

On meaning making in mathematics education: social, emotional, semiotic

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Abstract This paper is an attempt to add to the foundation of our understanding of meaning making in mathematics education. This attempt seems to be necessary as a growing body of research, primarily in developmental psychology, begins to change our view of early human development. Empathy, reciprocity, and implicit understanding seem to be more suitable concepts to describe human development and learning than the ones previously employed, based, e.g., on aggression as natural instinct or competitiveness as genetically wired basic drive.

Keywords Attachment · Consciousness · Culture · Deep play · Emotion · Empathy · Implicit learning and understanding · Meaning-making · Mind · Pragmatic maxim · Pretending · Reciprocity · Semiotic · Shared intentionality

Learning mathematics has been almost exclusively understood as a rational cognitive process of acquisition more or less along the lines the structure of mathematics makes available—with some notable exceptions (e.g., Zan, Brown, Evans, & Hannula, 2006). Only when learning failed, when it was difficult to understand why it failed due to some assumed misconceptions blocking the way, it was felt necessary and appropriate to think about the social/emotional aspects of learning mathematics. Reconsidering fundamentally the empathic foundation of learning is, however, not only an imperative given by our wish to be as complete and comprehensive in our understanding as possible. It is also a corollary of our responsibility to contribute to the care and precaution in upbringing and educating the coming generation (Holzkamp, 1984; Stiegler, 2010).

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In this paper, I shall wander through the territory of the foundation of mathematics education—in light of what might be called nothing less than a paradigm change in our view of human nature. Research on human and non-human primates has, over the years, accumulated evidence that the view of human nature as *homo homini lupus* (man is a wolf to [his fellow] man) has to be corrected. Helping, altruism, cooperativeness, and empathy seem to be as typical for the behavior of humans as the often applied aggressiveness. And this is true not only for grown-ups but especially for human infants and the members of the evolutionary infancy of humankind, the great apes. This growing body of literature has a distinctive evolutionary orientation (see, e.g., Tomasello, 1999, 2004, 2008, 2009; Warneken & Tomasello, 2009; Greenspan & Shanker, 2004; Rifkin, 2010; see also Hildebrand-Nilshon & Seeger, 2006 and Seeger & Hildebrand-Nilshon, 2010 for an application of these ideas), partly going back to altruism and reciprocity as non-Darwinian and developmental concepts of evolution (see, e.g., Kropotkin, 2006/1902; Trivers, 1971; Youniss, 1986).

I take it that the task of upgrading the theoretical foundation of mathematics education is still with us. We have learned from previous debates, from the rise and fall of paradigms, and from the decay of theoretical promises that the solution, if there is such a thing, may not be as encompassing, global, and comprehensive as we felt it should have been. My walk through the territory, thus, is an attempt to mention all the important signposts that I will find on the way. But I will not be able to visit all the locations that they are pointing at, keeping in mind that it would be necessary to visit those locations in order to get a full picture.

In the struggle to overcome the rationalistic orientation mentioned above, it has been one of the major accomplishments of research in mathematics education in the last two decades to look more closely at the discourse in the mathematics classroom (see, e.g., Kieran, Forman, & Sfard, 2001). Apart from the advancement made possible through this change of focus, it became apparent, however, that we meet a paradoxical situation here: discourse is a necessary condition for understanding and meaning-making in the classroom but learning and meaning-making remain, at the same time, largely implicit or indirect. The implicitness of learning seems to arise as a big obstacle against more systematic and goal-oriented instruction and more effective teaching—just because teaching can only create and support learning contexts and learning culture and not actual learning.

The implicitness or indirect nature of teaching and learning is closely connected to the legacy of being and becoming a human being. It is based on the same ability as our empathic and altruistic behavior: the intuitive empathy is not only working in relation to other human beings, e.g., in a form of such extreme sensitivity that the slogan of “mindreading” has been coined (see, e.g., Nichols & Stich, 2003; Goldman, 2008). This sensitivity is also working on the own self as some form of a perceptual sensitivity to the intended actions of others.¹

Generalizing and situating knowledge can both be seen as social and as conceptual processes—and the words already signal this duality, in much the same way that

¹ In its bodily expressions, this sensitivity and awareness has always been the aim in the context of education, ritualization, and training, see, e.g., Norbert Elias’ cultural-historical account of the development of table manners (Elias, 1978). It has been a major theme in Pierre Bourdieu’s work on distinction (1984) and the theory of practice (1977). The dialectic of visible and implicit, of form and content, of essential and incidental is nicely expressed here: “The whole trick of pedagogic reason lies precisely in the way it extorts the essential while seemingly to demand the insignificant: in obtaining the respect for form and forms of respect which constitute the most visible and at the same time the best-hidden (because most ‘natural’) manifestation of submission to the established order ...” (Bourdieu, 1977, p. 94–95).

