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Taking away and determining the difference—a longitudinal perspective on two models of subtraction and the inverse relation to addition

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Abstract Subtraction can be understood by two basic models—taking away (ta) and determining the difference (dd)—and by its inverse relation to addition. Epistemological analyses and empirical examples show that the two models are not relevant only in single-digit arithmetic. As curricula should be developed in a longitudinal perspective on mathematics learning processes, the article highlights some exemplary steps in which the inverse relation is discussed in light of the two models, namely mental subtraction, the standard algorithms for subtraction, negative numbers and manipulations for solving algebraic equations. For each step, the article presents educational considerations for fostering a flexible use of the two models as well as of the inverse relation between subtraction and addition. In each section, a mathe-didactical analysis is conducted, empirical results from literature as well as from our own case studies are presented and consequences for teaching are sketched.

Keywords Models of subtraction · Inverse relation · Arithmetic and algebra · Mental arithmetic · Standard algorithms · Negative numbers · Algebraic equations · Taking away · Determining the difference

1 Two models of subtraction

Many adults as well as school children understand subtraction solely as taking away. In this paper, we shall show the importance of the second model of subtraction (determining the difference) and the relevance of the *inverse relation* between addition and subtraction by adopting a longitudinal perspective.[1] Beforehand, some remarks are necessary with respect to the notions that we use.

[1]Note that we do not present an empirical longitudinal study where we followed a set of students or a programme over a longer period of time. We adopt a longitudinal perspective for conducting a mathe-didactical analysis (van den Heuvel-Panhuizen and Treffers, 2009).

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Operation and format First, it is important to distinguish between the arithmetic *operation* and the arithmetic *format* (Campbell, 2008). Starting from the subtraction $c - b = a$, each problem that presents c and b and requires the answer a is a subtraction. Consequently, $7 - 3 = ?$ and $? + 3 = 7$ are both subtractions, but the former is a subtraction problem in a subtraction format and the latter a subtraction problem in an addition format.

Different models of subtraction Usiskin (2008) distinguishes two models of subtraction, the take-away model and the comparison model, leading to different notions for the answer: remainder and difference. The interpretation of subtraction solely as taking away is too one-sided, as Freudenthal (1983, p. 107) or Usiskin and Bell (1983) already mentioned. In our paper, we follow van den Heuvel-Panhuizen and Treffers (2009) and term the two models of subtraction "taking away" (ta) and "determining the difference" (dd).

The ta model appears to be slightly more natural than the dd model. However, a deeper understanding of mathematics requires dealing with aspects that appear to be of theoretical nature, if they are being compared with perceived reality. Thus, it is important to interpret mathematical objects and operations in different ways, depending on the requirements of the problem.

Representation on the number line The inverse relation between addition and subtraction, the two models of subtraction and the operation–format distinction are illustrated in Table 1 by using the empty number line for the problem 4+3.[2]

Strategy Solving subtraction problems by using this inverse relation is sometimes called subtraction by (means of) addition (Beishuizen, 1997), counting up (Fuson, 1986), missing addend problems (Brissiaud, 1994) or indirect addition (Torbeyns, De Smedt, Stassens, Ghesquière & Verschaffel, 2009). These terms indicate that a *strategy* (counting or adding) is used where children start at 4 to solve subtraction problems like $7 - ? = 4$.

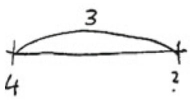
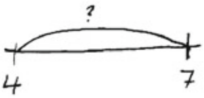
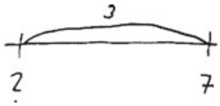
Finally, it should be noted that Table 1 indicates within its three columns that format, representation, model and strategy are somewhat closely related to each other. However, children can "switch between the columns", e.g. when they are given a problem in the take-away format $(7 - 3 = ?)$ which they solve in the dd model $(3 + ? = 7)$.

Efficiently, but rarely used Mathe-didactical analyses as well as empirical studies suggest that using the dd model and counting/adding-up strategies facilitates solving basic subtraction combinations (Baroody, 1983; Baroody & Ginsburg, 1986; Putnam, deBettencourt, & Leinhardt, 1990; Thornton, 1990; Campbell, 2008) as well as mental subtraction problems (Beishuizen, 1997; Menne, 2001; Treffers, 2001). In her review Fuson (1992, p. 256) cites different studies that showed that the relative ease of *counting up* leads children to choose it in preference to *counting down*, even on word problems suggesting to use the ta model. In recent years, the Leuven group has conducted several studies showing for example that young adults flexibly and efficiently used the dd model for solving three-digit subtractions (Torbeyns et al., 2009).

Despite its computational efficiency, the dd model and counting/adding-up strategies are rarely used for subtraction problems, if they are not explicitly being asked for (Baroody, 1999; Stern, 1993; Blöte, Van der Burg & Klein, 2001; Klein, 1998; Selter, 2001; Torbeyns et al., 2009).

[2] For the sake of clarity, we did not include the problem 3+4 here as well. For the same reason, we restrict ourselves to these different syntactic structures and do not primarily deal with numerical or semantic structures (Carpenter and Moser, 1982; De Corte and Verschaffel, 1987; Fuson, 1992), which would have made an analysis even more complex.

Table 1 Format, representation on the empty number line, model, strategy and operation

Subtraction Format	**?** – 3 = 4	7 – **?** = 4	7 – 3 = **?**
Addition Format	4 + 3 = **?**	4 + **?** = 7	**?** + 3 = 7
Answer	**7**	**3**	**4**
Representation on Number Line			
Model of Subtraction		Determining the difference	Taking away
Strategy	starting at 4	starting at 4 (or starting at 7)*	starting at 7
Operation	addition	subtraction	subtraction

*Besides, it is also possible, but not very common to start at 7 to determine the difference between 7 and 4, which is called indirect subtraction (Torbeyns, De Smedt, Stassens, Ghesquière, and Verschaffel, 2009).

