, while new mathematical objects can be constructed out of the present one.

2.2 Fractions

A factor contributing to students' difficulties in learning fraction refers to its multifaceted construct (Behr, Harel, Post, & Lesh, 1992; Kieren, 1993; Lamon, 1999). In this respect, five subconstructs have been identified: part–whole, ratio, quotient, measure, and operator. For instance, the fraction 3/4 can be conceived as a part of a whole (three out of four equal parts), as a quotient (three divided by four), an operator (three quarters of a quantity), a ratio (three parts to four parts), and finally as measure (as a point on a number line).

In this study, we focus on two of these subconstructs, as grounds to examine students' level of conceptualization, namely the part–whole and the measure, together with the notions of equivalence, comparison and addition of fractions. The choice of these two subconstructs was based on research (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003; Ni, 2001) revealing consistently that students' performance on tasks measuring the part–whole subconstruct was higher than their performance on tasks measuring the other subconstructs; on the other hand, students' performance on tasks related to the measure subconstruct was the lowest. In addition, in Kieren's theoretical model of the fraction concept the part–whole subconstruct is considered fundamental for developing understanding of the other four subconstructs. In this model, the measurement subconstruct is considered necessary for developing proficiency in additive operations on fractions (Charalambous & Pitta-Pantazi, 2007).

The choice of the equivalence and comparison of fractions was based on research (e.g. Behr et al., 1992; Ni, 2001) suggesting the importance of order and equivalence to the understanding of a fraction as an entity (a single number). The use of fraction addition items was used to investigate students' approaches in solving fraction addition activities classified as procedural (Kerslake, 1986) or conceptual (Byrnes & Wasik, 1991). Moreover fraction addition assesses students' ability to conceive the fraction as both a process (3/4) in an isolated context, and a fraction in process (3/4+1/6; Herman et al., 2004). In the specific social context in which the study was conducted, the fraction as part-whole subconstruct dominates mathematics textbooks and instruction, while the measurement subcontract appears less frequently. The other three subcontracts are not taught in a uniform way in this social context. Equivalence, comparison and addition of fractions are emphasized in the mathematics textbooks for grades 5 and 6.

2.2.1 The part–whole subconstruct and the measure subconstruct

The notion of fraction is frequently introduced via the part–whole subconstruct, which requires an ability to partition a continuous quantity or a set of discrete objects into equal-sized parts or subsets (Lamon, 1999). Several progressive activities have been suggested to help students master the part–whole subconstruct. Students have to recognize when an area or a set of models are divided in parts of equal size or not, they have to be able to proceed with partitioning and unitizing, that is, to construct a fraction when a whole is given and to conceive of a whole when a fraction is given. Students should realize that the relationship between the parts and the whole is conserved, regardless of the size, shape and arrangement of the equivalent parts and that the more parts the whole is divided into, the smaller the produced parts become. Students are also expected to repartition the whole and reconstruct the unit (Boulet, 1998). Research has shown that the part–whole subconstruct is quite easy for young students. For instance, Charalambous and Pitta-Pantazi (2007) found that the mean score for fifth and sixth grade Cypriot students in part–whole items was 75%.

On the other side, the measure subconstruct could be regarded as an indication of convergence of several subconstructs (Hannula, 2003). In the measure subconstruct, a measure is assigned to some interval on the number line (or region in the case of two dimensional models), a unit fraction is defined and used repeatedly to determine a distance from a preset starting point (Charalambous & Pitta-Pantazi, 2007). Lamon (1999) considered the following three capabilities as an indication that students understand the measure subconstruct: (a) perform partitions other than halving; (b) find any number of fractions between two given fractions; and (c) use a given unit interval to measure any distance from the origin.

Several researchers have reported on students' difficulties with the number line. For instance, Hannula (2003) found that only 20% of the fifth graders could locate the fraction 3/4 on a number line marked with the interval 0–1. Students have more difficulties when the unit measure is not equal to the denominator of the fraction, than when it is equal to the denominator. The reason for students' difficulties in these tasks may be found in the two different representations involved (the symbolic and the iconic; Charalambous & Pitta-Pantazi, 2007).

2.2.2 Fraction equivalence

Understanding of fraction equivalence constitutes one of the most important mathematical ideas in the primary school and a major difficulty for students (Ni, 2001). This difficulty has been ascribed to the multiplicative nature of the concept and also to the various subconstructs related to the fraction concept. Specifically, the difficulty concerns two related aspects of operative thinking, the multiplicative thinking and the conservation of the whole and the parts. According to Kamii and Clark (1995) multiplicative thinking constitutes a hierarchical structure, for example to think of 4×3 involves thinking on two hierarchical levels at the same time: "one 3, two 3s, three 3s, four 3s", and "four 3s" at once. Moreover, conservation of the whole and the parts in the case of fraction equivalence refers to students' ability to think of the same part of a whole as being for example three-twelfths and one-fourth.

Parallel to this, Ni (2001) argued that students' conception of the equivalence of fractions depends on the subconstruct involved in the process; the part–whole subconstruct is the easier since to perceive equivalence of two fractions in their regional area embodiments, one can make the judgment with the aid of perceptual prompts. In addition, English and Halford (1995) contended that equivalence of fractions represented in set embodiments is not suitable for noticing fraction equivalence since an understanding of equivalence is a prerequisite to identify equivalence of fractions. In this situation, the ratio remains constant while the number of counters increases or decreases. Moreover, perceiving equivalent fractions represented on a number line requires the conception of equivalence of two quotients together with the recognition of the denseness and order of rational numbers (Ni, 2001). In support of the above, Kamii and Clark (1995) found that only 44% of the fifth graders were able to perform a task requiring multiplicative reasoning, and Ni (2001) found that fifth and sixth grade students performed best on the area items, less well on set items, and the poorest on the number line items.

2.2.3 Comparison of fractions

The comparison of fractions includes finding of the order relation between two fractions (Arnon, Nesher, & Nirenburg, 1999). One of the key elements in comparing fractions is to understand that the greater the number of parts into which the unit is partitioned, the smaller the fraction size (English & Halford, 1995). To succeed in the ordering of fractions, children need to conceive that among fractions with the same denominator, the larger the numerator, the larger the fraction, (for example $2/4<3/4$), while among fractions with same numerator, the larger the denominator, the smaller the fraction, (for example $3/4>3/5$; Nunes et al. 2004). The difficulty that students face with the comparison of fractions can be seen in several studies. For instance, in the NAEP study (Kenney & Silver, 1997), only 12% of the fourth graders could compare two fractions with the same numerator and justify their answer, while only 30% of the eighth graders could order three fractions from the smallest to the biggest. Likewise, in the TIMSS study (Christou, Papanastasiou, & Philippou, 2003) only 25% of the Cypriot fourth graders could compare the fractions 1/3 and 1/4 and justify their answers.