“representing” has a social and conceptual meaning. In this paper, I will argue with Vygotsky that the social meaning is the precursor to the conceptual, individual meaning. I will also try to illustrate what it means to understand teaching and learning as processes of sharing and of shared intentionality. On the one hand, sharing must be seen as a cognitive function—where we look at processes of generalization, of concept formation, of logic, and of making sense of objects and how they work. But sharing can also be seen as an emotional function—where we look at processes of understanding the intentions of others, of helping each other, of solidarity, and of making sense of how to participate, communicate, and cooperate. In earlier theories of development, these social–emotional skills always appeared only as an accomplishment on later developmental stages quasi triumphing over the earlier egocentric stages. Today, we witness that evidence is accumulating that human babies and infants exhibit altruistic behavior, empathy, cooperativeness, and so forth (see, e.g., Tomasello, 2008, 2009; Warneken & Tomasello, 2009).

Apart from being a pretty ambitious list of research questions and programs, the above looks like the old confrontation of the rational vs. the emotional. In contrast to a recital of this often confusing contrast, I shall rather try to elucidate how these two functions depend on each other in making sense of the world—and making sense of mathematics. While trying to avoid the Cartesian dualism, I will start an attempt to recast the old problem of rational vs. emotional using a semiotic approach and rephrase it as the difference between signs referring to objects and signs referring to human beings. This approach has been elaborated elsewhere (Seeger, 2005, 2006; Hoffmann, 2007). I shall discuss some of Peirce’s semiotic arguments and also Vygotsky’s developmental reasoning. My suggestion is to see *shared intentionality* as a point of convergence of these two views—rather than as a point of growing distance between them. The context of shared intentionality will be taken as convergence point for the context of practice, on the one hand, illustrated through Peirce’s semiotic and the pragmatic maxim, and, on the other hand, for the context of development, illustrated through Vygotsky’s view on cultural development.

Meaning-making in mathematics classrooms, I claim, is not accomplished through some kind of mental organization or reorganization. Rather, meaning making happens in the world outside the head—and this is a world where the mind, the person, the mathematics, the classroom, the city, the country do not exist in isolation but are connected in expanding and interacting circles. The practice of meaning-making is a symbolic practice, a development of systems of signs borne out of the interaction and attachment of the human infant to her mother creating the self and situating action (for a fascinating account of this perspective on phylogenetic and ontogenetic development see Greenspan & Shanker, 2004). The present paper argues that the focus on practice is essential for a study of meaning-making—not a cold and soulless practice, but one full of real human beings, with thoughts, emotions, fears, wishes, and needs.

1 Making sense of the zero: an example

In a wonderful example from primary education, Heinrich Bauersfeld (1990) has presented a case from the massive data that had been gathered in his numerous studies on interaction in the mathematics classroom. This study was meant to follow one class through the primary years. This example can illustrate some of the crucial points I shall make, especially: (1) how the activity of school going as a precursor to school learning (Leont’ev, 1979) is mediated through signs and the playful interpretation of signs; (2) how in the

interpretation of signs the object can be some form of deep play; (3) how deep play and pretending are fundamentally related to meaning making and personal sense; and, last but not least, how all these streams of semiotic webs come together into some form of personal identity and personal control.

The example is taken from a series of transcripts made in one primary classroom (Voigt, 1989; see also Voigt 1998) and has the title “Thomas, our custodian of the zero.” It is about a boy who had exhibited a distinctive interest in the “zero” during the course of the first school year. His behavior had not been particularly noticeable until on one day the teacher introduced the ordinal numbers. This introduction was carefully prepared and staged as a surprise—as she went out with six children who afterwards entered the classroom one by one carrying tablets with the first six ordinal numbers: 1.—2.—3.—4.—5.—6., where the point is to learn to make a difference between ordinality and cardinality.

Everything went well with the presentation of the numbers—students exchanged and reordered tablets and so on—until Thomas who had shown some signs of growing discomfort yelled out loud: “The Zero is lacking.” This statement triggered some heated discussions among the students leading to a formulation of the obvious paradox that the first student entering the classroom with the tablet “0.” would be the first, and the one entering with the 1 would be second and so on—which clearly would not be correct. Thomas suggested a compromise as a solution to the dilemma: a “0” should be written on a piece of cardboard and should be put down on the ground (not being carried by a child) just before the first child. The class agreed, and it was done accordingly. From this moment on, Bauersfeld describes, Thomas felt himself responsible and called upon in all matters concerning the “0”—all through the first year. He became the expert in question with the “0”: “the custodian of the Zero.”

Later in the school year, another occasion came up to demonstrate that Thomas was seriously reflecting and supervising all matters regarding the zero.

In this lesson, the teacher had planned to illustrate that visualizations of a certain number expression could be read in multiple ways. She wrote $4=1+3$ on the blackboard and added a cardboard picture of five green apples (see Fig. 1): three apples having one worm, one apple with two worms, and one apple without a worm. The first student asked to read out loud the number expression said: “Four equals one plus three.” Other students arrived at different formulations like “Two worms in one apple, then three with one and one has no worm” or “Four apples with a worm and one has no worm.”