What are possible explanations for the observation that the dd model seems to be *efficient, but rarely being used?* Possibly, it is *efficient* because counting on/up is a natural strategy and children are practicing to count on as well as to add quite often in different situations, whereas counting down as well as to subtract are usually not being practiced to a similar extent (Fuson, 1992, p. 256). We hypothesise that it has *rarely been used* as the ta model is often seen as *the* model of subtraction in daily-life situations as well as in school, whereas the dd model does not occur that often in both domains. Especially in formal subtractions which are not embedded in a real-life context, the minus sign seems to be closely connected to the ta model for many children.

Another possible explanation is linked to linear representations like a row of counters, a bead string or the empty number line: as we usually proceed from left to right in reading and writing, adding some counters on the right might be more natural for children than adding them on the left, as well as undoing this operation by removing them on the right than on the left.

Whereas the illustrating example (3, 4, 7) used refers to single-digit arithmetic, the next sections show the relevance of the inverse relation and of the two models for other topics. In each section, the educational considerations for fostering a longitudinal flexible use of the two models as well as of the inverse relation between subtraction and addition are substantiated by a mathe-didactical analysis, by empirical results from literature as well as from our own case studies and by sketched consequences for teaching.

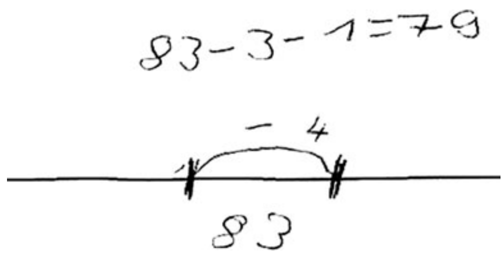
2 Mental subtraction[3]

Julius' and Jonathans' explanations Two second-grade students were given the problem 83−79.[4] The task suggests using the dd model and an addition strategy because the subtrahend is close to the minuend. However, Jonathan and Julius describe their solutions differently (Figs. 1 and 2).

[3] In this contribution, mental arithmetic is used as an umbrella term for all non-algorithmic methods.

[4] Both students attended a qualitative field study conducted by one of the authors.

Fig. 1 Jonathan's solution



It seems as if only Jonathan has noticed the small difference between the two numbers. He used the dd model, but unexpectedly he started from 83 and not from 79 (see Fig. 1). Then, Jonathan was asked to explain his strategy by means of the empty number line.

> J: I have calculated. 79 to 83 is 4.
> I: Oh, but I see you have put down minus 4, not plus 4.
> J: Yeah, that's because it is a minus problem. When there is a minus we have to use subtraction and not addition.

Here we see the connection between both operations to a "two-way traffic" interpretation (van den Heuvel-Panhuizen & Treffers, 2009, p. 108): On the one hand, Jonathan focuses on the relations between the numbers by using the inverse *operation* (addition). On the other hand, he puts down his own strategy in a subtraction *format*. His interpretation indicates "sociomathematical norms" (Voigt, 1995; Yackel & Cobb, 1996) of the culture of his mathematical classroom. Here, it is distinguished between addition and subtraction strategies and the literal meaning of the symbol "+" or "−" (the format).

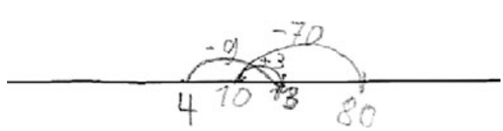
In contrast, Julius' strategy (see Fig. 2) following the ta model can be traced in three steps:

(1) Take away the tens of the subtrahend from the tens of the minuend.
(2) Add the ones of the first number.
(3) Take away the ones of the second number.

Even if the empty number line invites the use of a sequential or a shortcut strategy, Julius seems to combine a sequential strategy with a decomposition strategy (these strategies are discussed on the next page). His solution is a typical example that children tend to select the strategy that—from their point of view—leads safely and quickly to an accurate answer, even if the number characteristics suggest—from the mathematical point of view—to choose another strategy and to use other models (here: the dd model; Benz, 2005; Selter, 2001; Torbeyns, De Smedt, Ghesquière & Verschaffel, 2009).

Different mental strategies Mental arithmetic means to connect mathematical understanding and arithmetical processes by linking number and operative conceptions with relations. In the last two decades, several authors have provided different analyses of mental arithmetic strategies children use to solve multi-digit subtraction problems (e.g. Beishuizen,

Fig. 2 Julius' solution



1993; Benz, 2005; Blöte, Klein & Beishuizen, 2000; Carpenter et al., 1997; Fuson, 1992; Heinze et al., 2009; Selter, 2001; Torbeyns et al., 2009; Torbeyns et al., 2009).

Even if the mathematical discussion of the strategies is complex and the different categorisations of strategies exist with different names, one can distinguish three central strategies for mental arithmetical subtraction problems: decomposition, sequential and short cuts (like auxiliary task or balancing). These types of strategies are idealised because children are able to use a broad range of individual variants and combinations of the different strategies (see Fuson et al., 1997). The combination of decomposition and sequential is used to quite some extent (Selter, 2001), as for example illustrated by Julius' solution.

In Table 2, the relation of the ta and the dd model on the one hand and of the aforementioned main strategies on the other is analysed. Unlike other authors (Klein, 1998; Padberg, 2005), we do not consider adding up using the dd model as a strategy of its own, but as part of the sequential strategy. We shall also show that all main strategies can be interpreted by using both models, although it is rather uncommon to use the dd model in one case.

Results from research As already mentioned before, most children hardly make use of the dd model, and their choice of the strategy rarely depends on task or number characteristics of the given problems (Torbeyns et al., 2009). On the one hand, there is no clear empirical

Table 2 Strategies for mental arithmetic subtraction (The selected problem (83−79) is an example for tasks in which minuend and subtrahend are close together. In a similar fashion, one can discuss tasks with other typical relations such as 83−4, 83−14 or 100−79.)