2.2.4 Addition of fractions

Students' difficulties with the concept of fraction extend also to fraction operations; students may well apply the algorithm, but they lack any conceptual understanding and have limited idea of the sensibleness of their action. One possible reason is the way fractions are presented in many textbooks, through meaningless procedures (English & Halford, 1995). Students may not get involved in forming connections between the symbolic procedures and their manipulations with concrete analogs. This lack of conceptual understanding makes students develop only procedural competence. A second cause may be the prerequisites of this operation. Kieren's theoretical model supports the view that the notion of fraction as measurement is considered necessary for developing proficiency in additive operations on fractions (Charalambous & Pitta-Pantazi 2007). Since the measurement subconstruct is the most difficult for students to understand and it is rarely met in mathematics textbooks (Delaney, Charalambous, Hsu, & Mesa, 2007), it may lead students to limited understanding of the addition of fractions. In the study by Herman et al. (2004), students could find the sum of two fractions applying routine procedures using LCM but were unable to represent fraction addition.

2.2.5 Students' development of the fraction concept

Structural operation and the action–process–object models (Sfard, 1991; Dubinsky, 1991) indicated that the process conception precedes the object conception; "a process solidifies into object, into a static structure" (Sfard, 1991, p.20). Similarly, other researchers (Charles & Nason, 2001; Gray & Tall, 2007) agreed that the concept of fraction begins as a procedural activity in which different procedures may produce different sized fractions with the same quantity. The process of abstraction starts when students realize that different sharing situations can end up in equivalent fractions, thus the attention shifts from the sharing process to the outcome as an object, the fraction. In the same vein, Kerslake (1986) reported that many children were able to add fractions correctly but they were unable to explain the procedures they used, supporting the view that procedural precedes conceptual understanding.

Contrary to these views, other studies suggested that conceptual understanding is the basis of development and that procedural understanding is only a helpful means to be applied after the acquisition of conceptual understanding. In Byrnes and Wasik (1991) study, fourth and sixth graders were asked to answer questions assessing their conceptual understanding and, in a second study, their procedural understanding. Conceptual understanding was measured by the recognition of equivalent fractions and ordering of fractions and procedural understanding by addition of fractions with different denominators and multiplication of fractions. They found that to succeed in fraction addition required students to apply the same type of knowledge as that used to compare and order fractions using approaches other than the common denominator procedure.

Hallett et al. (2010) suggested that in understanding fractions, students may use a mixed approach of procedural and conceptual understanding according to their individual differences, and that the two kinds of understanding may be viewed as separate, without one necessarily leading to one another. In their study with fourth and fifth graders, they found five distinct clusters of students as regards their success with procedural and conceptual fraction problems. That means that students have relied on one or both types of understanding and these differences may not be developmental. An important finding of this study was that students who possessed both conceptual and procedural understanding outperformed the other students.

This finding is in line with Sfard's belief that "certain mathematical notions should be regarded as fully developed only if they can be conceived both operationally and structurally" (p. 23). Herman et al. (2004) investigated if fraction in process (1/2+1/4) and fraction as process (1/4) are interrelated according to Sfard's theory. They found that students could represent fraction as process but only few students could produce the image for the addition of fractions even though they could find the sum of the fractions in their symbolic form. In line with Hallett et al. (2010) they concluded that the routes of these two parts may be cognitively separate. A sound explanation of these results could be that few students can conceive this duality and these are the students who succeed in developing the notion of the fraction concept.

2.2.6 Aims and research questions

Based on the above analysis, we focused on students' conceptual understanding of fractions according to Sfard's theoretical framework. The prime aim of the study was to seek further insight into the way through which students proceed in the process of fraction understanding. Instruction based on learners' previous knowledge, through the difficulty levels, would thus be more meaningful (Noelting, 1983), providing a "prescription for teaching" (Sfard, 1991).

More specifically, the aim of this study was to develop and validate a test measuring students' mathematical performance in fractions, especially part–whole and measure subconstructs, along Sfard's framework. In particular, the study sought answers to the following questions:

1. To what extent does a test developed along Sfard's theoretical framework satisfactory assess students' performance in fractions?
2. Are there three distinguishable difficulty levels concerning students' abilities in fractions, in accordance with the characteristics of interiorization, condensation, and reification (Sfard, 1991)?

3 Methods

3.1 The development of the test

To answer our research questions, we progressively devised a test assessing the characteristics of each one of the stages proposed by Sfard (1991). Most of the tasks comprising the test were adopted from published research (Hannula, 2003; Herman et al., 2004; Kamii & Clark, 1995; Lamon, 1999; Saxe, Taylor, MacIntosh, & Gearhart, 2005). The tasks were further developed using specific representations that, in line with other studies (Ni, 2001) we assumed and tested in the pilot study, would play a determining role in students' understanding of fractions. Eventually, the tasks reflected the characteristics of each of the model's stages and suited the curriculum for fifth and sixth graders in Cyprus as evidenced by the official mathematics textbooks. Many of the activities created to assess the characteristics of the condensation and the reification stage are missing from the mathematics textbooks; they were planned to require students to apply conceptual approaches similar to other studies (Hallett et al., 2010). The test consisted of 21 tasks in seven triads reflecting understanding at each stage of Sfard's scheme in the two fraction subconstructs and the three operations respectively. Some items required only a step-by-step procedure, while others required the application of both conceptual and procedural understanding of the concept. The rationale for choosing these tasks follows (all tasks are shown in Appendix A).

Task A1 (Lamon, 1999) requires students' awareness that a fraction represents equal shares of a quantity. This task was created to assess characteristics of the interiorization stage as it requires a step-by-step procedure, i.e., count all the parts of each shape-denominator, count the shaded parts-numerator and find the fraction, which is the key towards understanding of the part–whole subconstruct (Boulet, 1998). In parallel, task B1 (Saxe et al., 2005) reflects characteristics of the condensation stage because students might combine different processes to reunitize the whole. In task C1 the shape is divided into parts of unequal size and different shape; students should reunitize the whole, a final step in the process of understanding the fraction as part–whole (Boulet, 1998). This task mirrors traits of the reification stage, since the students should be able to perceive the fraction as an object and conceive the representations of the parts.

