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The emergence of mathematical structures

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Abstract We present epistemological ruptures that have occurred in mathematical history and in the transformation of using technology in mathematics education in the twenty-first century. We describe how such changes establish a new form of digital semiotics that challenges learning paradigms and mathematical inquiry for learners today. We focus on drawing analogies between the emergence of non-Euclidean geometry with recent advances in technological environments that are dynamic and interactive both visually and haptically. This analysis yields a new digital semiotic theory.

Keywords Space · Structure · Euclidean and non-Euclidean geometry · Haptic · Dynamic · Multimodal · Environment · Epistemological ruptures · Digital semiotics

1 Background

1.1 Epistemological ruptures

We focus on epistemological ruptures in mathematical knowledge that have occurred because of the existence of various epistemological obstacles (Bachelard, 2002) in the past. We present an epistemological analysis not to present any new historical facts but instead to help us think about the nature of mathematical knowledge and hence learning today.

We present two epistemological ruptures from pre-twenty-first century history, which could be defined as the knowledge of mathematicians and how mathematicians questioned

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the work of previous mathematicians. We then present two analogous epistemological ruptures from the twenty-first century, which focus on the affordances of new technological environments and the knowledge of mathematics learners. Figure 1 outlines these four ruptures. We claim that these later ruptures will cause similar perturbations in how mathematics is conceived of in the future and how we apply it to learning.

The first pre-twenty-first century rupture describes Kant's conception of space as the space around us and the inflexibility of Euclidean geometry to analyze physical space. Gauss and others challenged the Euclidean structure of space and new mathematical structures were established to explore space. These new structures created a tension between cognition and logic. The second pre-twenty-first century rupture focuses on Euclid's conceptions as synthetic and not as analytical, and on the need to develop a more formal system with intuitive images in non-Euclidean geometry that link to reality.

The first twenty-first century rupture describes the evolution of dynamic modeling environments that can modify our natural ontology to accommodate change and variation. This "dynamic movement" illustrates a representational re-description of mathematics (Tomasello, 2000). The second twenty-first century rupture is in the evolution of new technologies that create not just visual but haptic experiences to analyze the mathematical structure of space. Such technologies become new mediating artifacts to explore space overcoming the need to abstract attributes of mathematical structures at a visual or literal level.

The direct correspondence with the historical mathematical structure was broken after the discovery of non-Euclidean geometry. We demonstrate that this is what is also occurring with the implementation of dynamic mathematical software, which breaks the correspondence with static mathematical objects in turning to new digital semiotics. This demonstration is the heart of our semiotic argument.

1.2 Theoretical framework

These technological inventions allow such ruptures and a new kind of modernism that we have seen in the past. Gutenberg's printing press and Koenig's flatbed cylinder press accelerated social and technological change, which established broad educational literacy campaigns (Krug, 2005) and impacted social identity and knowledge:

> By the beginning of the nineteenth century, all of the social apparatus was present for the creation of a mass, shared, language and literature which would help to produce new definitions of social identity and relations as well as an ideological viewpoint consistent with capitalism, empiricism, and the abstractions of time and place which would fit well with the modern, urban, disjointed social and which legitimated the roles of experts of all kinds in running the social. (p. 106)

Pre-21st Century	**21st Century**
Epistemological Rupture 1: *Emergence of an embodied mathematical structure*	Epistemological Rupture 1: *Emergence of an embodied mathematical structure through dynamic modeling environments*
Epistemological Rupture 2: *Reformalizing the mathematical structure of physical space*	Epistemological Rupture 2: *Sensing the mathematical structure of physical space*

Fig. 1 Epistemological ruptures

In mathematics education, the notion of *epistemological obstacle* has been a useful construct to analyze errors and student difficulties (Sierpinska, 1987; Janvier, 1998) in learning mathematics and the didactical consequences of acknowledging their existence (Brousseau, 1997). But in the mid-1990s it was criticized for not addressing socio-cultural factors. To quote Sierpinska (English & Sierpinska, 2004):

> And then, in the mid-90s, from the post-modern perspective, epistemological obstacles became a bad word; the philosophy underlying epistemological obstacles was criticized for being "recapitulationistic and parallelistic", for not sufficiently taking into account the socio-cultural factors. (p. 466)

Herscovics (1989) has also criticized the notion since it was not established in terms of individual learning experiences but the development of scientific thinking more broadly. We believe that the notion can be used to discover epistemological ruptures but the notion of culture needs to be redefined to address the co-evolution of technology and learning today. For us, culture is the co-evolution of digital media and expressive communicative actions. In other places (Moreno-Armella, Hegedus, & Kaput, 2008), we have begun to develop a digital semiotic theory that unpacks the nature of this co-evolution.

We adhere closely to the *pragmaticism* of Peirce (Peirce Edition Project, 1998) in analyzing the interaction of the (digital) sign, the artifact (its medium or mediated object) and its *interpretant* (the effect of the digital sign on the interpretation process). This view is based on Peirce's well-known triad of icon-index-symbol where signs can evolve through a series of referential associations (Deacon, 1997) to establish a meaning-making system and symbolic thinking.

In contrasting epistemological ruptures pre-twenty-first century with present development we propose a shift in semiotics to focus on a digital semiotic theory that continues to support the study of semiotics in its focus on the interaction of signs with the users of those signs but extending it to new "users" or "agents" that are evolving in digital environments; many of these are digital representations of the contributions of such users (e.g., the expressive actions of students, students' graphical constructions or equations).

We establish this shift through various notions including *co-action* that we have introduced previously (Moreno-Armella & Hegedus, 2009). This notion explains how technology users are involved in more than just a traditional semiotic enterprise when interacting with a digital "interface"; instead, it allows us to think and explore by guiding and being guided by that same environment. It is a reciprocal relationship—co-action—and not a one-sided arrangement—interaction. An environment that allows users to build things and see the interplay between their de-constructive and re-constructive actions presents a physically dynamic interchange between *force-push* of stylus or mouse to programmatic action. We can see the actions of our intentional force-push as being a structured thoughtful action. So listening to someone, watching someone's bodily motion, analyzing exploration, contraction of muscles, tension in the self or voice, can all be necessary modifiers of the environment explored. As our actions become deeply embodied into the very substrate of the environment we are in, we modify this place for our future thinking—in the spirit of Baldwin (1896)—both for others but most importantly for us. And so, the environment guides our thinking and our future actions as we interpret the feedback in various evaluative ways. Such co-action is an organizing principle for the way we make sense of sign-ing, object-ing, and sense-ing the very essence of the structure.