Mental subtraction	**Taking away**		**Determining the difference**	
Decompo-sition*	83–79=10–6=? 80–70 3–9	Get the tens-remainder by taking away 70 from 80. Get the ones-remainder by taking away 9 from 3. Add/subtract the two preliminary results.	79 + ? = 83 70 + *10* = 80 9 + (*–6*) = 3 10 – 6 = *4*	*rather uncommon*: Determine the tens-difference by adding 10 to 70. Determine the ones-difference by adding –6 to 9. Add/subtract the two preliminary results.
Sequential	83 – 79 = ? 83 – 70= *13* 13 – 9= *4*	Take away the tens of the subtrahend from the minuend. Take away the ones of the subtrahend from the preliminary result.	79 + ? = 83 79 + *1* =80 80 + *3* =83 1 + 3 = *4*	Determine the difference by adding 1 to 79 and then adding 3 to 80. Add the two preliminary results.
Shortcuts	83 – 79 = ? 83 – 80 = *3* 3 + 1 = *4*	*Auxiliary task* Take away a round number (80 instead of 79). Compensate (as 79 and not 80 has to be subtracted).	79 + ? = 83 80 + *3* =83 79 +*1*+3=83	*Auxiliary task* Use a round number to determine the difference (80 instead of 79). Compensate.
	83 – 79 = ? 84 – 80 = *4*	*Balancing* Add/subtract the same number to/from both numbers in order to arrive at an easier problem.	79 + ? = 83 80 + *4* = 84	*Balancing* Add/subtract the same number to/from both numbers in order to arrive at an easier problem.

*Children in primary schools are not familiar with negative numbers. However it is discussed at least in Germany, if and how the decomposition strategy is useful for subtraction problems with bigger digits in the second number than in the first one (An alternative strategy, used by some children is 83–79=__: 80–70=10, 9–3=6, 10–6=4 or 70–70=0, 13–9=4). A representation of the decomposition strategy with the empty number line is not possible.

evidence for the effectiveness of alternative instructional approaches fostering strategy flexibility from the beginning of primary school. Here, the interpretation of subtraction problems with reference to the dd model does not seem to be a direct consequence. As a result of their intervention study, Heinze et al. (2009) emphasise that students choose efficient and flexible strategies with respect to the instructional approach. But the "strategy indirect addition which could be used efficiently for two subtraction items in the test played only a marginal role" (Heinze et al., 2009, p. 598).

On the other hand, Blöte et al. (2000) point out that children who took part in an investigative, problem-solving approach are able to apply different strategies on different tasks (thus thinking with the dd model is malleable by appropriate instruction). In contrast, children from skills-oriented classrooms preferred only one standard strategy on all types of problems (see Klein, Beishuizen & Treffers, 1998; Torbeyns et al., 2005).

The results from research are thus not unequivocal. Most of the cited studies on multi-digit subtraction focus on children's preferences and different strategy uses. Only a few of them investigate how the "ambiguity" (e.g. Nührenbörger & Steinbring, 2009) of interpreting an illustration as well as a symbolic representation of a subtraction problem can be fostered. Here, we see a clear indication of the need for carefully designed teaching experiments.

Consequences for teaching The introduction of subtraction in the first grades is traditionally based on an empirical form of explanation: numbers are visualised by real objects and operations by concrete activities as taking away, going back, etc. Using only this standard way is too one-sided. It probably leads to restricted mathematical thinking: mental arithmetic is learned as an algorithmic arithmetic. An alternative interpretation of subtraction with reference to the dd model requires a modified interpretation and explanation of the character of numbers and of operations. The use of manipulatives like the empty number line can foster reflection about differences and commonalities between solutions (Klein et al., 1998; Blöte et al., 2000; see Fig. 3).

In this sense, representations are used as epistemological tools for thinking and communicating (Nührenbörger & Steinbring, 2008). Also, flexibly seeing addition and

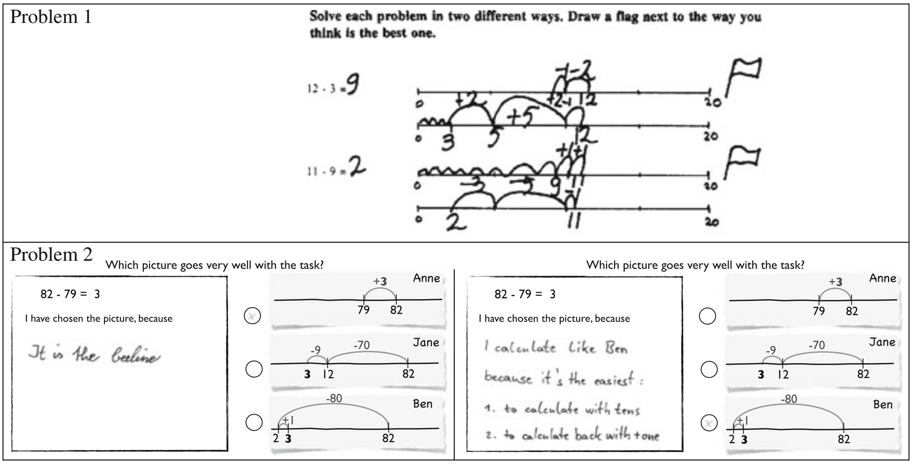


Fig. 3 Two problems to foster different interpretations of subtraction. (Problem 1 is proposed by Blöte et al. (2000), problem 2 is an example of a qualitative field study by the authors.)

subtraction as intertwined operations widens restricted sociomathematical norms in the classrooms. It can foster insights in and argumentations of mathematical relations, e.g. by explaining the relations between "inversion numbers" like $76 - 67 = 9 = 65 - 56 = \ldots$, $75 - 57 = 18 = 64 - 46 = 53 - 35 \ldots$. The difference is a multiple of nine or $10-1$ in relation to the difference of the digits.

Being able to use both models is, however, important not only for flexible mental arithmetic, as will be shown in the following.

3 Standard subtraction algorithms

David's explanation In a small piece of empirical research supervised by one of the authors, Greshake (2010) interviewed 32 fourth grade students individually while they solved six problems where they had to subtract according to the standard algorithm being taught. Afterwards, they were asked to explain the algorithm using the example

$$\begin{array}{r} 60932 \\ -19641 \\ \hline \end{array}$$

The children's explanations were categorised according to Mosel-Göbel's (1988) categories into mechanical understanding (17 children), partly substantiated understanding (nine children) and substantiated understanding (six children). Less than one fifth of the children were thus able to explain the procedure, more than one half were just able to describe what they did (but not why they did it), like David who was taught the equal addition subtraction algorithm.