Now, Thomas could no longer hold back: “Here are two worms (pointing) and in the other three is only one, and here (pointing) is one without. A zero has to be put on top of that apple, because there is no worm in it ... zero worm.” Two groups spontaneously emerged in the classroom discussion and a real fight began between the followers of the Zero and the students advocating that a One had to be put on top of the empty apple. The two groups, accordingly, arrived at different formulations of the corresponding numerical expression. The “One”-Group presented “ $4+1=5$ ” as their solution while the “Zero”-Group, counting the worms, presented “ $2+1+1+1+0=5$ ” as their solution. With this, both



Fig. 1 Apples and worms (from Voigt, 1998)

groups were satisfied. At the end, students applauded, clapping their hands—an extraordinary expression of their deep satisfaction—as if they had been witnessing an excellent play or performance.

Again, Thomas had proven to be an attentive “custodian of the zero” which continued to be his preoccupation whenever it turned out to be necessary. On the first day of the following school year, he turned up with a brand new sweatshirt. On the front Alf, the alien creature from the TV series was depicted crying out: “Null Problemo,” possibly the German version of the American original “Zero Cool.”

This example demonstrates several points—not all of them can be elaborated here. First, it is a demonstration of the pervasiveness of signs and their importance in getting to get around with school mathematics. It marks exactly the difference between mere “school-going” and “school-learning,” and its beginning, that children play around with the signs to watch how the meaning changes and how sign, interpretant, and meaning are related. The semiotic perspective, in a way, helps to describe how learning to learn is basically functioning and how it is related to deep play. It appears that serious learning and playfulness are not two mutually exclusive things: In pretending to know and in acting as if one already knows the solution, meaning making is taking place through discussion, argumentation, turn-taking.

It is obvious that this example would not have functioned in a classroom that was not based on reciprocity and mutual support: that the teacher was open and experienced enough to allow all these deep play activities.

The case of Thomas also illustrates how intimately personal all these experiences of meaning making are. Those discussions and argumentations are not something that is only superficially touching the students—it is touching their personal identity which is typical for deep play. Children are born as super social creatures and their sociality is making it possible that they become human beings in the proper sense. They do not stop being extremely sensible and receptive to all social matters—although they might develop defense strategies for not being touched too much.

For this child, Thomas, the zero was something very personal. Peirce once emphasized the fundamental importance of signs by saying: “Man is a sign” (Colapietro, 1989). I could not think of any better way to express how Thomas identified with the zero wearing the sweatshirt with Alf on the front.

This example demonstrates the importance of indirect learning and meaning making. In what follows, I take indirect learning as a starting point for discussing the issue of meaning making, because it is a basis, a focal and starting point for intentional learning and learning from teaching, the embodiment as the groundwork of learning. In indirect learning we find a unique blend of social, cognitive, emotional, and motivational processes. Indirect learning is, however, not always benign and productive, as in the case of Thomas. It can also lead to learning barricades; it can trigger fear of failure or hope of success as the primary forces of learning motivation. For a start, however, I shall discuss the issue of implicit learning and thinking in order to show what this important question has to do with the remainder of the paper. It will also lead directly into the heart of the debate on how learning, culture, mind, and consciousness are related.

2 Implicit understanding, indirect learning, tacit knowing

Putting these diverse forms of understanding and learning into one heading means to point at the common ground of these concepts—and their common ground can be seen as the

culture of learning and the learning of culture. Now, there is an apparent paradox in the fact that culture is meant to be typically implicit or even tacit on the one hand, and culture being shared and public, as Bruner has put it:

By virtue of participation in culture, meaning is rendered *public* and *shared*. Our culturally adapted way of life depends upon shared modes of discourse for negotiating differences in meaning and interpretation. ... The child does not enter his or her group as a private and autistic sport of primary processes, but rather as a participant in a larger public process in which public meanings are negotiated. And in this process, meanings are not to his own advantage unless he can get them shared by others. (Bruner, 1990, p. 12–13)

Meaning making, as it were, is based on a public and shared participation in discourse. Quite in agreement with Bruner's starting point, Bauersfeld begins in the following quote, ending up with an emphasis on the indirect character of learning:

Participating in the processes of a mathematics classroom is participating in a culture of mathematizing. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks of the building, but the design of the house of mathematizing is processed on another level. As it is with culture, the core of what is learned through participation is *when* to do *what* and *how* to do it. ... The core part of school mathematics enculturation comes into effect on the meta-level and is "learned" indirectly. (Bauersfeld, 1988 as quoted in Cobb & Bauersfeld, 1995, p. 9)

It is obvious that the partial contradiction of these two quotes cannot be eliminated in a rush. We have to look for more aspects of what "implicit" or "indirect" could mean. This contradiction is, obviously, not superficial. It expresses, so to speak, two complementary positions of theorizing: the exteriority of mind and interiorization as a developmental process.

It is often claimed that the common ground for these phenomena is the fact that the capacity of the human mind of having simultaneous conscious thoughts and feelings is limited. It is claimed that this limitation is, however, extremely functional for adaptive behavior. This is expressed, e.g., in the work of Arthur Reber, who has coined the expression *implicit learning* (Reber, 1967, 1993).