In this mode, we are examining the evolution of how mathematical sign systems evolved and how the symbolization process was transformed through representational re-descriptions over time. We are analyzing mathematical epistemology and applied epistemology in the present day through a semiotic lens (Otte, 2006).

And so, we propose that mathematical structure is the semiotic essence of a new digital semiotic theory. Without an understanding of mathematical structure and emerging structures that we as users or agents can modify through the affordances of new technology, we are not in a new era. As such, the embodiment of our prior knowledge allows us to design and think about mathematical ideas, and new mathematical modeling activities, hitherto un-heard of. For example, later on in our analysis, we shall discuss the idea of circle-ness and see how triangle-ness is a very plastic term that stretches across paradigms within one technological platform. It is no longer necessary to think about deficiencies but affordances to see through the structure that Kant and Euclid painted. Mathematical structures have historically been invisible and we shall illustrate how such ideas can become accessible and perceivable through our interaction with digital technologies. Structure no longer needs to be an abstract quality known only to the inventor.

2 Epistemological ruptures of mathematical knowledge: pre-twenty-first century

2.1 Epistemological rupture 1. Emergence of an embodied mathematical structure

2.1.1 Kant: forms of sensibility

In the first lines of the Introduction to his *Critique of Pure Reason* (1781/1787), Kant explains (Wood, 2001, p. 24) that although our knowledge always begins with experience, this does not mean that it *arises* from experience. This would happen because the knowing subject actively contributes to the production of knowledge through his/her faculty of knowledge. That everyone possesses an innate cognitive toolbox (forms of sensibility) that transforms those perceptions into pieces of knowledge is one of the keystones of Kant's epistemology. Of course, nobody is aware of the workings of these forms of sensibility that transform the original perceptions into knowledge and so we could have the conviction that what we experience is reality in itself. Kant decided to live an epistemological life different both from empiricism and the modern rationalist school of Descartes and Leibniz. His was a third way. He even severely criticized Plato because he considered the Greek philosopher was too far from the solid soil of sensuous experience. He expressed his criticisms with these words:

> The light dove, piercing in her easy flight the air and perceiving its resistance, imagines that flight would be easier still in empty space. It was thus that Plato left the world of sense, as opposing so many hindrances to our understanding, and ventured beyond on the wings of his ideas into the empty space of pure understanding. He did not perceive that he was making no progress by these endeavors because he had no resistance as a fulcrum on which to rest or to apply his powers, in order to cause the understanding to advance. (p. 27)

Kant was not an empiricist. His epistemology is based on those a priori forms of sensibility that wisely transform the empirical material into knowledge. It is as if the empirical data were a liquid entering a pitcher that conceded a shape to those empirical data. Knowledge is then, for Kant, the result of giving shape to data through the innate forms of sensibility. That is how space, for instance, was conceived in Kant's magnum opus. All we see around us can be seen, uniquely, as in Euclidean space, and there is no other possibility according to Kant. We filtrate the sensuous experience of space through this geometric form of sensibility. At the end, the only possibility for the experience of space is Euclidean: this is the only conclusion we can reach through our (innate Euclidean) glasses. In his words:

> Space is not an empirical concept which has been derived from external experience… Space is a necessary representation a priori, forming the very foundation of all external intuitions…Space is therefore regarded as a condition of the possibility of phenomena, not as a determination produced by them; it is a representation a priori which necessarily precedes all external phenomena. (p. 45)

Space is thus the pitcher and, furthermore, a Euclidean pitcher. Reality then, is imbued with a deep subjective dimension that makes us conceive it unavoidably as Euclidean space. Thinking this way led Kant to assert that human intellect imposes its laws on nature. Certainly, this view was a kind of Copernican revolution.

2.1.2 Letters of Gauss

When non-Euclidean geometries began to make their presence felt, the custodians of Kantian orthodoxy erected their opposition. They saw the new geometries as perhaps interesting intellectual exercises but not impacting the true nature of space (Torretti, 1984, p. 33). In fact, the order and rationality we perceive in the external world are imposed on it by our forms of sensibility. As our sensibility is Euclidean, Kant taught that the nature of space—its rationality—was in full agreement with *The Elements* of Euclid. Sir Arthur Eddington, the British physicist, followed Kant's teachings almost verbatim. In 1916, he wrote (Eddington 1916/1979, p. 213):

> The space and time of physics are merely mental scaffolding in which, for our own convenience, we locate the observable phenomena of nature.

Nevertheless, the gates were already open and new conceptions of space began demanding attention. Firstly, a clear-cut distinction was to be made between space, as described and explored formally by Euclidean geometry and physical space. This distinction was necessary as Kant had declared that physical space was shaped by our Euclidean sensibility. Consequently, according to Kant, there was no difference. But there was. During his university days, Gauss had heard the fuss about the fifth postulate of geometry. It is well known how from Proclus to Legendre, for over 2,000 years, people had fought with the postulate trying to establish it as a theorem. Because *it had to be true*. Very young, Gauss tried to leave his mark on this old problem. Eventually he realized why his attempts would be in vain. In his letter of 1817 to Olbers (Gray, 2007, p. 91), he wrote:

> I am becoming more and more convinced that the necessity of our [Euclidean] geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable.

The voice of Gauss is particularly interesting because, besides being an extremely deep thinker, most of his meditations—as he called them—with respect to the nature of space can be read in the letters he wrote to his friends. These letters reveal such undertones, as if Gauss had Kant in mind as an interlocutor with whom he was discussing secretly and intensely. Along these revealing lines, we can read his realization that the nature of physical space can escape the Euclidean almost iconic determination dreamt by Kant. At the same time, Gauss was reluctant to go against such an old tradition that this corner of the universe had supported so well. It is natural to grant global legitimacy to local experience. Consequently, Gauss was in the middle of a vortex, whirling around tradition and a new deep idea: the separation of reality from mathematical space. Iconicity would be broken. Gauss continued thinking on this problem through his friends. The letters he wrote and received from Olbers, Schumacher, Bessel, and some other confidants, played a role of mediators to his thought. Thinking through his friends' thinking was his personal approach that, at the same time, avoided all risk to his immense shadow.