> D: I add 10 to the 3 [in the tens-column of the minuend] and write 1 below the 6 [in hundreds-column of the subtrahend].
> I: What is the reason for doing that?
> D: I think, if I add 10 to the 3, I have to carry the 1.
> I: In order to be able to go further?
> D: Yes!
> I: But why are you allowed to add something?
> D: To be honest: I don't really know. We were just taught to do so.

Standard algorithms have developed over the centuries for efficient, fast and accurate calculation. Often they are removed from their conceptual underpinnings that lead to students poorly being able to explain how and why they work (Treffers, 1987; see also Verschaffel & De Corte, 1996). David's explanation is a representative example for the "cognitive passivity" they are prone to lead to, as many decisions like how to set out a calculation, where to start, what value to assign to the digits, etc. are taken out of the individual's hand (Thompson, 1999, p. 173).

Different subtraction algorithms We shall not discuss the pros and cons with respect to teaching standard algorithms in this paper and thus we shall start our further argumentation from the starting point that the teaching of at least one standard algorithm for each of the four operations—which the children are able to conduct efficiently—is generally accepted as an element of curricula worldwide. The most commonly used subtraction algorithms can be found in Table 3. It is distinguished on

Table 3 Different subtraction algorithms

Algorithm	Taking away		Determining the difference	
Decomposition	4 0 ~~51~~2 –397 115	decrement 1 to 0, as 1 T is changed to 10 U 12 – 7 = 5 decrement 5 to 4, as 1 H is changed to 10 T 10 – 9 = 1 …	4 0 ~~51~~2 –397 115	decrement 1 to 0, as 1 T is changed to 10 U 7 + 5 = 12 decrement 5 to 4, as 1 H is changed to 10 T 9 + 1 = 10 …
Equal addition	10 10 512 –397 1 1 . 115	add 10 U and 1 T 12 – 7 = 5 add 10 T and 1 H 11 – 10 = 1 …	10 10 512 –397 1 1 . 115	add 10 U and 1 T 7 + 5 = 12 add 10 T and 1 H 10 + 1 = 11 …
Turning the counter			512 –397 1 1 . 115	from 7 onwards 5 is 2 carry 1, as the meter moved one further from 0 onwards 1 = 1 ...

the one hand between the model used (ta or dd) and on the other hand between the method applied.[5]

The decomposition method is used in many countries, and it can be conducted by using the ta model as well as the dd model. The same is true for the equal addition method.[6] Besides, there is a third method, described by Lietzmann (1916). However, during recent years it was put to the fore again (Wittmann & Müller, 2007, p. 87). Here, a counter is used to work out 567−439 by determining the difference, starting from 439 (old) aiming at reaching 567 (new).

In order to make connections to their mental strategies, children first add eight units on the empty number line and arrive at 44**7**. Then, two tens are added in order to make the tens digit as it should be (4**67**). Finally, one hundred is added (Fig. 4).

The standard algorithm works similarly. The little 1 below the 3 means that the counter has moved one further at the tens digit (Fig. 5).

Several authors (e.g. Gerster, 1982; Padberg, 2005; Schipper, 2009; Thompson, 2007; or Wittmann, 2010) discuss pros and cons of the different algorithms. They all more or less agree that the equal addition method should not be used, but there is no further consensus.

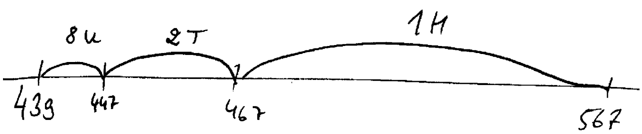


Fig. 4 Sequential strategy and dd model

[5] This distinction is made, although quite some children might proceed entirely mechanically, using basic facts without any understanding. However, at least in Germany, it is a goal of mathematics teaching that children also understand the algorithm they apply.

[6] Note, that this method—at least as it is taught in Austria and Germany—does not use decomposition, where the amount "borrowed" from a given order of the minuend is added to the corresponding order of the subtrahend using the equivalence between the differences $(x-1)-y$ and $x-(y+1)$, like Fiori and Zuccheri (2005, pp. 324–325) claim. Instead, the law $x-y=(x+a)-(y+a)$ is used, namely by adding ten ones to the minuend and compensating this by adding one ten to the subtrahend.

Gerster, Thompson and Wittmann offer valid arguments for *turning the counter with the dd model* as well as Padberg and Schipper do for *decomposition with the ta model*. For us, no clear decision can be made on the basis of these arguments.

Results from empirical research Given the importance algorithms (still) have, it is a bit surprising that in contrast to single-digit or multi-digit arithmetic, ascertaining research on algorithms is rather scarce (Verschaffel, Greer & De Corte, 2007, p. 574) and often concentrating on pupils' systematic errors (Brown & van Lehn, 1980; Cox, 1975; Gerster, 1982; Huth, 2004; Kühnhold & Padberg, 1986; Radatz, 1980).

However, there are some studies on "understanding algorithms". Fuson (1990; 1992) has reported on teacher-directed instruction that links exchanges of base-ten blocks to the decomposition algorithm. Alternative approaches were described by Kamii, Lewis & Livingston (1993), Madell (1985) or Labinowicz (1985) where the students were encouraged to invent their own mental (left to right) strategies.

Another line of research deals with the comparison of different algorithms. Ross and Pratt-Cotter (1997) report on research that was conducted in the late 1800s and the early 1900s in the USA on this issue giving no clear picture on which algorithm to favour. This is also true for the studies by Johnson (1938) and Brownell and Moser (1949) which came to contradictory results—by the way, the latter dealing solely with its algorithms.

In Germany, Mosel-Göbel (1988) compared third graders who were taught different methods: decomposition with the ta model, equal addition with the dd model and two variations of turning the counter with the dd model. She found no relevant differences with respect to success rates, but with respect to understanding: the former method could be explained by considerably more students and could more easily be related to the existing pre-knowledge in comparison to equal addition and to one of the counter-methods (whereas this was not true for the other one).