Task A2 (Lamon, 1999) reflects features of the interiorization stage, since students have to follow a certain procedure—count all the objects (rectangles and triangles), count the triangles and write the fraction. Task B2 (which was constructed for the study) requires alternation between different representations and combination of processes with other processes—both characteristics of the condensation stage. Task C2 (Lamon, 1999) assesses characteristics of the reification stage, since the various representations of the concept are expected to be unified in the learner and the construct is no longer dependent upon any process.

In task A3 (Lamon, 1999) students have to follow a procedure, count all the objects (18), divide the number of objects by the denominator and multiply by the numerator. Students' achievement in B3 requires a degree of interiorization of the concept of fraction and also the combination of different processes. In C3, students have to combine the various relations of the categories of the fraction (unit fraction, fraction, whole, improper fraction).

Task A4 is designed to assess students' understanding of the measure subconstruct. This task (Hannula, 2003) evaluates characteristics of the interiorization stage, "a process has been carried out through mental representation" and number line representations are representations of mental objects corresponding to mental operations (Ni, 2001). Task B4 was created for the study and assesses students' ability to combine various processes and to alternate between different representations, a characteristic of the condensation stage (Sfard, 1991). Students can achieve success in task C4 if they can conceive the concept of fraction as an object, a static structure and detach it from any process, while at the same time they are expected to have a sense of equivalence, of density, of order, and relative magnitudes of fractions (Lamon, 1999).

Task A5 assesses students' understanding of the equivalence of fractions requiring the application of a routine procedure. In task B5, students are called to alternate between different representations and to reason for fraction equity, characteristics of the condensation stage. Task C5 asks students, in the context of a problem, to represent the equivalence of fractions using the symbol x. To succeed in this task, students should perceive the invariance of a multiplicative relation between the numerator and the denominator or the invariance of a quotient (Kamii & Clark, 1995) and turn it into a static entity—the symbol x, a new-born object.

Task A6 refers to the comparison of two fractions with like denominators, an application of a routine procedure. Task B6 involves comparison of two fractions with unlike denominators using two different ways. The flexibility in the approaches used we believe starts from the condensation stage. Task C6 requires students to find a fraction between two fractions with consecutive denominators. To succeed in this task students should conceive of each fraction as an object, and the density of fractional numbers. All three tasks were developed for the present study.

In A7, students are asked to apply a routine that leads to the sum of two fractions with like denominators, an application on lower-level mathematical object. In B7, they have to find the sum of two fractions with unlike denominators. This task is considered to measure characteristics of the condensation stage since even if students apply the common denominator procedure this is not a straight forward procedure. It involves awareness of the fraction as a concept and the combination of various processes such as the equivalence of fractions. Moreover, other studies showed that students may use also conceptual approaches to solve this task (e.g., Byrnes & Wasik, 1991). Finally, the parallel task C7 (Herman et al., 2004) consists of three steps. The first step is associated with the sum of two fractions with unlike denominators, the second asks students to represent the process of the sum by making a drawing, and the last step asks students to pose a problem from the addition of the two fractions. These three steps of task C7 assess students' ability in the operation of addition as both a process and an object in process.

The tasks of the test were content and face validated by ten experienced primary school teachers and two university tutors of Mathematics Education. Based on their comments, minor revisions were made.

3.2 Participants

The test was administered to sixth graders by the researchers. In the pilot study, it was completed by 302 participants. After the analysis of the data, modifications were made to three of the tasks in order to correspond to the difficulty and the characteristics for which they were developed for. The final version of the test was administered to 321 students from 15 classes of an economically homogeneous school district and it lasted from 40 to 60 min.

3.3 Data analysis

The specific construction of the test with the items reflecting certain characteristics of the framework under examination led us to mark students' responses with 0 if the answer was wrong and 1 if the answer was correct. We performed two types of analysis on the 321 students' responses, Rasch model and cluster analysis: the first to examine the reliability of the test as a whole and to specify the sequence of items by difficulty and the second to search for groups of students in each of the difficulty levels.

The Rasch model is appropriate for the specification of this scale because it allows the researcher to test the degree to which the data meets the requirement that both students' performances on the test items and the difficulty of the items form a steady order (within probabilistic constraints) along a single scale (Bond & Fox, 2001).

Identifying an individual's position on this scale provides information about this individual's probability of success on items below (high probability) and above (low probability) this position. At the same time, identifying an item's position on the scale informs precisely about the individuals who can succeed (those scoring higher than this item's position) or fail (those scoring lower than this item's position) on the scale.

Specifically, in the Rasch model, the probability that a person j will give a correct response to an item i is modeled on the basis of his/her ability θ_j and on the item difficulty β_i. These two parameters are both estimated on the same continuous scale. It is hypothesized that the probability of a correct response is a logistic function of the difference between θ_j and β_i. This S-shaped function converts any value of the real line into a value between 0 and 1. Moreover, the parameters θ_j and β_i refer to the same latent range. This latent range is described in two ways: as the dimension on which the participants are located in increasing order regarding their ability and as the dimension on which the items are located in increasing order regarding their difficulty. The Rasch model converts these likelihoods into quantitative estimates of item difficulty and person ability formed in the same equal-interval metric, the logit scale, in reference to the log odds unit used (Wright & Masters, 1981).

In addition, reliability and validity assessments require the following criteria: First, that item difficulty and person ability estimates should be related with an error term that would allow the researcher to set up confidence intervals for all item and person ability estimates. Second, it requires having model fit statistics that would allow the researcher to examine if both items and participants fit the requirements of the model. The fit statistics used are (a) infit (weighted) and (b) outfit (unweighted) mean square. Fit statistics are used to assess whether a person's ability (an item's difficulty) is coherent with other persons' ability (other items' difficulty) and are based on the differences between the expected and observed scores. Outfit statistics are based on the disparity between observed and expected scores. In the calculation of infit statistics, extreme persons or items are excluded. Likewise, the fit statistics can be approximately normalized using the Wilson–Hilferty transformation. The normalized statistics are called infit-t and outfit-t and they have a mean close to zero.

Cluster analysis was used to search for the existence of difficulty levels, i.e., whether the items could be systematically grouped into levels of difficulty along a continuum that reflects Sfard's framework. We used the procedure for detecting pattern clustering in measurement designs developed by Marcoulides and Drezner (1999). This procedure enables us to segment the observed measurements into constituent groups (or clusters) so that the members of any one group are similar to each other, according to a criterion (in our case difficulty). Next, examining students' responses, we proceeded in determining a pattern in each level according to the pre-described characteristics.