Regarding his attitude, one comes to think of what Wenger (1998) has called *communities of practice*. That is, a group of people sharing a serious interest—in this case geometry—that they learn better as long as they interact regularly. However, interactions took place exclusively through Gauss' letter with each one of them. This is an especial community of practice with a kind of implicit interaction through Gauss' guidance.

By 1817, his interests were not just theoretical but practical and experimental as well. Euclidean geometry was a main tool in his present geodetic work (Breitenberger, 1984), thus for long years, he was having diverse opportunities to test the old knowledge. Geodetic triangles measured 180°—even the famous (and controversial) BHI triangle that Gauss used to link his triangulation of Hanover with contiguous ones (p. 277). Nevertheless, in 1829, Gauss sent this *probe* into Bessel's mind: "My conviction that we cannot base geometry completely a priori has, if anything, become even stronger" (Gray, 2007, p. 95).

A year later he added that his inner conviction (*always convictions, never proofs…*) was that number had an a priori nature not shared by space:

> Our knowledge of the former [space] is missing that complete conviction of necessity (thus of absolute truth) that is characteristic of the latter [number]; we must in humility admit that if number is merely a product of our minds, space has a reality outside our minds whose laws we cannot a priori state. (p. 95)

Gauss's perception was not that of his community of practice. It was his mediating artifact and his sowing field for such an audacious point of view. Earlier, in a letter to Taurinus, Gauss had explained some crucial results from a non-Euclidean geometry he was investigating. He wrote, for instance, that in such a geometry, the area of the triangle can never exceed a definite bound; the sum of the angles of a triangle can be as small as wished if the sides are taken large enough. And he added:

> All my efforts to discover a contradiction, an inconsistency in this non-Euclidean geometry have been without success…But it seems to me that we know, despite the say-nothing word-wisdom of the metaphysicians, too little, or too nearly nothing at all, about the true nature of space, to consider as absolutely impossible that which appears to us as unnatural. (p. 95)

He was fully aware of the implications from the existence of a non-Euclidean geometry. It would be a definite rupture with the past, the emergence of a new mathematical epistemology. There was no way to test the truth of geometry experimentally. We cannot let be unnoticed this remark: that what is possible from the logical viewpoint can be unnatural from a cognitive viewpoint. In fact, Gauss is saying—although implicitly—that below the collection of results, we have as a body of mathematical knowledge, a structure emerging that creates a tension between cognition and logic.

2.1.3 Lobachevsky

Lobachevsky's conception of geometry is similar to that of Gauss. In the *New Principles of Geometry* (1825), he says (Bonola, 1955): "The fruitlessness of the attempts made, since Euclid's time, for the space of 2,000 years, aroused in me the suspicion that the truth, which it was desired to prove, was not contained in the data themselves" (p. 92).

Thus, Lobachevky arrived at the conviction that it was not possible to prove the fifth postulate as a theorem *from the remaining postulates*. This conviction fueled him to explore the consequences of the new system that included as a postulate: *through a point, not on a given (straight) line L, there are at least two lines that never meet the line L.* Let us call this the *hyperbolic postulate*. By adopting the hyperbolic postulate, Lobachevsky made explicit his rejection of Kant's thesis that space is an a priori form of sensibility and consequently, that it does not arise out of experience. Lobachevsky went quite far in his development of this postulational system and never met any logical contradiction. But he knew that a contradiction could be lurking ahead. Then, he tried a different approach to study the coherence of his system. It is interesting to observe that there was never any doubt with respect to the coherence of the Euclidean system. Everyone was convinced of its coherence. In his *Imaginary Geometry* of 1835 (see Efimov, 1980), Lobachevsky wrote:

> Based on astronomical observations, I proved in my works on the elements of geometry that in a triangle whose sides are almost as long as the distance between the Earth and the Sun the angle sum can differ from two right angles by an amount not exceeding 0.0003 of a second…The propositions of practical geometry must therefore be viewed as having been rigorously established…. (pp. 34–35)

Lobachevsky developed the corresponding hyperbolic trigonometry and found that, translated into this analytic language, the theorems of the hyperbolic system were coherent. In a sense that was as far he could go at that time with his available tools.

When Farkas Bolyai, the father of Janos Bolyai, learned of his son's attempts to solve the problem of parallels, he urged him to publish as soon as possible because "when the time is ripe for certain things, these things appear in different places in the manner of violets coming to light in early spring" (Kline, 1962, p. 559). The image that these words convey is that ideas circulate throughout the communicational infrastructures of cultures; ideas resonate and become crystallized by means of the diversity of representational media that cultures provide. And in fact, that happened to Janos Bolyai's world of Hyperbolic Geometry that he thought of as a world he had created out of nothing.

The distinction between the objects of mathematics and the objects of natural science was gaining momentum. Gauss' student, B. Riemann, in his famous essay *On the Hypotheses Which Lie at the Foundations of Geometry* went further in this line of thinking. Poincaré (1905/1952) in his classic *Science and Hypothesis* made explicit his view when he said: "geometrical axioms are neither synthetic a priori intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts" (p. 50).

The idea of mathematical model was taking on a new epistemological status. We can define mathematical objects as very slow processes. When we represent them symbolically, we are making them stable; by crystallizing them, we make them objects of a discourse. Nevertheless, this has not always been the case. Geometry has traveled a long journey beginning with iconic drawings and arriving at the conscious realization that mathematical structures, in their role as models, are the new mediating artifacts to explore space. Having a structure, seen as a mediation-artifact, is having a determined mode of action that eventually will produce knowledge; a knowledge dependent on the artifact—the structure. Artifacts are never epistemologically neutral.

2.2 Epistemological rupture 2. Reformalizing the mathematical structure of physical space

2.2.1 Euclid: synthetic vs. analytic

Frequently, modern authors criticize the *Elements* for "lack of rigor." For instance, in passages when Euclid uses "the point of intersection" of a circle with a straight line, those critics react saying that the axioms Euclid used did not guarantee the existence of the said point—with these or similar words. We propose that these authors are misunderstanding the role and the nature of the drawings in the *Elements*. Let us explain with an example taken from Book III, Proposition 2 in the *Elements*. This proposition reads: "If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle" (Heath, 1956, pp. 8–9).

The proof is by *reductio ad absurdum*. But then Sir Thomas Heath, the translator, adds a commentary explaining that such a cumbersome proof is not necessary.