Implicit learning is the acquisition of knowledge that takes place largely independently of conscious attempts to learn and largely in the absence of explicit knowledge about what was acquired. One of the core assumptions of our work has been that implicit learning is a fundamental, "root" process, one that lies at the very heart of the adaptive behavioral repertoire of every complex organism. (Reber, 1993, p. 5)

Implicit learning has also been defined as "learning complex information without complete verbalizable knowledge of what is learned" (Seeger, 1994, p. 163). In both definitions, the relation to language is central: while Reber's discovery was made during his research on artificial grammars, Seeger's experimental study uses a language-specific definition.

"Implicit" is one of the most often used categories to refer to processes that operate below, beneath, or behind those processes that are accessible and visible. If a person presents a mathematical argumentation while gesturing with hands and body, there is

obviously an implicit level of processing that is connected to gesturing as well as to speaking. We can suitably say that this level of processing is unconscious. However, we will not use “unconscious” in the strict Freudian sense of being the place where the true driving forces of a person’s life activity can be found. For the time being, it can be said that only a small portion of a person’s ongoing activity is conscious—operating on the background of unconscious levels of processing in perceiving, speaking, calculating, hearing, and so on. This means that it is crucial to understand how implicit and explicit knowledge and understanding, how unconscious and conscious activity, in general, are working together as complementary processes. Until recently, discourse on the unconscious had been overshadowed by an understanding heavily influenced by the Freudian original version which meant that the properties of the unconscious as a reservoir, as a container for displaced drive energy were emphasized. Implicitness, as it were, touches upon the unconscious, the body-related signs and techniques, the habitual.

3 Limits of consciousness?

The above definitions of consciousness and verbalization raise a couple of questions: We find here an idea of “conscious” that is strongly connected to speaking: We know only what we can verbalize explicitly. This is, however, an objectionable view, as has been shown by Merlin Donald (2001). It seems important to find a view of consciousness that is neither restricted to processes inside the head nor restricted to language and speaking.²

The usual view of consciousness, hidden behind the notion of implicit learning, tacit knowledge, and, for that matter, inert knowledge (see, e.g., Bereiter & Scardamalia, 1985), is putting an emphasis on the limited capacity of consciousness. Ever since the influential paper of Miller (1956) on our limits of processing information, the limited capacity of the human consciousness is drawn upon as an explanation in educational contexts and discussions.³

Of course, the “narrow window” (Donald, 2001) of what can simultaneously be present in consciousness is a fact. But the preoccupation with demonstrating this again and again is remarkable. A possible explanation could be that the conception of humankind is heavily influenced by the idea that it was consciousness that provided the decisive selective advantage in human evolution and that the obvious narrowness of consciousness contradicts this view. Just because it is felt that humans are the product of their own history, a self-made existence created through the “openness” of human consciousness due to their lack of specialization that has thrown them into what Gehlen (1988) had called a “deficient being,” it seems so hard to believe that consciousness is so narrow. It has led to a debate in the philosophy of mind whether consciousness is a genuine philosophical concept at all (see Donald, 2001).

Critics, like Geertz (1973), Donald (2001), and Gebauer & Wulf (2009), have pointed out that the problem of the narrow consciousness is a result of thinking about the evolution of human consciousness in terms of biology—and then adding on the cultural context. And this is the fact for phylogeny as well as for ontogeny. This also means a neglect of what

² The discussion on “bounded rationality,” initiated by H. A. Simon (1955, 1956), will not be touched upon here; for a more recent state of discussion, see Gigerenzer and Selten (2001).

³ We find the *narrowness of consciousness* already (1892) mentioned by James (2001, p. 84 f.), and much earlier (1689) Locke (1997, p. 147) mentioned the *narrow mind* of humans.

Bruner (1990, p.19) had called attention to: the *ordinary conduct of life*. Donald (2001, p.18) has expressed the same concern with strong words:

Our examination of the microdetails of behavior over the last three decades has yielded at least one very consistent impression: Under these conditions, human consciousness measures out as a fleeting and narrow window. When people are removed from their familiar settings—work, school, home, city, street—and subjected to bare-knuckled scrutiny, their conscious capacity reveals its severe limitations. This includes everyone. There are no geniuses in the psychological laboratory. In real life you may be the local Einstein or the village fool, but it does not matter.

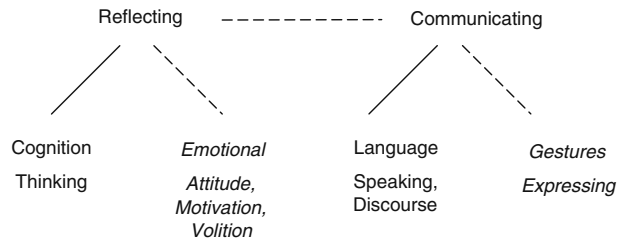
In simple words, this means that we direct our efforts to understand conscious, self-directed learning and shared understanding in the classroom at the practice of the classroom itself—and not at the attempt to localize arithmetic and/or algebraic functioning in the human brain through neuroimaging technology, even though the results might be quite entertaining.