4 Findings

4.1 Item fit

The fit of the Rasch model was examined first by investigating the overall fit of the items in the fraction test and also by investigating the fit of individual items. The infit and outfit mean square were the fit statistics used in order to detect any discrepancies between the Rasch model prescriptions and the fraction data. Fit analyses were run in Quest (Adams & Khoo, 1996) and the results are shown in Table 1.

Table 1 presents the overall fit of the items as reported by infit and outfit fit statistics. According to Table 1 both item fit statistics have means very close to 1 which indicates that pupils' responses to test items are generally in accordance with the Rasch model. The analysis also revealed that all test items have item infit with the range 0.83–1.21. Usually items are considered to fit the Rasch model if they have item infit in the range 0.77–1.30 (Adams & Khoo, 1996).

4.2 Person and item descriptive statistics

This section presents and examines person and item estimates as reported from the Rasch model, the reliability of these two estimates and the relation between them, through an item-person map. Table 2 presents a summary of item difficulties and person performances.

Table 1 Overall item fit

Statistic	Infit fraction test (21 items)	Outfit fraction test (21 items)
Mean	1.00	0.99
SD	0.10	0.23
Minimum	0.83	0.51
Maximum	1.21	1.31

The mean item difficulty is located at 0 by default. The mean person ability estimate is 0.31 logits which shows that the test was slightly easier for the particular sample. Reliability estimates are interpreted in the same way as Cronbach's alpha, i.e., on a 0 to 1 scale. According to Table 2 the reliability of both item and person ability estimates is quite high. According to Bond and Fox (2001), the higher the number of reliability estimates, the more confidence we can place in the replicability of item placements across other samples or of person placements across other tests like fraction test.

Figure 1 illustrates the scale for the 21 items of the fraction test with item difficulties and student measures adjusted on the same scale. Person's performances are represented by an "X" on the left hand side of the map while items are indicated by their item number on the right hand side of the map. The vertical line in the middle of the map represents the logit scale as both item difficulties and person abilities are measured in logits. Equal distances on the map, either up or down, have equal value as the logit scale is an interval scale.

The difficulty scale extends from easy items (negative logits) at the bottom of the column and moves on to harder items at the top. The distance between logits has a probabilistic meaning; in our study an ability estimate for a given student means that the probability that he/she will succeed on an item at the same level is at least 50%.

Clearly, all test items have a good fit to the measurement model, indicating an agreement among the 321 students located at different positions on the scale, across all 21 items.

Figure 1 presents items developed to meet the characteristics of the interiorization stage (1–7) to be located at the bottom of the scale as easier than the items created to assess characteristics of the other two stages. The items developed based on the characteristics of the condensation stage were located in the middle of the scale (8–14), while the items reflecting characteristics of the reification stage were found to be the hardest (16–21).

Item C1 (15) was the only item which was planned to assess characteristics of the reification stage and in practice appeared to be easier. The task asked students to write the fraction presented in a shape divided in unequal size and shape parts expecting them to reunitize the whole; it might be the upper step in the understanding of the fraction as part–whole according to Boulet (1998) but it nevertheless refers to part–whole subconstruct, which is among the easiest for students to understand.

It appears that the easiest items of the scale were 7, 5, and 6, requiring the application of procedures, the addition of similar fractions (A7), the equivalence of fractions (A5), and the

Table 2 Summaries of items and persons estimates

Statistic	Item estimates (21 items)	Person estimates (N=321)
Mean	0.00	0.31
SD	1.64	1.35
Reliability of estimates	0.99	0.81