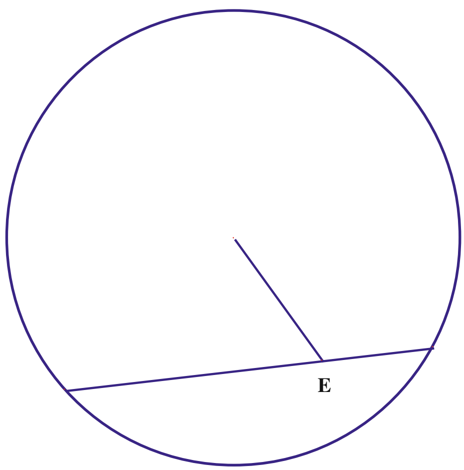
He says that to prove the proposition, it is sufficient to show that any point E (see Fig. 2) in the segment joining the two points taken on the circumference, defines a segment (with the center of the circle) that is shorter than the radius.

Why did Euclid take such a complicated route? Very simple. Because the Euclidean conception of segment (and line) is synthetic, not analytic. This means, for instance, that a point is produced at the intersection of two lines without implying that the lines "are made" of points. A line and a segment are holistic objects—they have points but are not built from the points. Consequently, Euclid had to consider the segment itself, not just its collection of points.

These remarks are intended to demonstrate that the proof of a result depends on how the objects involved are conceived. Heath's misinterpretation, we believe, comes from ignoring this fact.

The problem of ontology is ever present in Greek mathematics where the geometrical objects from Euclid's geometry result from empirical abstractions. This is not a passive

Fig. 2 Circle construction



process on the part of the subject but results from observation and activity upon the material world. In particular, the continuity ascribed to these objects reflects the synthetic nature that these objects have in Euclidean geometry. The notion of a circle is abstracted from *active* observation of round plane objects as well as from feeling round objects following the border with your finger, that is, by *feeling the shape*. The transfer to abstraction takes this experience aboard. Euclid's drawings reflect the primary intuition of physical objects. The axioms of geometry were adopted as self-evident truths about physical space. In this line, we can understand why objections to the Fifth Postulate had to do with the supposed lack of intuitive content of the assertion as it describes what happens far away.

2.2.2 Gauss and Lobachevsky: truth about physical space

Gauss and Lobachevsky claimed that Euclidean geometry was similar to other experimental sciences—that it did not contain the truth about physical space. Lobachevsky explored the nature of physical space using astronomical triangles to test if the sum of their angles was more than two right angles. That this sum was less than two right angles had been proved formally—assuming that there was more than one parallel to a given line—but they did want to confront this formal system with physical space. Unfortunately, the experiments did not provide conclusive evidence. Nevertheless, this way of thinking represented a giant step ahead: mathematics could model the material world. But now, the internal coherence of the model was independent from former (that is, Euclidean) ontological considerations. Nevertheless, the ontological conception of the Euclidean objects was too deep-rooted to easily get rid of it. In fact, Euclidean geometry could be *seen* in the world around people whilst non-Euclidean geometry—developed as a formal system—was still invisible. A geometer could draw a Euclidean straight line (what they *believed*) but they could not draw a genuine non-Euclidean line. It lacked an intuitive image. A new epistemology of mathematics was about to arrive but its last step was yet frozen in the air.

2.2.3 Beltrami: representational re-descriptions

By 1868, the Italian mathematician, Eugenio Beltrami, proved the *local* consistency of hyperbolic geometry. Beltrami showed that this geometry is locally equivalent to the *intrinsic* geometry of surfaces of constant negative curvature (see Fig. 3).

The sides of the triangle are geodesic lines. These geodesics were interpreted as corresponding to the straight lines of non-Euclidean geometry. This illustrates a triangle with a sum of angles less than two right angles. Thinking in terms of intrinsic geometry led geometers to interpret the Euclidean plane as a surface of zero curvature. The surface introduced by Beltrami was the *Pseudosphere* (see Fig. 4) with constant negative curvature: $\frac{-1}{R^2}$.

We insist this is a *local* model of Hyperbolic geometry because the geodesics cannot be extended beyond the rim (thus, the second Euclidean axiom is only partially satisfied). Perhaps it is well known that Hilbert proved years later that it was impossible to embed a complete surface of constant negative curvature in the ordinary 3-dimensional space. Hyperbolic geometry could be *seen* although just locally. The model changed the perception of this geometry from something logically possible to something actually existent on a surface. The system was abstract however *intuition was provided* for it. It is not possible to exaggerate the importance of this work for mathematics. Bolyai and Lobachevsky had considerably developed the formal structure of non-Euclidean geometry

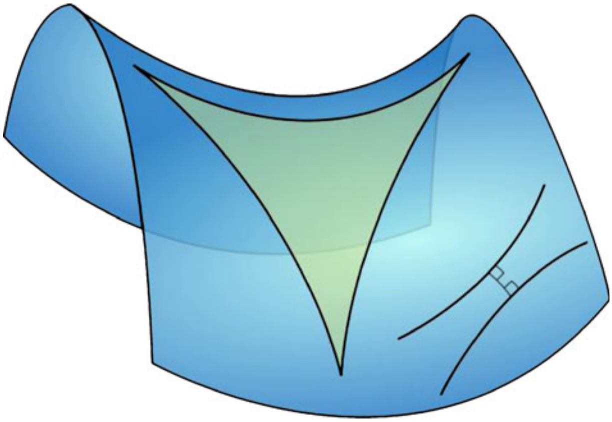


Fig. 3 Negative curvature

convinced that it was free from contradiction because the "Euclidean system itself did not contain the whole truth of the physical space" and consequently could not become a genuine obstacle to the existence of non-Euclidean geometry as a description of physical space. They were convinced that the hyperbolic system was internally coherent but could not provide a formal proof. Beltrami did this locally. His interpretation of non-Euclidean geometry as the intrinsic geometry of a surface *embodies* the hyperbolic system in the surface. So far, non-Euclidean geometry was a formal system; now it could be represented and become visible. This kind of embodiment is what, in general, a model provides and with this embodiment, the model provides as well an interpretation, a sense of truth for the formal system.

Trying to understand human cognition, Karmiloff-Smith (see Tomasello, 2000) explained that: "a specific way to gain knowledge is for the mind to exploit internally the information it has stored by re-describing its representations" (pp. 194–195).