Taking Stanislas Dehaene's *The Number Sense* (1997) as such an entertaining example, I cannot say that, after having studied it, I really have an understanding of *How the Mind Creates Mathematics*—what the subtitle promises. The attempt to demonstrate where mathematics processes are localized within the head obviously does not contribute to a real understanding. A remarkable step in the theoretical discussion on the development of the number concept has been the *triple code model* Dehaene had presented in 1992. This model, and with it the theoretical standard, does no longer appear in 1997, possibly because the fact that the three modules of the model are located within the brain is theoretically not so productive. The problem is that the development of the number concept, as given in the characterization of the three modules, and the results of imaging methods of localization have nothing to do with each other.

If we are talking about *tacit knowledge* or tacit knowing, as I would prefer, we meet another aspect of implicitness or indirectness. The term is going back to Polanyi's well-known book (1967) and is meant to denote some form of "pre-logical" knowing. "Pre-logical" suggests that this knowledge is not language-bound, non-conscious—expressing Polanyi's idea that *we can know more than we can tell*. Later this concept became a prominent feature in research on expert knowledge used to express the obvious fluency of expert action (see, e.g., Dreyfus & Dreyfus, 1986). This knowledge and understanding is necessarily tacit, non-accessible to conscious recall and non-representable in the narrow window of consciousness. The obvious problem is that we cannot conclude that "tacit" is a paradigmatic quality of thinking, feeling, and remembering in learning situations and develop a philosophy of the human condition based on the fact that intuitive acting and decision is based on non-accessibility. Let me illustrate this point with an example from the pioneer studies in expert problem solving. In his studies on the psychology of expert chess masters, Adriaan de Groot (1978) presented chess positions briefly (2–15 s) to chess masters and novices. Even after such brief exposures, the chess masters answered questions about the positions of the pieces to about 93% correctness. More generally, performance in this task systematically varied as a function of skill. Parallel studies revealed that chess masters and novices were not very different in the number of moves they were thinking ahead. De Groot concluded that perception and memory were more important differentiators of chess expertise than was the ability to think ahead in the search for good moves. These results were corroborated in later studies (see, e.g., Chase & Simon, 1973; Charness, Reingold, Pomplun, & Stampe, 2001).

From here on, several other routes could be taken to expand on the issue and role of acting consciously—one could lead to the enormously important role of attention and how attention could be educated in mathematics education, another one could lead to a

Fig. 2 Several aspects of the relation of communication and reflection treated in Hiebert et al. (1997); concepts in *italics* are not discussed



discussion on the role of habit formation and the concept of habitus as proposed by Bourdieu. For reasons of space, I will not be able to touch upon these issues. Rather, I will present an example from mathematics education research on the relation of meaning making and understanding (see also Fig. 2).

4 Meaning making and understanding

Teaching students to make sense of mathematics and helping them making sense of mathematics by themselves seem of overriding importance to mathematics education in theory, research, and practice. This is, of course, not a new and surprising statement. In 1997, a couple of well-known researchers in mathematics education came out with *Making Sense: Teaching and Learning Mathematics* (Hiebert et al., 1997) that was felt to be a consensus on the essential features of classrooms for understanding mathematics. The book is unique in the sense that it does not deliver panaceas for promoting the understanding of mathematics, but tries to explicate how understanding can be taught (and learned) under different teaching paradigms and different classroom contexts.

I shall not go into too much detail about this work which is to be seen in the context of the *Standards* of the NCTM being under development at that time. I will rather focus on the understanding of sense making and understanding put forward in this work. Discussing this understanding will help me to contrast my own approach in the present chapter, and it will help to grasp the differences between now and then and, of course, see that the problems are still very much the same.

The approach of Hiebert et al. (1997; see also Hiebert & Carpenter, 1992) to understanding can be called relational as it places a heavy emphasis not on the access to content but on relating different contents: "... we understand something if we see how it is related or connected to other things we know" (Hiebert et al., 1997, 4).

This definition yields at least two questions: How do we relate things? How do we relate to things we know and how do we relate old and new things?

For Hiebert et al. (1997) the answer to these questions can be given quite straightforwardly: through reflecting and communicating we can find relations, we can relate things and we can relate the known and the unknown. This means that we can, in fact, relate to the unknown by relating (it) to the known things. They illustrate their point through an example:

... a student *understands how* to add 35 and 47 if she can relate this problem to other things *she knows about* addition and about the *meaning* of the numerals 35 and 47. *Knowing that* 35 is 3 tens and 5 ones and that 47 is 4 tens and 7 ones helps her understand several ways of combining the numbers. In both cases, evidence for understanding is often provided in the form of explanations for why things are like

they are, ... why 35 and 47 is 82. Explanations are usually filled with connections, either *implicit or explicit*, between the target situation and other things that the person knows. (Hiebert et al., 1997, p. 4–5; my italics)

Scrutinizing the issues addressed in the italicized words one finds a bunch of problems poured together into one single explanation: (1) the necessity is neglected of differentiating between “understanding how,” “understand that,” and “understanding why” that is familiar since the discussion of Ryle (1949) and has been taken up in a broad range of educational discussion (see, e.g., Skemp, 1976, Schön, 1983, 1987; Seeger, 2001); (2) the benefit of a relational perspective on understanding and knowing is given up as it is seen as a genuine alternative to “substantive” understanding in the sense of Cassirer (1953); (3) the most interesting difference between making implicit vs. explicit connections is not discussed and is being treated as basically the same thing.