However, as just four different classrooms were participating in the study and as it has considerable methodological weaknesses, the conclusions rather have to be drawn with caution. The study by Fiori and Zuccheri (2005) does not lead to a clear decision as well, as it is mainly focused on error patterns and, in addition, did not use the equal addition method as such (see above).

To sum up: Neither theoretical considerations nor results from empirical research give a clear picture which algorithm and which model should be favoured. Thus, we clearly see the need for further research on this topic. However, we want to show in the following that *turning the counter* is at least a coequal method to *decomposition*.

Consequences for teaching As Verschaffel et al. (2007, p. 575) summarise, teaching algorithms should actively involve children in devising them, starting from or relating them to their knowledge about numbers, single-digit and multi-digit computation, following the

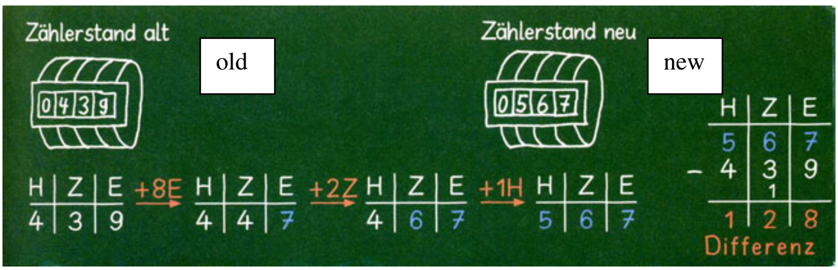


Fig. 5 Turning the counter (Wittmann & Müller, 2007, p. 87). (H(underter), Z(ehner), E(iner) means H, T, U.)

principle of progressive schematisation (Treffers, 1987). But which algorithms do relate to which mental strategies?

Taking into account the *auxiliary task strategy*, there seems to be no relation to one of the algorithms that can easily be comprehended. The *balancing strategy* has some connection to equal addition, as both use the same arithmetical law $x - y = (x + a) - (y + a)$. However, while using the mental strategy, you try to reach "easy" numbers by adding or subtracting the same amount to minuend and subtrahend, whereas using the algorithm always means adding ten, hundred, thousand, … to both.

The *decomposition strategy* requires that the students work column-wise. On the one hand, it is possible to connect the decomposition algorithm by means of Dienes blocks to a variation of the decomposition strategy where the children change for example one ten to ten ones. This is not a natural mental strategy (Thompson, 2007); however, it seems to be possible to make it a topic of teaching as it bridges the way to the decomposition algorithm. Treffers, Nooteboom & de Goeij (2001) propose to also take into account an old Italian subtraction method (Treviso subtraction) which is an abbreviation of what they call column subtraction which is also based on the decomposition strategy.

The *sequential strategy* can easily be related to turning the counter, as demonstrated above. Let us finally illustrate this approach with an example from teaching (PIK AS, 2010). Third graders had learned the algorithm according to turning the counter. After the children had gathered some experience with the algorithm, they were given the following worksheet (Fig. 6). On it one child had worked out 526−283 by using the empty number line, whereas another child had used the algorithm in order to solve the same problem. The children were asked to track both ways and apply it to other problems. Finally, they were asked to reflect on similarities and differences, at first on their own, then in a small group conducting so-called maths conferences and finally, in a whole class discussion being moderated by the teacher.

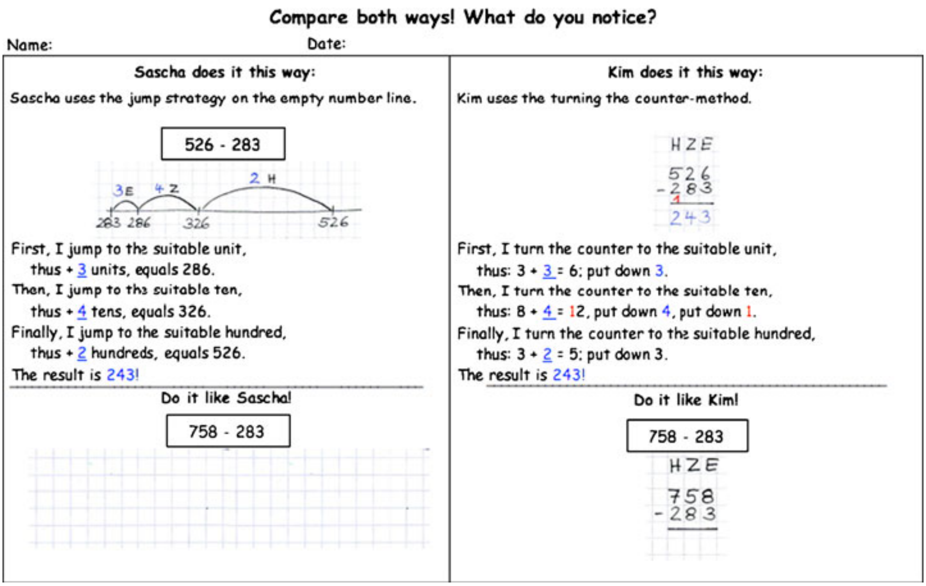


Fig. 6 Compare both ways!

4 Subtracting negative numbers

Jana's explanation After seeing the solution of the task 70−(−50), Jana, a 13-year-old student (Grade 7), expresses her surprise: "This can't be true. If you take away something, the solution can't be bigger".[7] Jana tries to assimilate calculations with negative numbers to her preconceptions of natural numbers. It seems as if Jana uses the ta model for solving this problem. In doing so, she ignores the second minus sign or she does not perceive its meaning.

Several studies showed a noticeable increase of errors as soon as negative numbers appear in problems (e.g. Vlassis, 2004). They offer descriptions and analyses of the erroneous use of integers due to the students' experiences with natural numbers (Gallardo, 2002; Glaeser, 1981; Hefendehl-Hebeker, 1989; Thompson & Dreyfus, 1988; Vergnaud, 1989). One of the main difficulties is seen in the double nature of the minus sign: it has to be understood not only as an "operating" but also a "predicative" sign (Glaeser, 1981).