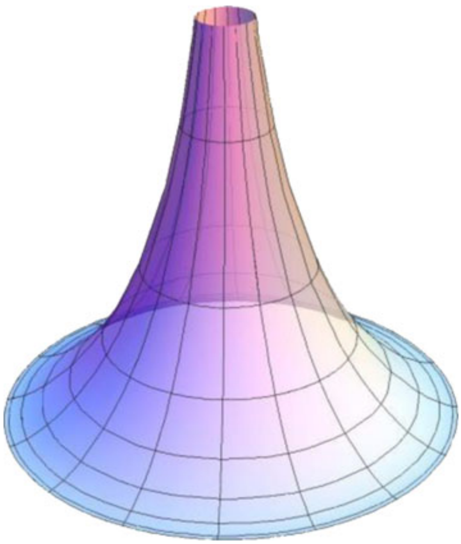


Fig. 4 The pseudosphere

Of course, once this process has taken place, it is crucial that it be externalized through new semiotic representations. This is what Beltrami did. His embodiment of hyperbolic geometry as the geometry of a surface of constant negative curvature corresponds to a *representational re-description* of the geometric elements pertaining to the surface itself.

With Beltrami's model the human eye recovered its role as an interpreter. A new and previously non-intuitive geometrical formal structure had a concrete realization. Perception and abstraction turned out to be the sides of the same coin.

From this moment on, the rupture that had been taking momentum through the works of Gauss, Lobachevsky, and Bolyai was finally crystallized. The coexistence of different systems of geometry—Euclidean and non-Euclidean—was established and with this move, the nature of mathematics was deeply transformed. Mathematicians realized that through the *art of mathematics* they could provide symbolic models, not iconic models. Of course, they could explore how well the model corresponded to reality under study. This is a modern attitude and Lobachevsky's work was probing this attitude.

With Euclidean ontology a mirror was placed between the world and mathematics. Non-Euclidean geometry broke the mirror.

But an end is always a beginning. Mathematics is a human activity; an activity mainly pursued with symbols. Now, this activity is about us, not about immaterial and atemporal objects whose existence is pre-semiotic. We do not see ourselves reflected on a natural mirror, we see ourselves *refracted* on a world we have saturated with our own activity. This is our mirror, not the former, now broken, iconic mirror.

Truth is gone from mathematics. This helps explain the famous dictum of Einstein (Pesic, 2007): "In so far as mathematical theories are about reality, they are not certain; so far as they are certain, they are not about reality" (p. 147).

In summary, when we speak of the deep consequences of Euclidean ontology, particularly how it hindered the conquest of non-Euclidean geometry, this does not mean we had to transcend our senses and become pure thinkers. This stance would mean a back step into Plato's realm of pure ideas. What we say is that when a partial experience of space is taken as a universal experience, the search for the new is closed. This is reflected in over 20 centuries of a quest for a proof of the fifth postulate. Mathematicians became blind to the possibility of a new geometry.

Euclidean ontology made a segment a synthetic object. So, the reasoning was based on the diagrams taken as absolute. The diagrams became the mediational artifacts for Euclidean epistemology.

What we introduce in the next section is a perspective based on a post-non-Euclidean geometry. This perspective entails a deep consequence that force is part of geometry, as we shall demonstrate. We are living in a post-Einstein world so space cannot be conceived of anymore as an empty vessel. Physics and geometry are blended and this is what we refer to as the rupture.

From the perspective of education, geometry cannot be taught as static. It cannot be taught as *cinematic* geometry either. A genuine dynamic geometry has to incorporate *force* and elastic dynamic visuals.

3 Epistemological ruptures of mathematical knowledge: twenty-first century

3.1 Emergence of embodied mathematical structure through dynamic geometry

Becoming symbolic beings means, broadly, that we have developed the strategy based on the ability for something to take the place of another thing. Because this is essentially a social, communicative strategy, "taking the place of another thing" had to emerge as an agreement within a community. The result—the symbol—is a cognitive artifact that enables us to go beyond the biological frame to find the way to produce more objective versions of human knowledge. The symbol crystallizes intentional human activity. Human knowledge results from human activity and the new artifact, the symbol, radically altered the nature of the activity that produced knowledge. Everything was re-described within new representational settings.

Dynamic geometry, as a semiotic medium, provides examples of representational re-descriptions in geometry. We shall illustrate this point by introducing a model for hyperbolic geometry. Beltrami's model enabled us to *see* a hyperbolic straight line. Indeed, what we saw on that surface, the pseudosphere, was a local embodiment of the hyperbolic system. There is still a difference we see if we compare this embodiment with the Euclidean plane. The latter has a naturalness lacking in the former, as we have discussed in the previous pages. We *can* draw a segment on a piece of paper and we do not see this drawing as an embodiment of the concept of Euclidean segment: It *is* the segment. But we cannot draw, the same way, a hyperbolic segment on a piece of paper or using a dynamic medium like, for instance, Cabri software. Our lives flow in a local, flat space. Hyperbolic space is not flat, it has negative curvature. Beltrami gave the evidence. To draw a hyperbolic segment or line, we need to do it through an explicit embodiment, that is, through a model of the hyperbolic segment. The semiotic mediation of the model enables us to *see* with new eyes, the segment, for instance. This has a deep consequence: we are constructing a way of being for these hyperbolic objects and consequently, we develop a way of perceiving them. In short: *we are creating an ontology.*

Consider the following example. The hyperbolic world is the interior of the disk (see Fig. 5). The (straight) lines are the segments of orthogonal circles to the border of the disk. Observe that diameters fulfill this condition.

The border of the disk is constituted by the points that are at infinite distance from the center of the circle. In a famous chapter of his book *Science and Hypothesis*, (1952, pp. 65–68), Poincaré described a (possible) world whose physical conditions led to a hyperbolic geometry as the most "natural" geometry for that world. The following figure shows two parallels and the fact that they are not equidistant (See Fig. 6).