In the light of the previous debate, however, it must be said that the answer Hiebert et al. (1997) were giving is far from satisfactory. Of course, any attempt to understand will need reflecting and communicating, how else could it be attained? Understanding will not be handed down from above. At least, not in learning situations!

Doubtlessly, the assertion that “we” learn to understand through reflecting and communicating has an intuitive persuasiveness. But it is also alarmingly incomplete and is reiterating the old dualisms of body and mind, inside and outside. It is not enough, and far from being a problem solution, to separate reflecting and communicating.

Figure 2 puts together in a diagram how the concepts are discussed and which relations are not discussed: while reflecting is seen primarily as a cognitive activity and communicating is seen primarily as a language-bound discourse, the emotional aspects of reflecting and the gestural aspects of communicating are completely neglected—and there is also no idea on how reflecting and communication stick together, influence each other, and develop together. The dashed lines point at relations neglected in Hiebert et al. (1997), while the bold lines mark relations leading to concepts usually attributed to reflection and communication.

The key for our better understanding of sense making lies in the appreciation of the fundamental roles of the cognitive and the emotional, mind and body, consciousness and unconsciousness, shared, reciprocal and self-controlled activity.

When we try to be somehow more systematic on a relational perspective on understanding, the first thing that comes to mind is that “relational” must be extended to the social situation, because we can only hope to understand understanding better if we extend the perspective beyond cognition and language to the social–emotional and to bodily gestures and expressions. It seems heuristically fruitful to introduce three planes or three domains of relations and relationships:

- the domain of *practice*
- the domain of *development*
- and the domain of *shared intentionality and attachment*

These domains are analytically separated, although they coexist (see also Fig. 2).

5 Meaning-making as relational

A semiotic perspective on meaning making can focus on the sign–sign relations and on sign–people and even on people–people relations. Typically, a study on sign–sign relations

has been done as a reconstruction of the system of signs creating mathematical themes in a rationalist spirit, as an epistemological or historical issue (see, e.g., Rotman, 2000; Duval, 2008; Otte, 2008), while a study on sign–people relations has been done as a social–emotional reconstruction of what makes mathematics hard or easy for people—inside and outside the classroom (see, e.g., Arzarello & Sabena, this issue; Radford & Roth, this issue). Both perspectives can be understood as developmental processes, e.g., as a certain mathematical notion develops in the course of history, and—as an example for the social–emotional perspective—how learning mathematics develops over time in ontogenesis.

On the other hand, the use of discourse is the necessary condition for understanding and manufacturing meaning—that we do not understand *through* discourse what is in the real world, but that we only understand *in* discourse. This creates the difficulty of determining if we understand the discourse or understand what the discourse is about. In a sense, this is the old problem Gadamer expressed as: “Being that can be understood is language” (Bubner et al., 2001; see especially Rorty, 2001).

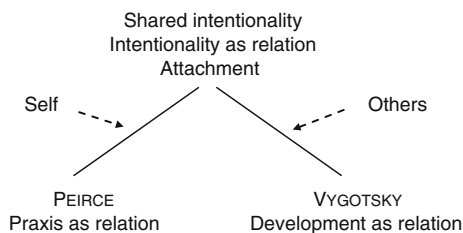
In the ensuing two sections, I shall present two aspects or dimensions of this developmental process—following the general line of argumentation that the cognitive and the emotional should not be opposed. One line of argumentation is going back to Peirce’s pragmatism understanding practice as relation, one is referring to Vygotsky’s theoretical program. The two lines are bracketed by the ideas of shared intentionality, attachment, and empathy. For reasons of space, I will not be able to equally expand on all three themes, but I will give at least a figure (see Figs. 2 and 3) to illustrate how the themes are juxtaposed.

We have discussed above Polanyi’s idea that *we can know more than we can tell* and mentioned also how this implicitness has to do with habit formation and the formation of the habitus. Now, it is important to contrast this with the idea that *we are telling more than we can know*. This is going back to Nisbett & Wilson (1977) where they found in a study on self-report that subjects were telling more than they could possibly know. In related studies, Langer even found that people were acting *mindless* (Langer, 1978)⁴—meaning that people were giving justifications for their actions that were clearly at variance from what they had actually done.

In a way, both formulations express something like a “surplus meaning”—on the one hand a surplus meaning in the implicit database, on the other hand in the verbal exchange between speaker and listener.