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Item Estimates (Thresholds)
all on all (N = 321 L = 21 Probability Level= .50)
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  4.0                          XXX |
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                                XX |
  3.0                              |
                                   |
                                   |
                            XXXXXX |       19
                                   |       21
                                   |       20
                           XXXXXXX |
  2.0                              |
                                   |
                        XXXXXXXXXX |       16      17
                                   |
                        XXXXXXXXXX |       18
                                   |
                      XXXXXXXXXXXX |       11
  1.0                              |       13
                XXXXXXXXXXXXXXXXXXX |
                                   |
                   XXXXXXXXXXXXXXX |
                                   |        9
                      XXXXXXXXXXXX |
   .0                              |       10      12
                    XXXXXXXXXXXXXX |
                                   |
                 XXXXXXXXXXXXXXXXX |       15
                                   |        8
                                   |
                         XXXXXXXXX |       14
 -1.0                              |        4
                          XXXXXXXX |
                                   |        1       3
                           XXXXXXX |
                                   |
                              XXXX |        2
                                   |
 -2.0                              |        6
                               XXX |
                                   |
                                   |        5
                             XXXXX |
                                   |
                                   |
 -3.0                              |
                                 X |        7
                                   |
                                   |
                                   |
                                   |
                                   |
 -4.0                            X |
--------------------------------------------------------------------------------
  Each X represents    2 students
```

Fig. 1 Scale of pupils ability and item difficulty

comparison of fractions with the same denominators (A6). Item A1, which asked students to circle the representations of the fraction 1/4, proved to be harder than the other tasks also planned to reflect characteristics of the interiorization stage. Students were expected to be aware that fractions represent equal parts of the whole. Not surprisingly, the hardest item of this stage was A4, which referred to the measure subconstruct (A4).

In the same vein, the hardest among the items reflecting condensation stage traits was B4 (11), which required understanding of the fraction as measure subconstruct (Fig. 1). The second most difficult item of this group was B6 (13), which asked the comparison of two fractions in two different ways followed by item 9 (B2) asking students to alternate between different representations of a fraction as part of a set of discrete objects. Next came item 12 (B5) in which students had to write the fraction represented in a continuous quantity-rectangle and choose among the representations of continuous quantities-circles the one that represented the fraction. The easiest task of this group was B7 (14), which required the addition of fractions with unlike denominators.

At the top of the scale were the items developed to reflect characteristics of the reification stage. The hardest item of all C5 (19) corresponded to the equivalence of fractions using the variable *x*, requiring a level of abstraction. The second most difficult item C7 (21) corresponded to the addition of dissimilar fractions through various procedures. Another item that proved to be hard for students was the C6 (20), which asked for a fraction between the fractions 1/9 and 1/8. The difficulty of this item might be due to the various sub procedures students should engage in order to complete the task. For instance, to perceive these fractions as objects each one representing a set of equivalent fractions and realizing the denseness and order of fractions. Next in the facility order was item C2 (16), which asked students to circle from various representations the ones that corresponded to a certain fraction followed by C3 (17) asking students to find an improper fraction based on a given fraction, and C4 (18) that corresponded to the measurement subconstruct.

4.3 Using cluster analysis to specify levels of concept formation

Having established the reliability of the test, we moved to the second research question, to investigate if the items were systematically grouped into levels of difficulty corresponding to the three stages developed by Sfard. Appendix B provides a description of the procedure used to identify clusters (Marcoulides & Drezner, 1999).

Applying this method to segment the 21 test items on the basis of difficulty that emerged from the Rasch model, the optimal clustering resulted in six clusters, explaining 61% of the variance, indicating six difficulty levels in the process of fraction concept development. Table 3 summarizes the results of cluster analysis.

Level 1 refers to the item A7, which proved to be the easiest item. Level 2 consisted of the items A5, A6 and A2, developed according to the characteristics of the interiorization stage. Level 3 can be characterized as a transition level from the interiorization to the condensation stage as it includes items A3, A1, and A4, the hardest items of the interiorization stage, and items 14 (B7) and 8 (B1), the easiest items of the condensation stage. Item 15 (C1) was also placed in this group due to its difficulty as already justified. Level 4 encompass items 10 (B3), 12 (B5), and 9 (B2), all of them reflecting condensation traits. Level 5 can be conceived as a transition from the condensation to the reification level; it consists of the hardest items built on the features of the condensation stage, 13 (B6), 11(B4) and the easiest items reflecting features of the reification stage, 18 (C4), 16 (C2), and 17 (C3). Finally, in Level 6, the hardest items reflecting characteristics of the reification stage were placed. These were items 20 (C6), 21 (C7), and 19 (C5).

5 Discussion

5.1 Investigating the reliability of the test

On the basis of the outcomes of this study, 20 of the 21 items of the test could be consistently placed on a scale concerning difficulty, reflecting characteristics of the three stages described by Sfard in understanding fractions. The role of representations in

Table 3 Cluster analysis of the 21 items into six groups

Clusters	Rasch	Tasks
1	–… to −3.11	7 (A7)
2	−2.37 to −1.69	5 (A5), 6(A6), 2 (A2),
3	−1.27 to −0.39	3 (A3), 1(A1), 4 (A4), 14 (B7), 8 (B1), 15 (C1)
4	0.09 to 0.33	10 (B3), 12 (B5), 9 (B2)
5	0.9 to −1.65	13 (B6), 11 (B4), 18 (C4), 16 (C2), 17(C3)
6	2.25 to …	20 (C6), 21 (C7), 19 (C5)

students' way to the understanding of fractions was apparent; the presence of some representations in the items was a decisive element in turning the concept on a higher level regarding conceptual understanding. The role of graphical embodiments was also stressed in the study of Ni (2001).