Fig. 5 Hyperbolic model

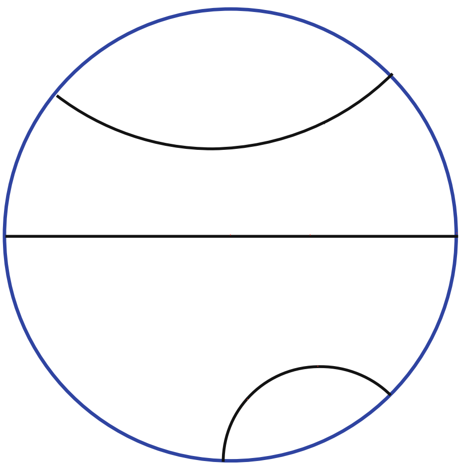
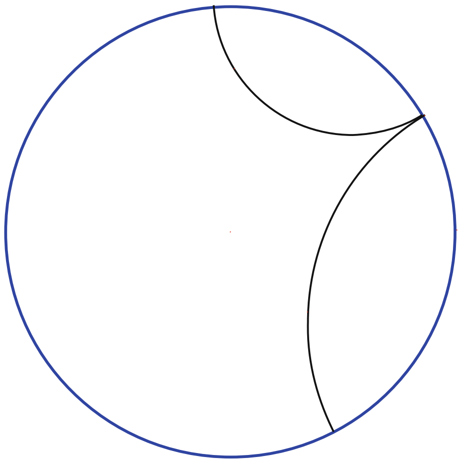


Fig. 6 Parallel lines

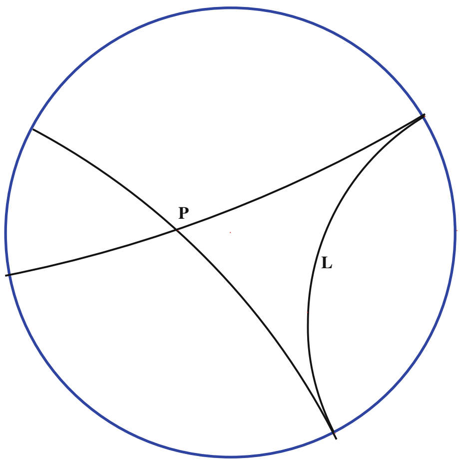


Given a line it is very easy to see how to draw at least two parallel lines. For instance, through the point P, two lines parallel to the line *l* are shown (see Fig. 7).

Perhaps the most interesting movie we could film in this hyperbolic world is the morphing from a "Euclidean" into a non-Euclidean triangle (see Fig. 8). If we select a very small triangle it looks Euclidean in the sense that beings living in that world could find the sum of the angles equal to 180° (within experimental error). But when the triangle is growing, this Euclidean appearance (and nature) continuously changes. This is a lesson in perception as well.

What can we learn from this? Structure as a mediating artifact can modify our natural ontology to accommodate change. Modification is unavoidable; ontology is always there as an emergent phenomenon from the semiotic object, which leads to the overcoming of the epistemological obstacles that came as a barrier to past geometers.

Fig. 7 More parallel lines



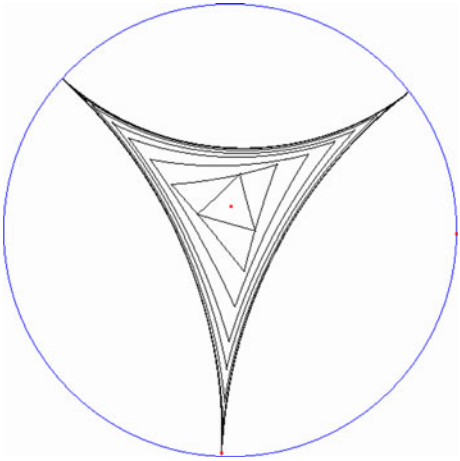


Fig. 8 Perceiving Euclidean triangles in non-Euclidean space

3.2 Epistemological rupture 2. Sensing the mathematical structure of physical space

Haptic technology is another example of representational re-description, especially when used with dynamic (interactive) geometry. Haptics have evolved over the past 10 years—particularly out of a focus on virtual reality in the 1990s—and haptics have become more available in a variety of commercial and educational applications—particularly 3-dimensional design and modeling, medical, dental and industrial applications. In a meta-analysis of the use of haptic devices, Minogue and Jones (2006) found over 1,000 articles on haptics in the electronic databases ERIC and PsycINFO, but only 78 relating haptics to learning/education. A large proportion of these were focused on "haptic perception"—a major field in psychology focused on haptic sense—and the second main set was focused on multimodality. Multimodality reaches into education in various ways, intersecting deeply with a multimedia approach. Multimodal approaches have also focused on the role of gesture with increasing interest recently (Wagner Cook, Mitchell, & Goldin-Meadow, 2008) with particular focus on this mode as a form of mathematical expressivity (Hegedus & Moreno-Armella, 2008; Hegedus & Penuel, 2008).

Historically, this shift has been translated as a way to create multiple learning pathways for students to work within, and predominantly these were dominated by auditory and visual modalities. Visual perception is inextricably linked with haptic or tactile-kinesthetic perception (Scheerer, 1986). In fact, this perception is historically said to precede formal language development and speech (Piaget & Inhelder, 1969). More recently, research in brain sciences has confirmed the overlap between perception and physical action. Our own movements and proprioception (continually sensing what our body is doing) allows us to regulate and make sense of the physical world. In addition, Iacobini, Woods, Brass, Bekkering, Mazziota, and Rizzolatti (1999) have highlighted an overlap between perceiving a dynamical pattern and physically enacting it. In these dynamical situations, perception occurs in many forms. Kozhevnikov, Hegarty, & Mayer (2002) investigated the relationship between mental imagery and problem solving in physics, specifically in kinematics, and reported that while spatial imagery (encoding of spatial relations among objects) may promote problem-solving success, the use of visual imagery (pictorial imagery that encodes the literal appearance of individual objects) presents an obstacle to problem solving in kinematics. Similar findings have been discovered in mathematics education where certain kinds of imagery can create obstacles (Presmeg, 2006).

Studies have shown that students can experiment with everyday objects in the 3-dimensional world in which they live and create spatial graphs that portray the actual motion, as well as time-based graphs using the variables of position, velocity, and acceleration through a more sensory immersive experience (Dede, 1996, 2000).