One could also say that people are acting *as if* they know what they are talking about and *as if* they talk about what they are knowing. To pretend to know something or to know how to do something can be seen as something close to a lie or as basically false. To pretend is, however, a basic mechanism in ontogenetic development assisting the growth of fantasy as well as the growth of a realistic approach. Imitation, pretense play, and acting-as-if are such

Fig. 3 Three themes of argumentation as they are connected to each other



⁴ Interestingly, Langer later has become popular on a large scale for advocating *mindfulness* (Langer, 1989, 1997).

core features of childhood that an adult's *acting as if* is denounced as "childish" or even malicious deceit. In contrast to such philosophical approaches as the one by Vaihinger (2009/1911), I do not see this primarily as a faulty procedure demanding remedial treatment and logical propaedeutics, it is rather a characteristic of ongoing situated thinking and acting, of feedback and feedforward, depending heavily on a semiotic environment and a social situation.

Acting as if is a fundamental principle whenever it has to be explained how something *new* can be learned with an old mind and old habits:

- the educational slogan of "performance before competence"(see, e.g., Cazden, 1981) expresses very well that one has to pretend to know what the new is and how to do it, before the new is at one's disposal;
- in abductive reasoning (see, e.g., Hoffmann, 1999; Shank, 1998) and problem-solving, the discovery of the new is often attained if we pretend to know what the next step is;
- acting as if often seems to give birth to authenticity, as discussed in the famous essay of the paradox of acting (Diderot, 2007; Vygotsky, 1999b).

I shall try to show that this surplus meaning is a feature of sign systems and semiotic mediation. I shall, however, not so much concentrate here on the epistemological foundation of this perspective, e.g., in terms of a Peircean or Vygotskian terminology. I shall rather try to show two exemplary and complementary fields of application where the idea of a surplus meaning is an important and underlying mechanism for learning. First, I shall try to show how this principle works in the field of practice and discuss Peirce's *pragmatic maxim*, and then, I shall discuss Vygotsky's *zone of proximal development*. As it were, the ideas of practice as relational and development as relation will be elucidated.

5.1 Surplus meaning in the pragmatic dimension

Peirce has made diverse and repeated attempts to define the meaning of a concept and the meaning of a thought. Two strands of his thinking will be taken to show his exemplary importance: the consequences of his pragmatic maxim and the three types of sign-mediated meaning known as Firstness, Secondness, and Thirdness.

There have been different versions of the pragmatic maxim in Peirce's writings. The first version appeared in 1878 and reads as follows:

Consider what effects, which might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object. (Peirce, 1878, p. 293)

Peirce notes himself (1906) that the line of words "conceivably—conceive—conception (three times)" seems hard to get used to for the reader. In several attempts, Peirce gave extensions and clarifications of the original maxim from 1878, but these were meant as interpretations not as changes of the original version. The simplest version is this: "we must look to the upshot of our concepts in order rightly to apprehend them" (CP 5.4).

Peirce was, in all his versions, putting the emphasis on the role of the practical consequences as in the following quote from 1905:

Such reasonings and all reasonings turn upon the idea that if one exerts certain kinds of volition, one will undergo in return certain compulsory perceptions. Now this sort of consideration, namely, that certain lines of conduct will entail certain kinds of inevitable experiences is what is called a "practical consideration". Hence is justified

the maxim, belief in which constitutes pragmatism; namely:

In order to ascertain the meaning of an intellectual conception one should consider what practical consequences might conceivably result by necessity from the truth of that conception; and the sum of these consequences will constitute the entire meaning of the conception. (Peirce, 1905)

In a version from 1908, Peirce puts an emphasis on the habitual foundation of meaning:

The only way to complete our knowledge of its nature is to discover and recognize just what general habits of conduct a belief in the truth of the concept (of any conceivable subject, and under any conceivable circumstances) would reasonably develop; that is to say, what habits would ultimately result from a sufficient consideration of such truth. It is necessary to understand the word “conduct,” here, in the broadest sense (CP 6.481–482).

The search for meaning should not be directed inside the skull or on the power of thoughts, Peirce recommends, meaning is at our fingertips—the practical consequences provide the key to understand understanding. Meaning making is a habit—and with that all dichotomies become redundant that entertain a contradiction between understanding and the formation of habits.

It is, however, obvious that the “sum of the practical consequences” indicates a limit, something that can only be approached by approximation. But the relation to practice is providing an interpretation that allows pretending the final sum total has been reached.

In an interesting way, Peirce’s idea of Firstness, Secondness, and Thirdness cuts across the usual dichotomies and categories. In Table 1, the three categories are presented as they reflect different dimensions.

The three categories seem to form some sort of developmental or evolutionary sequence: from simple, primitive sign-relations to more complicated ones. If we try to apply this sequence to ontogenetic development, it becomes, however, apparent that from the earliest phase of human development all three categories apply. This means, for instance, that Thirdness is already present and functional in early infancy: basically, this is what has been shown in recent research on shared intentionality and similar fields. We shall come back to this in the next section.