Different functions of the minus sign Gallardo and Rojano (1994) classified three functions of the minus sign: *unary*, *binary* and *symmetric function*.

The minus sign in its *unary function* (as a predicative sign) characterises the number (−50 € means 50 € debts or −50 m can be interpreted as 50 m under sea level), whereas in its *binary function*, it appears as operational signifier that can be interpreted by the ta model as well as the dd model. In the *symmetric function*, the minus sign is used as an operational signifier to get the opposite number, i.e. the additive inverse. According to Bruno and Martinon (1999) many students are not aware of the distinction between the unary and the binary function of the minus sign.

To understand students' difficulties, we analyse the consequences of the extension from natural numbers to integers in both models of subtraction. We shall show that for operating with integers in the ta model, the conceptional challenges are considerably larger than in the case of natural numbers. The dd model is much easier to convey from natural numbers to integers than the ta model.

Operating with negative numbers in the ta model For interpreting subtractions with negative numbers in the ta model, the unary as well as the binary function of the minus sign is needed.

In the subtraction $a - b = c$, minus signs as unary functions can appear for negative values of a, b and c. As a consequence, the representations have to become more sophisticated for distinguishing positive or negative values of a, b and c. On the number line, the values of a and c can be interpreted as positions left or right to zero (see Fig. 7), but how to take away a negative value, if $b<0$? The number b is represented by an arrow that visualises what is taken away. In the case of $b<0$, the arrow is directed to the left and for $b>0$ to the right. The operating sign determines the starting point of the arrow: if the sign is negative, it starts in a, otherwise it ends in a. This representation makes sense as it matches interpretations in different contexts, for instance understanding the predicative sign as debts and assets and the operating sign as the variation between two states: $7 - (-3) = ?$ can be interpreted as "Tom has 7€ assets, he has weekly costs of 3€, how many assets did he have last week?" (Hußmann, 2010).

Focusing only on b there are four different cases, for instance: $7 - (+3) = ?$, $7 - (-3) = ?$, $7 + (+3) = ?$ and $7 + (-3) = ?$. Figure 7 shows how they can be represented on the empty number line within the extended ta model.

[7] The data are taken from a qualitative field study by one of the authors (Hußmann, 2010, p. 1).

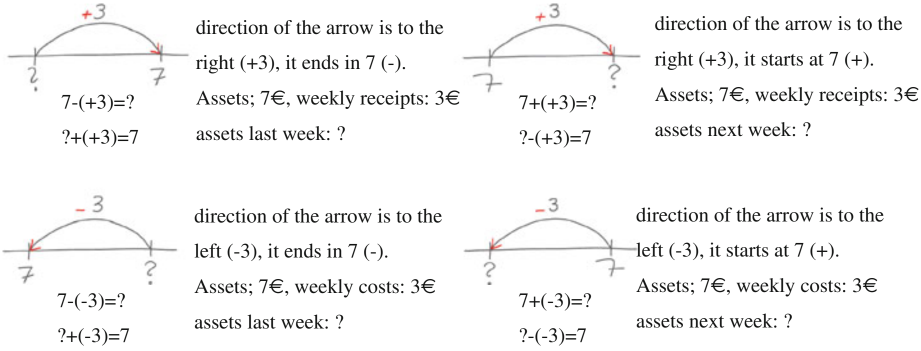


Fig. 7 Representing subtractions of negative numbers in the extended ta model

In contrast to addition and subtraction with natural numbers, we have to introduce directed arrows to represent the minus sign of the value of b in its unary function. In addition, we have to take the position of the arrowhead into account. Using the inverse relation may help us here to understand the used representation. In the addition format, it is a little bit easier to understand how the arrow has to be drawn: it starts at the first summand and ends at the sum. This is a great discontinuity compared to the experiences with natural numbers, where for instance $7-(+3)=?$ is interpreted by means of the ta model as "starting at 7, operating sign is negative, thus move to the left".

Problems of the format $a-?=b$ do not necessarily have to alleviate these difficulties. For instance, to solve $7-?=10$, not only do the two functions of the minus sign have to be considered simultaneously; also the direction of the arrow is unknown (see Fig. 7). Here, too, the inverse relation could be easier to handle, but this type is conceptually more difficult than the former one because the sign of b and also the direction of the arrow are unknown. Tasks of the type $?-a=b$ seem to be even more difficult because the usual starting point is missing. This could be an explanation for the results that Bruno and Martinon (1999) pointed out in their research. They showed that tasks of the kind $?-a=b$ and $a-?=b$ are more difficult to solve than tasks in the format $a-b=?$

Operating with negative numbers in the dd model Whereas the ta model has to be extended for being applied to interpretations of subtractions with negative numbers, the dd model can be applied without introducing new meanings. In additions with negative numbers, the unary function and binary function are considered not *simultaneously, but one after the other*. For instance, Fig. 8 shows how to solve the equations $7-3=?$ or $7-(-3)=?$ by using the dd model.

The numbers have to be subtracted or added (binary function), depending on the constellation. In the case of subtracting negative numbers, one first has to check the position of the numbers (unary function). We distinguish two cases (see columns in Fig. 8): (1) the numbers are located on the same side relative to zero, then one has to subtract the minor from the major number; (2) the numbers are located on different sides relative to zero, then one has to add the amounts of the numbers. Regarding the starting point on the number line, we either distinguish two cases (see rows in Fig. 8): (1) Subtrahend is smaller than minuend, thus look from left to right, predicative sign is positive. (2) Subtrahend is bigger than minuend, thus look from right to left, predicative sign is negative.