Our recent NSF-funded work has built upon these established theoretical perspectives of the role and use of dynamic mathematics software. We have worked on the incorporation of certain haptic devices with The Geometer's Sketchpad®. In particular we have focused on the use of Sensable's PHANTOM Omni® (http://www.sensable.com/haptic-phantom-omni.htm). This controller is a desktop haptic device with 6 degrees of freedom for input (x, y, z, pitch, roll, and yaw), and 3 degrees of output (x, y, and z). The Phantom's most typical operation is via a stylus-like attachment that includes two buttons. The Phantom has a very robust community and Software Development Kit (SDK) behind it. The SDK (OpenHaptics-Academic Edition) allows for two levels of programmatic control: precise programmer created feedback—such as vibration—and pre-programmed feedback—such as springs and dynamic/static friction. The PHANTOM provides up to three forces of feedback for x, y, and z. The PHANTOM is primarily used in research, with a significant presence in dentistry and medicine.

We have investigated whether the device could be integrated in a mathematically effective way so that it would be used as a semiotic mediator (as a dragged mouse might be used in stand-alone dynamic geometry environments to mediate visuals). Our preliminary work that we report on here incorporated four activities. The first was a point-activity where connected points could be moved around to assess the ease-of-use of the device in coordination with the visuals on a screen. The second presented two spheres that could be pushed around a "universe." Participants included two young children, two undergraduates and a mathematics professor. Each participant was associated to project staff. All participants quickly adopted the use of the device and made the connections described. The third activity had no visual screen representation but the device was programmed to be constrained by the forces of a spring. All participants described the forces exhibited correctly without a visual. The fourth was a dynamic triangle exercise with 4 inputs we report on here (see Figs. 9 and 10).

Figure 10 illustrates the geometric figure designed with constraints similar to those in most dynamic geometry environments. A triangle is constructed on a pair of parallel lines. The participants interacted with the figure by moving a cursor via the haptic device to one

Fig. 9 Experiment with a 6-year old

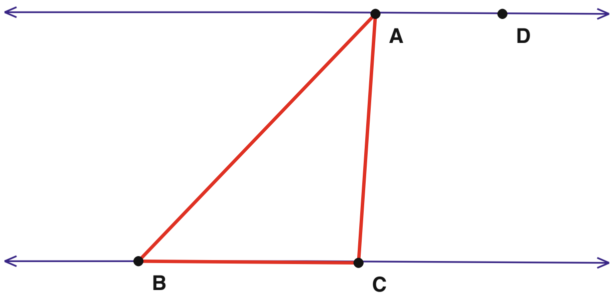


Fig. 10 Dynamic triangles

of the four hotspots (A through D). When selected, the force feedback of the device was linked proportionally to the area of the triangle. The aim of the exercise was for the users to understand that moving point A would not change the area of the triangle (since the height and base of the triangle would be the same) and so the force was constant, hence the force and area were invariant under this action. But moving point D (which moves the parallel line) or points B or C (which changes the "base" of the triangle) would result in different forces, as the resulting triangle would have a different area.

Our initial findings with young children, adults and professors provided evidence that the haptic device could be very quickly utilized (within minutes). Participants used a variety of dynamic metaphors in describing what they saw (a deformable triangle with various degrees of freedom) and in terms of what they felt. The hotspots (directly linked to the feedback from the device when dragged) were described by the children to be pulling or pushing them. These expressive descriptions were translated during the activity from informal to formal descriptions of the properties of the triangle (it was bigger or smaller), yet invariance was less well formulated. The children in the study offered similar stories of how the family of triangles could be interpreted in a playground context (i.e., as slides). Adults in the study used the device in a more sensitive way, not moving the arm through space as much as the children. In summary, we found that even a 6-year old could use the device and begin to describe the structure, the various properties of the triangle (without any formal introduction to geometry) and link his haptic experience (in terms of force feedback) to these properties. Our ongoing work aims to build on this work in creating activity structures (constructions that yield a wide set of activities) based upon similar design principles from dynamic geometry (e.g., hotspots).

In this preliminary design work we have established an environment for young children to sense mathematical structure through new media, both physically and visually. The intersection of these senses challenges the role of perception in mathematical thinking, and if our thinking is challenged through the transformation of the mediator (the controller, the dynamic visuals) and the co-action with the environment then a new kind of semiotics emerges. Why? The sign system is no longer engrained in a historic ghost form. Instead it exists in the very medium of the new environment. We can use metaphors such as plastic and elastic to describe the deformability and the accessibility of such forms, but it goes deeper, as the very mathematical structure (form, shape, and attributes) can be explored since we as researchers and developers can establish environments with specific curricular- and pedagogically refined constraints, for example, the triangle example where the specific attribute of the triangle was linked to the haptic sensation. It is this very turn in the representational structure and re-description that establishes our proposed epistemological rupture; just as occurred with the models of Gauss, Lobachevsky, Bolyai, and especially Beltrami.

4 Reflection: a new digital semiotic theory

The non-Euclidean model devised by Beltrami showed that this new geometry was locally the geometry of a surface with constant negative curvature. Being in existence at the same time, non-Euclidean geometry shattered the Euclidean ontology and offered the vision of a geometry as a possible model of space.

It also offered a pre-digital semiotic theory, i.e., it established the request for a new form of semiotics that was saturated within the medium. If we could have a new medium, the representational affordances might allow us to "see" what we previously could not perceive; to sign what we could previously not signify; and to symbolize within a consensual domain what we previously could not agree upon as a field.

Mathematical structures are about human experience; they mirror human deeds, not a pre-given reality. We wish to underline that this point of view crystallizes the epistemological revolution that came hand in hand with the new geometries.

The experience Gauss obtained through his geodetic labors was instrumental for his research on the intrinsic geometry of surfaces. His *Theorema Egregium* was the key to intrinsic geometry as it establishes curvature (in the sense of Gauss and later of Riemann) as an isometric invariant. If Gauss had realized that the Minding–Beltrami surface with constant negative curvature was a model for hyperbolic geometry, this geometry would have been accepted earlier. But that was not the case. Euclidean geometry was a very special case from this viewpoint: it corresponded to the geometry of a flat surface.

The transformation of this discipline was profound; the new epistemology could be synthesized this way: *Mathematics provides maps of the material universe; it is not the terrain*. This realization facilitates understanding of mathematics as a semiotic activity.

In 1854, Riemann gave a lecture: *On the Hypotheses Which Lie at the Basis of Geometry* (see Pesic, 2007, pp. 23–40). He explained that, in this context, hypothesis refers to what the mind adds. The most fundamental breakthrough Riemann made was to determine that space is not an empty receptacle that eventually is filled with bodies; that bodies are not simple passive presences of this space but actors that bend and shape the space itself by their very presence.