All items assessing characteristics of the interiorization stage proved to be within most students' reach. The addition of fractions with same denominators requiring the application of a procedure proved to be the easiest of all items, revealing that students initially develop procedural understanding—a finding in line with Kerslakes' (1986) study on fraction addition. Next along the difficulty sequence appear to be the items evaluating students' understanding of the equivalence and comparison of fractions, requiring the mastery of a certain rule. Success in tasks of this kind has been predicted by Sfard (1991) who argued that students may get through using step-by-step procedures, without necessarily understanding the fraction concept. The recognition of the part whole subconstruct in a regional area involving a conceptual trait, proved to be more difficult than the three pre-mentioned items. This clearly indicates that in students' understanding of fractions, procedural learning precedes conceptual learning, unlike the results of other studies (Byrnes & Wasik, 1991; Hallett et al., 2010). In line with other studies (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003), the most difficult item mirroring characteristics of the interiorization stage was the item assessing the measure subconstruct. According to Lamon (1999), students' difficulty with the measure subconstruct may be due to the absence of relevant activities in the mathematics classrooms; there appears to be an overdose of part–whole subconstruct activities at the expense of the other fraction subconstructs, something observed in the specific context in which the study was conducted. Consequently, teachers should be aware of and pay equal attention to all the different fraction subconstructs.

Item B4 was the most difficult among the items developed to assess characteristics of the condensation stage (Hannula, 2003). This task asked students to find the fraction presented in a regional area and place it on a number line. Unbalanced instructional emphasis concerning this fraction subconstruct together with the alternation between two representations, an activity rarely met in mathematics classroom, might be the causes for this result. The second hardest item of this group proved to be the one asking students to compare two fractions in two different ways and to justify their reasoning. Difficulties in the comparison of fractions were also traced in other studies (Christou et al., 2003; Kenney & Silver 1997) while asking students to use two ways to approach the item made it even harder. The easiest item developed for the condensation stage was the sum of two fractions with unlike denominators. This finding may again indicate students' facility to acquire procedural knowledge at the expense of conceptual knowledge. Even though Sfard (1991) states that the precedence of the operational conceptions over structural is invariant regarding the instruction process, the overdose of activities on fraction operations at the expense of activities for the development of conceptual understanding may facilitate this sequence of understanding.

The task referring to fraction equivalence using the symbol "x" proved efficient in assessing characteristics of the reification stage, as it was found to be the hardest of all the test items. The dose of abstraction included in this activity functioned as an obstacle, despite the fact that those students had some introductory lessons in algebra. In line with other studies (Hallett et al., 2010; Herman et al., 2004), the addition of fractions with unlike denominators and the graphic representation of this process assessing students' conceptual and procedural understanding, proved very difficult for students. This finding supports the results of Hallett et al. (2010) that students possessing both conceptual and procedural understanding perform the best of all the students. The third most difficult item reflecting traits of the reification stage was found to be the task requiring students to find a fraction between two given fractions with consecutive denominators, just as found in other studies (Kenney & Silver, 1997; Christou, et al., 2003). Many factors might have contributed to this difficulty, such as students' lack of ability to perceive the density of fractions with respect to two given fractions, or the lack of ability to perceive the two fractions as mental objects.

In conclusion, the results of the study reveal that students' diverse knowledge of fractions, procedural or/and conceptual, leads to students' diverse performance in solving fraction tasks. Moreover, the results may be a first indication that students' ability to understand fractions and apply their knowledge for solving fraction tasks can be modeled as a one-dimensional construct. The term "first indication" is used since the study concentrated on two subconstructs of the fraction concept, the three stages by Sfard and certain representations. The results also provide useful information for the instructional process (Noelting, 1983) regarding students' knowledge of fractions.

A deeper insight into students' approaches through the way from the procedural to conceptual understanding of fractions follows.

5.2 The difficulty levels in the way to acquire a fraction concept

In this section, we refer to and illustrate the six difficulty levels found in the study with regard to Sfard's three stages of concept development. We describe the characteristics of each level, examining the problems involved and the strategies used by students, in an attempt to shed some light on the way from the procedural to conceptual understanding of fractions.

Level 1 is characterized by procedural understanding. Students placed in this level are able to perform a simple step-by-step procedure for computing the sum of fractions with like denominators. In Level 2 similar characteristics as in Level 1 portrayed the items grouped together. Students could also apply a step-by-step procedure to fill the missing numerator in two equivalent fractions and to find the largest fraction among two fractions with same denominators. Figure 2 shows that a student used multiplication to fill the missing term.

The third item of the group, concerned with writing the fraction of a part of discrete objects, could also be accomplished by students applying only procedural understanding. A similar item was used by Hallett et al. (2010) for the evaluation of students' procedural understanding.

We consider Level 3 to be a transitional level from the interiorization to the condensation stage in the development of the fraction concept. This level illustrates Sfard's (1991) phrase: "the squeezing of lengthy sequences of operations into more manageable units" (p. 19). This level differs from the former two levels since it includes items requiring a high degree of procedural understanding and a low degree of conceptual understanding. Such a group was also identified by Hallett et al. (2010). Specifically, students placed in this level were able to find the fraction of a set of discrete objects (A3) and to select the correct representation of a fraction as a part of the equally divided whole (A1), a first step towards the conceptual understanding of fractions. Students were able to locate a fraction (3/5) on a number line from 0 to 1 divided into five equal parts, interiorizing the fraction as a mental object (A4). The ability to combine certain processes, a characteristic of the condensation stage, was also reflected in the next item of this group, the calculation of the sum of two fractions with unlike denominators. Many students applied the approach they were taught. They found a common multiple of the denominators of the fractions and converted the two fractions to ones with the same denominator using equivalent fractions, for example 2/5= 12/30. The last two items in this group refer to students' ability to divide a given representation of a fraction in equal parts (B1 and C1; Boulet, 1998) and find the fraction of the shaded part as a student did in Fig. 3.