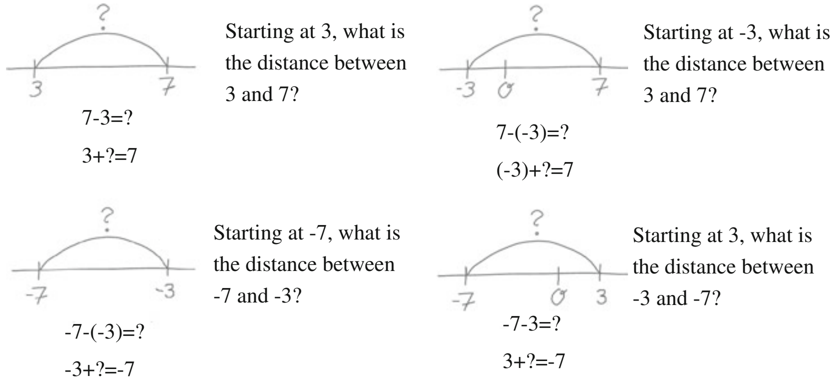


Fig. 8 Representing subtractions with negative numbers in the dd model

In both cases, the inverse relation between addition and subtraction in interplay with the empty number line helps to determine the result while focusing on the dd model. By using the dd model, Jana probably would not be surprised anymore. She would recognise that −50 and 70 are on different sides relative to zero, thus $70 - (-50) = 120$. The subtrahend is smaller than the minuend, thus the predicative sign of the result is positive.

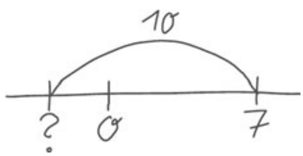
"Determining the difference" as a single model entails operating with only problems of the format $a - b = ?$. Figure 9 shows how to solve tasks like $7 - ? = 10$. Seven is a point on the number line, the distance is 10. But in which direction does it have to be drawn, to the left or to the right? The former proposition regarding the starting point may help: the predicative sign of the result is positive, thus the subtrahend is smaller than the minuend. Therefore, the subtrahend is to the left of the minuend (see Fig. 9). This interpretation gives reason to hope that the difficulties Bruno and Martinon (1999) pointed out might decrease with this model.

Recapitulating and comparing the two models, the following consequences may be stated:

(1) The dd model is much easier to convey to the negative numbers, whereas
(2) The ta model needs a sophisticated conceptual extension because directed arrows have to be introduced for taking into account both functions of the minus sign simultaneously.

Consequences for teaching Although the articulated preference is given to the dd model, this does not mean that no importance is ascribed to the ta model any more. There are many real-life situations that can be mathematised only by the ta model, for instance, giving and taking of assets and debts. Nevertheless, the dd model should be the first model to be taught because the number of possible obstacles is far smaller. In this approach, students have the

Fig. 9 $7 - ? = 10$



possibility of getting familiar with the new functions of the minus sign. In difficult or complicated situations, they are on firm ground with the dd model.

5 Understanding manipulations for solving algebraic equations

Walter's problem with solving equations:

> I: Can you solve this equation? $\frac{x}{8} = 9$
> W: (*after several minutes of silence*) I do not remember how this works. There is a rule for it, but I have forgotten it.

As for Walter (Malle 1993, p. 3), algebra appears as a senseless system of procedures for many people. Acquired in school only algorithmically, even equations like $x + 3 = 7$ cannot be solved by making sense of the symbols. This limited success of algebra in classrooms is documented in various empirical studies all over the world (e.g. Kieran, 1992; Stacey & Chick, 2004).

As a reaction, modern algebra curricula have aimed at acquiring algebra with understanding and therefore offered carefully planned opportunities for students to construct meanings for variables (e.g. Usiskin, 1988; Malle, 1993; Mason, 1996) and for the equal sign (e.g. Kieran, 1988) for more than 20 years. Solving equations is a further crucial step in such an algebra curriculum in which, firstly, the meaning of an equation and informal solving strategies are to be constructed, and, secondly, the corresponding formal procedures for manipulating equations are introduced and grounded in informal strategies.

In this particular step of learning algebra, the ta model and the dd model and the inverse relation between addition and subtraction again can play an important role as will be sketched in this section. We argue that those students who acquired deep understanding and experience with the inverse relation between addition and subtraction in both models in arithmetic can substantially build upon it for learning to solve equations algebraically.[8]

Mathematical and epistemological analysis How can equations like $x + 3 = 7$ or $4x + 17 = 77$ be solved? Kieran (1992, p. 400) summarises seven methods of solving equations; two of them are special strategies for restricted cases (use of recalled number facts and use of counting techniques). Table 4 shows the other five methods, here already classified (by the columns) as informal strategies and formal procedures. The lines in Table 4 show the connections between them. *Undoing* is the informal, model-oriented base for *transposing*. *Covering-up* the informal strategy that leads to the *performing the same operation on both sides* procedure (shortly "same on both sides"). *Trying by substituting* is grounded in the conceptual background, namely the definition of an equation. Conceptual understanding of formal manipulations requires the ability to trace back at least one of the formal procedures to its informal base (preferably both, as will be argued below).

Table 5 shows that *transposing* and *undoing* directly derive from the inverse relation between addition and subtraction (as mentioned in Usiskin, Peressini, Marchisotto & Stanley, 2003). It can be understood in both models of subtraction by reconsidering the primary school representations from Table 1 (and using the commutative laws if only one model will be applied).

[8] And analogically the inverse relation between multiplication and division and their models, cf. Prediger (2008).

Table 4 Informal strategies and formal procedures for solving algebraic equations

Informal strategy	Corresponding formal procedure
Trying by substituting 10 does not fit in the equation $4x+17=77$, as $4\cdot 10+17$ is 57, 20 does not fit, but $4\cdot 15+17$ is 77, thus 15 is the solution	*none, but raises from conceptual background*: the solutions of an equation are defined as those numbers that can be inserted and make it a true equation
Covering-up x is the unknown number in the equation $4x+17=77$. Make both sides of the equation easier by taking away 17, then get $4x=60$. Then make both sides easier by dividing by 4, and get $x=15$.	*Performing same operation on both sides* $4x+17=77 \quad / -17$ $\Leftrightarrow 4x+17-17=77-17$ $\Leftrightarrow 4x=60 \quad / :4$ $\Leftrightarrow 4x:4=60:4$ $\Leftrightarrow x=15$
Undoing (working backwards) The unknown was multiplied by 4 and then 17 was added. I can undo it by first subtracting 17 and then dividing by 4, thus the unknown is 15.	*Transposing* $4x+17=77$ $\Leftrightarrow 4x=77-17$ (inverse relation add/sub) $\Leftrightarrow 4\cdot x=60$ $\Leftrightarrow x=60:4$ (inverse relation mult/div) $\Leftrightarrow x=15$