In a now famous phrase (see Schwartz, 1979), John Archibald Wheeler said, "Space tells matter how to move and matter tells space how to curve." Curvature, at the end of the day, explained forces. Once again we meet an analogy: we have discussed the fact that an organism and its environment are coextensive. Now, we have that space and matter are coextensive and their *co-action* is crystallized in the curvature of space.

Euclidean geometry is a structure—an artifact of knowledge. It structures our local experience with space. As with every artifact, there is a Zone of Proximal Development of the Artifact (ZPDA) associated with Euclidean geometry (Hegedus & Moreno-Armella, 2010). Non-Euclidean geometry is the realization of this ZPDA. Moreover, the ZPDA corresponding to non-Euclidean geometry is *mathematical structure,* and *model* is a different word for expressing the fact that mathematics is a way of understanding the world with the artefactual mediation of formal, semiotic structures.

Human epistemic activity never stops. For instance, in Greek culture, the famous impossibility problems (duplicating the cube, squaring the circle and trisecting the angle) could be conceived of as showing the epistemological limits of ruler and compass. Mathematical knowledge is mediated knowledge; then the nature of the knowledge we can generate depends on the mediational artifacts.

We mention anew this epistemological problem: human activity is the producer of knowledge, but knowledge is neither arbitrary nor pre-determined in nature. Being in the world, for us, means that we have a nervous system that has evolved and became adapted to the evolving world. Our symbolic powers are a medium to produce representational re-descriptions, and that constitute genuine knowledge. As Merlin Donald (2001, p. 155) has explained:

> Animal brains intuit the mysteries of the world directly allowing the universe to carve out its own image in the mind. This is largely a receptive mode of knowing, and we share it with our animal cousins. In contrast, the symbolizing side of our mind is more aggressive in its approach. It creates a sharply defined, abstract universe that is largely of its own invention.

The links between dynamic, interactive images and haptic experiences are the essential attributes or properties of the mathematical structure. When we talk about essentials, we refer to the “ness” of the structure. So for example, straight-ness is not only a form but also something that we attribute to a particular feeling. It is interesting to note that we do not describe all figures like this, so what is circle-ness? Can we ascribe an essential attribute to a particular kinesthetic feeling? Certainly we can motion a circle by moving our arm where the center of the circle is our shoulder socket and the arm can be a radius of a circle that our hand traces out in space. But the circle is invisible. With a haptic experience, we believe that such essential qualities will begin to enter our conceptual domain. Feeling circle-ness or triangle-ness, or equilateral-ness can potentially allow us to access conceptual mathematical structures in different ways. We believe that integrating this approach with a dynamic visual environment increases the potential for learning and is one we are continuing to investigate at this time.

Abstraction is an emergent quality of knowledge, as something that occurs between one person and a piece of knowledge. Perception is an “artifact” or a channel for this to happen. The result of abstraction is an increased level of concreteness; one appropriates things better and this can be understood as embodiment. It does not mean something is “entering” one’s body but rather one is transforming one’s body including one’s brain. We can read this process along the transformation of geometry we have explained. The embodiment of the parallel postulate led to a representational re-description of the space, something that can be perceived in Gauss and Lobachevsky’s “experiments” with huge triangles trying—through extended human experiences—to detect the structure of those objects (the triangles). So the focus is on perception and cognition.

Feeling the shape, for instance, is a door to acquire the structures of geometry, to understand that the shape of space is in “you”—extended already in space. Mathematics is not outside and we need to embody it. Mathematics is not inside (static) and we need to make it visible. Mathematics, rather, is in our movement in the world as an emergent phenomenon for appropriating that world, which means to re-describe it.

Drawing a circle with a compass and feeling its shape with your finger, traveling along the border or a round object, are two ways of re-describing the circle. Doing that separately is part of the process to appropriate the shape. Addition is part of abstraction.

Artifacts are not epistemologically neutral so if one uses an artifact as a mediator, the nature of the knowledge produced is linked to the artifact and it is breaking one’s ontology. Structure as a mediator can modify one’s ontology to accommodate change—to address epistemological obstacles. Structure is an artifact to enter the secrets of geometry. We need to embody to help us deal with the challenges (and overcome the epistemological obstacles) in moving from Euclidean to non-Euclidean geometry. It is dependent on the chosen path and the affordances of the tools in dynamic geometry.

In Fig. 8, we saw the Euclidean triangle disappear into a non-Euclidean world and that structure, as a mediator (as a mediating artifact) can modify our ontology to accommodate change. We have attempted to defend the emergence of multimodal technologies that can establish an infrastructure composed of elements or tools that afford new ways to experience structure through dynamic visual or haptic experiences. We can envision new multimodal experiences that allow us to feel spherical or hyperbolic triangles through an embodiment of curvature and a radical move away from straight-ness to new forms of triangle-ness.

Feeling and seeing shapes can be an embodied experience yet the affordances of new technologies mediate structure that was not available before—in a sense, establishing an augmented reality confronting previous epistemological obstacles.

In our paper, we have illustrated how structure acts as a mediator in two ways. Firstly, structure acts as movement in geometry. It has always been there in geometry but has been "silent" evolving historically between Euclidean and non-Euclidean geometries, and emergent in dynamic representations. Secondly, structure acts as movement in a haptic domain both as force and velocity. This is extending a visual, interactive medium where dragging does not embody force or expressive change (i.e., a high-velocity drag does not mediate meaning any more than a low velocity drag), yet a haptic experience can utilize velocity (or acceleration) to ascertain certain uni-dimensional attributes (e.g., measures of length, area, or angular change) to a physical action. Feeling is an exploration of embodied structure. This structure can be a geometry of space.

In summary, this space is mediated by structure as functional relationships, both as measurement (e.g., area to dragging), overcoming the epistemological obstacles of geometry within static media and a new affine space to perceive and conceive the nature of mathematics (i.e., directly mapping physical space to structure). And so, traditional meanings of structure and construction (con-struere)—which implies a together-ness, conformity and rigidity of a pre-defined order and form—can be transformed to incorporate more crystallized or organic notions of symbol, object and processes that allow actions on these objects. We believe that this process can define new learning environments in the future and theories of teaching and learning that are focused on structure as a mediator through new technologies.

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