The role of representations such as the number line and the representation of the fraction shape divided into unequal parts can be regarded as a decisive step in forming this level. For example in Fig. 3, the representation of the fraction forced that student to apply more than a routine procedure showing his awareness that the whole should be partitioned in smaller equal parts in order to write the fraction of the shaded area.

In conclusion, the items of Level 3 assess students' ability to combine certain processes, and to exhibit signs of conceptual understanding to a certain extent.

Level 4 corresponds solely to the characteristics of the condensation stage. The relevant test items reflect students' ability to think of a process as a whole, combine various processes, make comparisons and alternate between different representations. Again representations hold an important role in the formation of this level. Students in this level combined various processes and proceeded in the reconstruction of the whole from a given quantity, 2/3 equals four objects, finding the value of 1/3 and then the 3/3 (B3). Moreover students were able to alternate between different representations and succeed in tasks B5 and B2. Most important is the fact that such activities are rarely met in the classroom. The student in Fig. 4 used fraction equivalence to make comparisons and to choose the correct representation of the fraction signified in another representation. Student's reasoning for the choice of the representation referred to the representation as being more than half but less than 3/4, meaning that the student had interiorized the fraction as object and could justify his/her answer.

Level 5 can be a transitional level encompassing features of the condensation and the reification stage. Students placed in this level exhibited a perception of the fraction as an abstract construct to a certain extent and a possession of flexible thought. Specifically students' flexible thought could be exhibited through task B6, in which students had to compare two fractions in more than one way. Three of the approaches students used are presented in Fig. 5, i.e., the comparison of each fraction with one-half, the conversion of two fractions to fractions with common denominators, and the application of rules of proportion.

In addition, students placed in this level could find the fraction represented as a continuous quantity (3/4) and place it on a number line divided into different parts (eighths) than the ones presented by the fraction's denominator (B4). This activity requires the alternation between

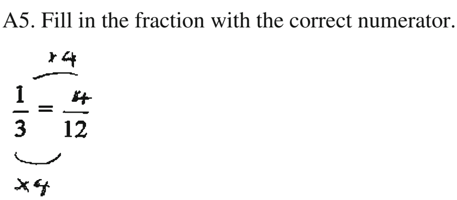


Fig. 2 A student's response to task A5

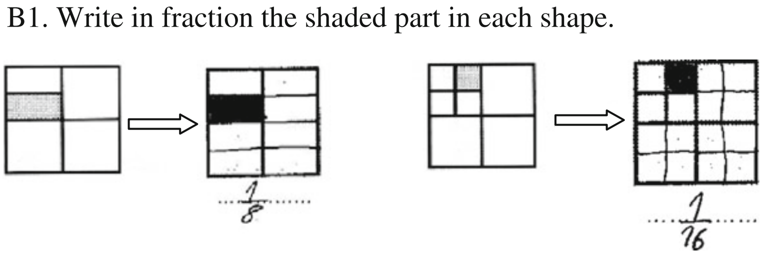


Fig. 3 A student's response to task B1

different representations and the perception of fraction as an abstract construct. In this level, students were able to fill in a number line marked from 0–1/2 divided in thirds (C4), with a missing fraction, an activity that entails the perception of a fraction as an imaginary construct, with its general properties (characteristic of the reification stage) like the equivalence, and order of fractions. Indeed, some students used fraction equivalence 1/2=3/6 and filled the missing fraction, 1/6, in the number line. Again, these two activities are rarely met in classrooms. Moreover, students could chose between various representations the ones that correspond to the same fraction (C2), disclosing a characteristic of the reification stage; the various representations of the concept are integrated in the learners' reified construct and the construct no longer depends upon any process. Last, students were able to find an improper fraction 7/6 from a given set of 12 objects which constituted the whole (C3), an activity which requires the perception of the fraction concept as on object, and a combination of the various relations of the categories of the fraction (unit fraction, fraction, whole, improper fraction).

Level 6 corresponds solely to the characteristics of the reification level. The results of this study show that the process solidifies into object, and students' are able to investigate general properties of fractions and various relations between their representatives. A distinct characteristic of this level was students' ability to move back and forth between an object and a process and thus understanding the mathematical idea both conceptually and procedurally (Hallett et al., 2010; Sfard, 1991). Specifically, students could find a fraction between two consecutive fractions (C6), an activity that requires the perception of the fraction as an object and the acquisition and coordination of various sub procedures and representation. Specifically, a student used the comparison of fractions with the same numerator to find a fraction between two consecutive fractions, 1/9=2/18 and 1/8=2/16 and

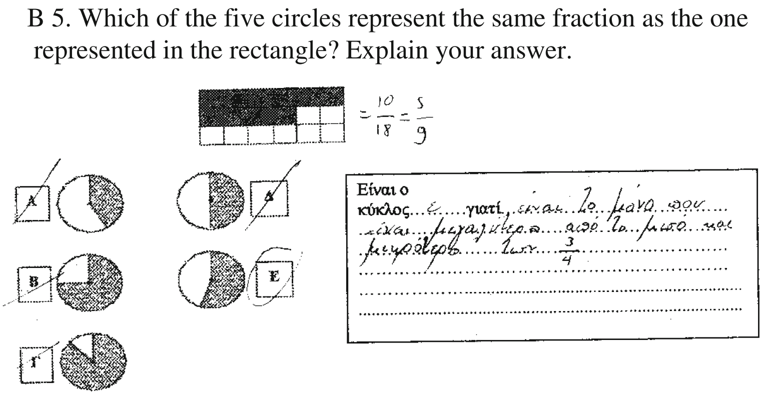


Fig. 4 A student's response to task B5

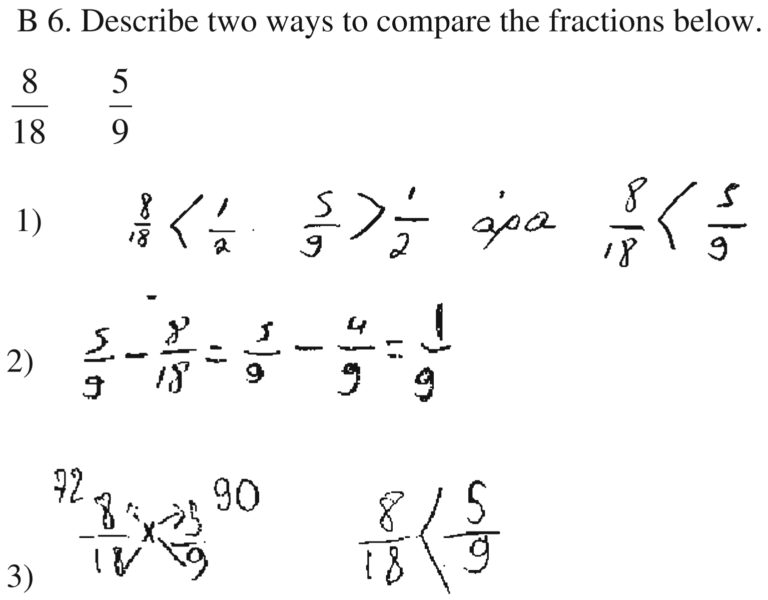


Fig. 5 Students' approaches in task B6

wrote 2/17 as an answer. Another student used fraction equivalence and turned fractions to fractions with the same denominators using 72 as the LCM in the first attempt and 144 in the second one, 1/9=8/72=16/144 and 1/8=9/72=18/144 and gave 17/144 as an answer. A representative item of this stage referred to the sum of two fractions with unlike denominators and the representation of the process (C7; Fig. 6) since in structural conception one knows both why and how to do something (Sfard, 1991). In line with the results of Hallett et al. (2010), this study reveals that students being able to apply both conceptual and procedural approaches perform best in mathematics.

Sfard notes that reification requires seeing the newly formed object in the context of processes that act upon it. Very few students managed to draw the process of the fraction addition and explicitly highlight how the two parts fit together in the process of adding. The most difficult item of this level (C5), which constitutes the point where an interiorization of

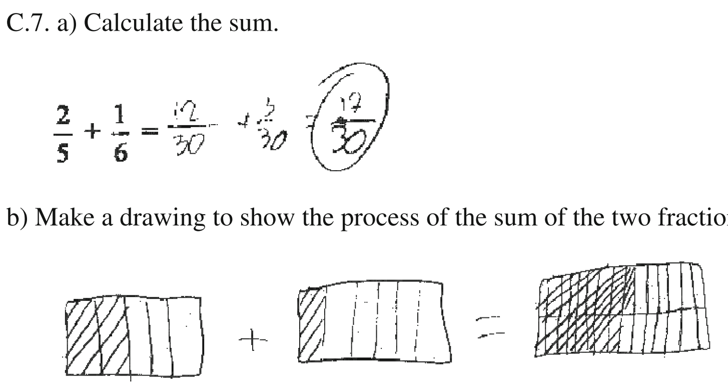


Fig. 6 A student's response to task C7

a higher level concept begins, is the one in which students had to use the symbol x and represent a problem referred to the equivalence of fractions as Fig. 7 sketches. An example of two approaches used by two students is presented in Fig. 7.

The first student chose to write only the generalized form of the pizza's pieces, providing the static relation between two magnitudes (response a) and the other explained his thoughts in a more sophisticated way (response b). Particularly, the second student explained that the numerator should be four times smaller than the denominator writing this fraction using the symbol x. This higher level item forced students to turn process into object (Sfard, 1991).