Starting from these basic equivalences between elementary equations with numbers, the right column in Table 5 shows how the transposing rule can be generalised and justified also for encapsulated sub-expressions that are treated like numbers or variables:[9]

$$A+B=C \Leftrightarrow C-A=B \,(\text{and analogically}\, A+B=C \Leftrightarrow C-B=A).$$

In contrast, the "*same on both sides*" procedure bases on the following transformation rule:[10]

$$A=B \Leftrightarrow A+C=B+C.$$

This rule is usually justified by the balance model that is introduced in classrooms for this purpose (e.g. Kieran, 1988 or Vlassis, 2002).[11]

It is the inversion principle (Baroody, Torbeyns & Verschaffel, 2009) that guarantees the mathematical equivalence of both procedures, transposing and same on both sides, and their underlying set of rules. Regardless of this mathematical equivalence, Kieran emphasises that they are not didactically equivalent, since "these two solving methods appear to be perceived quite differently by beginning algebra students" (Kieran, 1992, p. 400).

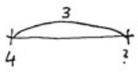
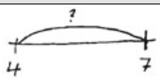
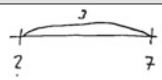
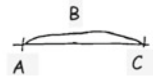
Consequences for teaching Kieran pleads for teaching the *same on both sides* procedure because *transposing* can (if badly taught) be conducted without any understanding (Kieran, 1992, p. 400). As this is equally true for the *same on both sides*, we plead (with Malle, 1993) for including the transposing procedure in the curricula. The major reason is the greater proximity to students' thinking: although Vlassis claims that the transposing

[9] Analogically, the inverse relation of multiplication and division is the foundation for the second transposing rule: $A\times B=C \Leftrightarrow C\div B=A$ (and analogically $A\times B=C \Leftrightarrow C\div A=B$).

[10] And accordingly: $A=B \Leftrightarrow A\times C=B\times C$

[11] The balance model only applies for positive numbers and values of the variable, not for equations like $4x+15=-3$ (Malle, 1993, p. 224).

Table 5 Equivalence for transposed equations derive for inverse relation between addition and subtraction

Operation and its model	Addition	Subtraction as determining the difference	Subtraction as taking away	Generalisation for sub-expressions
Number line				
Equation with two equivalent algebraic formats	4+3 = **x** ⇔ **x**–3 = 4	4+**x** = 7 ⇔ 7–**x** = 4	**x**+3 = 7 ⇔ 7–3 = **x**	A + B = C ⇔ C − A = B ⇔ C − B = A
Solution	7	3	4	(evtl. intermediate step to solution)

procedure “neglects completely any prior knowledge the students might have” (Vlassis, 2002, p. 342), we have shown that it can be nearer to arithmetical relations and hence easier deducible from arithmetical structural experiences if these are well founded (see Table 5). In this way, it can help to fill the often experienced gap between arithmetic and algebra. This mathematical argument is supported by empirical evidences that the transposing procedure is nearer to many students’ thinking. Many students think in terms of transposing although having officially learned the same on both sides (Kieran, 1988; Striethorst, 2004).

Of course, transposing cannot be simply derived from dealing with numbers without any reflection (cf. Usiskin et al., 2003, for mathematical reflections). With respect to student thinking, Kieran (1988, p. 400) especially emphasises the challenge of equations with more than one occurrence of variables like $3x + 4 - 2x = 8 - 7x$. For treating these cases, the commutative and associative laws have to be applied and the generalisation of the transposing rules for encapsulated sub-expressions A, B, and C has to be mastered.

These exemplary challenges make clear that deriving transposing from dealing with addition and subtraction must comprise more than calculating with numbers. It is the awareness of structural relations between additive and subtractive equations (not only the relation between the expressions left and right from the equal sign) that forms the important foundation for understanding the algebraic manipulations.

To sum up, it is less the different models of subtraction, but the inverse relation between addition and subtraction that is of crucial importance for understanding manipulations for solving algebraic equations.

6 Concluding remarks

In this article, we have discussed mathematical and epistemological issues regarding two models of subtraction and its inverse relation to addition on the basis of a review of results from empirical research. Let us finally sum up the main messages.

Using the dd model and the inverse relation between addition and subtraction seems to be efficient; however, it is rarely being used. Our mathe-didactical analysis suggests that it should become more prominent in teaching from grade 1 on, due to the following reasons.

- (Almost) all mental strategies can be applied in the ta as well as in the dd model. Seeing subtraction solely as taking away is too one-sided.

- All subtraction algorithms can be applied in the dd model. Turning the counter in the dd model relates to an informal strategy. So far, there seems to be no clear evidence for solely using an algorithm following the ta model.
- The dd model can much more easily be extended to subtracting negative numbers; switching between addition and subtraction is needed in order flexibly to handle equations and their representations.
- In algebra, it is the awareness of structural relations between additive and subtractive equations (not only the relation between the expressions left and right from the equal sign) that can form an important foundation for understanding the algebraic manipulations.

We have also presented some consequences for teaching, which were based on the following ideas (see van den Heuvel-Panhuizen & Treffers, 2009).

- Continuous attention *also* (!) to the dd model of subtraction
- Flexible use of addition and subtraction as intertwined operations with an inverse relation
- Stimulating interactive reflection about differences and similarities between models, strategies and representations
- Dealing with the ambiguity of subtraction tasks and illustrations
- Using the empty number line as an important representation.

Our analyses and suggestions were derived from a longitudinal perspective, since learning mathematics is a longitudinal learning process, constantly building on existing knowledge and always laying foundations for further learning. It is a major task of further didactical design of learning environments to take into account how previous learning environments were organised and must be oriented towards future learning environments. Longitudinal empirical research should investigate the long-term development of students' thinking in such a carefully planned curriculum.

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