5.3 Further consideration

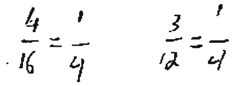
The results of the study, based on the specific social context, provided an indication that a difficulty scale across operational and structural understanding may exist. This study concentrated on two fraction subconstructs and some representations. More research using quantitative methods could investigate the fact that students' understanding of fractions can be described as a one-dimensional construct taking into consideration the fraction subconstructs, a framework's characteristics and the various representations.

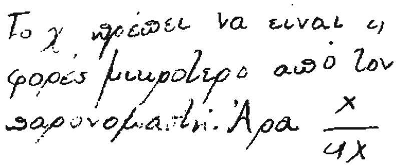
In addition, the cluster analysis revealed six levels, each of them including distinct characteristics regarding students' behaviors, attitudes and skills, as regards Sfard's framework in relation to fractions. Even though Sfard (1991) views the hierarchical sequence from procedural to conceptual understanding invariant of external intervention like teaching and textbook, we suggest that teaching and textbook can contribute to the acceleration of students' way through the stages. The findings of the study reveal that students who rely only on procedural knowledge have lower performance on fraction tasks than students who gain also conceptual knowledge. Moreover, the study suggests that the use of representations and the alternation between representations are substantial elements for the development of students' conceptual knowledge. The importance of teaching students more

Three friends ordered three pizzas of the same size and shape. George ate $\frac{4}{16}$ of his pizza. Andreas ate $\frac{3}{12}$ of his pizza. Costas ate x pieces of his pizza. If the three friends ate the same amount of pizza, write how many pieces might be Costa's pizza using the variable x.

Response a

Response b





(x must be 4 times smaller than the denominator. So $\frac{x}{4x}$)

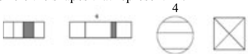
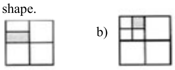
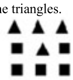
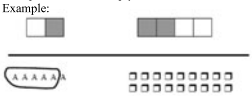
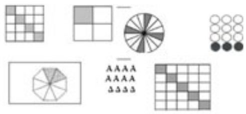
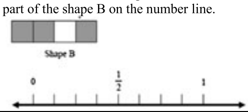
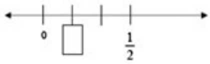
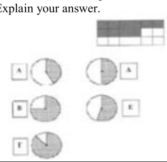
Fig. 7 Students' responses to task C5 than one approach for solving a task is also stressed. Finally, activities that demand explanation and justification using various means (words and figures) can also contribute to students' understanding of fractions. Concisely, the specific levels disclosed in the study can constitute the basis for a sequencing of instructional activities related to fraction understanding, to a certain extent, since the study focused on two of the fraction subconstructs.

No doubt, further research is needed to illuminate the six levels revealed in this study in the same and different social contexts including also other fraction subconstructs. It is also essential to investigate environmental or affective factors that may facilitate students' way from the procedural to conceptual understanding represented through these levels.

Acknowledgments We wish to thank Dr. Leonidas Kyriakides and Dr. Demetra Pitta-Pantazi of the University of Cyprus for their constructive comments during the research of this study.

Appendix A

Fraction test items

Task	Items-Interiorization stage	Task	Items-Condensation stage	Task	Items-Reification stage
A1	Circle the shapes that represent $\frac{1}{4}$. 	B1	Write in fraction the shaded part in each shape. a) b) 	C1	Write in fraction the shaded part of the shape. 
A2	Write in fraction what part of the shapes are the triangles. 	B2	The picture on the left represents a fraction. Use the picture on the right to represent the same fraction as the one on the left picture. The example below will help you. Example: 	C2	Circle the pictures that represent the same fraction. 
A3	Circle $\frac{2}{3}$ of the whole. 	B3	If the four As [A A / A A] represent $\frac{2}{3}$ of the whole, draw the whole in the box below.	C3	Draw in the box the fraction $\frac{7}{6}$ of the objects below. AAAAAA AAAAAA
A4	Place the fraction $\frac{3}{5}$ on the number line. 	B4	Place the fraction that represents the shaded part of the shape B on the number line. 	C4	Write the correct number in the box below. 
A5	Fill in the fraction with the correct numerator. $\frac{1}{3} = \frac{\quad}{12}$	B5	Which of the five circles represent the same fraction as the one represented in the rectangle? Explain your answer. 	C5	Three friends ordered three pizzas of the same size and shape. George ate $\frac{4}{16}$ of his pizza. Andreas ate $\frac{3}{12}$ of his pizza. Costas ate x pieces of his pizza. If the three friends ate the same amount of pizza, write how many pieces might be Costa's pizza using the variable x.
A6	Circle the bigger fraction. $\frac{4}{7}$ $\frac{2}{7}$	B6	Describe two ways to compare the fractions below. $\frac{8}{18}$ $\frac{5}{9}$	C6	Write a fraction bigger than $\frac{1}{9}$ and smaller than $\frac{1}{8}$. Explain your thought.
A7	Calculate the sum. $\frac{1}{6}+\frac{3}{6}=$	B7	Calculate the sum. $\frac{2}{5}+\frac{1}{6}=$	C7	a) Calculate the sum. $\frac{2}{5}+\frac{1}{6}=$ b) Make a drawing to show the process of the sum of the two fractions. c) Pose a problem using the equation $\frac{2}{5}+\frac{1}{6}=$

Appendix B

The method applied for finding clusters

Suppose that I_1, I_2, I_3, … I_n represent the items to be clustered into groups. First we find the range of the observed measurements that is (I_{max}−I_{2min}). Next, we change the item values to a standardized 0–1 scale, using the formula $S_i = (I_i - I_{min})/(I_{max} - I_{min})$, a transformation that conserves the relative item standing. We next sort the values S_i in ascending order and calculate the gaps between two consecutive items, using the formula $i\Delta = S_{i+1} - S_i$ (where i=1, 2, 3, … n). Finally, we sort the values of Δi in descending order ($\Delta 1$, $\Delta 2$, $\Delta 3$ …); the largest term $\Delta 1$ divides the items into two groups according to the largest gap identified among these items. The second largest term $\Delta 2$ further splits one of the two resulting groups into two subgroups based on the second largest gap, and so on. Hence, when the first k largest Δs are considered, the items are split into k+1groups. The number of clusters that can be formed is determined by exploring the contribution of each $i\Delta$ to the cumulative Δ, which is expressed as a percentage and represents the explained variance.

Table 4 Grouping of the 21 items into clusters based on the procedure of pattern clustering

Problem	β	Sorted S	Δ	Sorted Δ	Cumulative Δ
7	−3.11	0	0.13	0.13	0.23
5	−2.37	0.13	0.07	0.10	0.33
6	−1.98	0.2	0.05	0.10	0.41
2	−1.69	0.25	0.06	0.08	0.48
3	−1.27	0.32	0.01	0.07	0.55
1	−1.21	0.33	0.04	0.06	0.61
4	−1	0.37	0.04	0.06	0.66
14	−0.79	0.41	0.05	0.05	0.71
8	−0.48	0.46	0.02	0.05	0.75
15	−0.39	0.48	0.08	0.04	0.79
10	0.09	0.56	0.01	0.04	0.83
12	0.11	0.57	0.04	0.04	0.87
9	0.33	0.61	0.10	0.04	0.91
13	0.9	0.71	0.02	0.04	0.93
11	1.02	0.73	0.06	0.02	0.95
18	1.4	0.79	0.04	0.02	0.97
16	1.63	0.83	0.01	0.02	0.98
17	1.65	0.84	0.10	0.01	0.99
20	2.25	0.94	0.02	0.01	1
21	2.35	0.96	0.04	0.01	
19	2.56	1			

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