If we recall the example of Thomas and his relation to the Zero, we can see that three levels are present and have to be considered to understand his behavior: it is the attractiveness of the Zero as an *icon*, as an *index* the Zero creates a tension between Thomas’ intended action and the cardinality of Zero, and as *symbol* the Zero becomes, finally, a sign relating the object, the interpretation, and the social relations between subjects, the teacher and the other students.

We can say now that the implicit nature of learning as described in the paradox of *knowing too much and knowing too little* is a result of the semiotically mediated nature of

Table 1 Firstness, secondness, and thirdness according to Peirce (after Trevarthen, 1994)

Firstness			
Sign as such	Quality	Icon	Emotion <i>in</i> subject
Secondness			
Sign and relation to Object	Actuality	Index	Object <i>of</i> subject in intended action
Thirdness			
Sign and relation to Interpretant	Potentiality	Symbol	Cooperation, self, and value <i>between</i> subjects

human thinking and feeling. In order to secure the quality, the actuality, and the potentiality of action humans have to move constantly between the poles of knowing more than they can tell, as they reach beyond the situation given in some form of abduction, and telling more than they could know, as they try to grasp the given situation and express it in discourse. *Acting as if*, deep play (Bruner, 1972; Geertz, 1972) is the integrated employment of all three levels of semiotic activity.

If we look only at knowing and talking and cut the link of knowing and talking to ongoing action, we arrive at the paradox. If we keep this link into consideration, the need will arise to conceptualize thinking, feeling and talking also as actions and as actions reaching out into their potentiality in pretense and deep play.

5.2 The zone of proximal development in ontogenesis

The development of meaning making under the perspective of reciprocity has to do with the complicated interplay between the social and the individual which has been the dominant theme in Vygotsky's developmental psychology. There will be no presentation of Vygotsky's ideas, also because this has been done extensively over the past years (see, e.g., van der Veer & Valsiner, 1991, 1994; Daniels, Cole, & Wertsch, 2007).

Control and self-control have been an important motive for Peirce in formulating the pragmatic maxim. Equally, Vygotsky has taken great efforts in giving a vivid account of the transition from other-regulated to self-regulated control as it can be demonstrated in the development of volition and sign operations (see, e.g., Vygotsky, 1997, 1999a). To gain self-control is one of the great accomplishments in human ontogeny. It is especially remarkable on the background of the fact that no infant and child can develop normally if attachment and secure base are not provided by the mother and other caretakers. It is as if we have again two paradoxically opposing poles where development has to find its way—the self being neither completely attached nor completely self-directed. It is quite clear that real self-determination and autonomy is not a steady state but a process of becoming, a developmental process.

When we look at what research has found out about this development of self-control we find again that already at a very early age the infant is not a passive vessel controlled by the mother. Rather, the infant starts meaning making from birth on—and even earlier. What could be mistaken for a genetic predisposition turns out to be, at closer scrutiny, a result of interaction and preverbal communication. Beginning in the late 1960s, gradually intensified research has accumulated evidence of the fundamental nature of reciprocal interaction in early interaction (see, e.g., Bullowa, 1979; Field & Fox, 1985). The work of Andrew Meltzoff (Meltzoff & Moore, 1977; Meltzoff, Kuhl, Movellan, & Sejnowski, 2009); Hanus Papousek (Papousek & Papousek, 1974, 1977, 1981), Colwyn Trevarthen (Trevarthen, 1977, 1979, 1980, 1994; Trevarthen & Hubley, 1978), Daniel Stern (1971, 1985), to name only a few, has paved the way to a new understanding of the “competent infant”—a term coined already in a 1973 book by Stone and others (Stone, Smith, & Murphy, 1973).

One interesting result of research is that infants already know how to communicate long before they know how to speak. This means that the roots and the ground of communicating are not of a verbal nature—neither developmentally nor functionally. One could say that the common ground of communicating verbally and nonverbally is the semiotic nature of communication already present in an early age. This basis is sign operations and actions with signs, and this includes verbal and nonverbal activity. We can use *turn taking* as an example for this developmental context. It is a crucial feature of verbal exchange, conversation, and discourse that speaker and listener are taking turns—either in

listening or in speaking. As research has shown (see, e.g., Bateson, 1975; Beebe, Stern, & Jaffe, 1979; Beebe, Jaffe, Feldstein, Mays, & Alson, 1985) taking turns is already a feature of nonverbal, expressive and gestural interaction between mother and child long before the infant starts to learn to talk. This means that reciprocity is working as a fundamental principle of being in the world from birth on.

6 Summing up

The ideas presented above should not be taken as a comprehensive or new theoretical approach to the problems of teaching and learning mathematics. It has become apparent that a practical relevance and applicability might be better served if we are finding critical and sensitive points in the practice of the mathematics classroom. New embracing and comprehensive approaches often do not seem to deliver what they promise.

However, there has to be a minimal complexity and comprehensiveness in research approaches to meaning making if the essence of the “language game” is to be captured—which means nothing less than to capture how meaning is made in the practice of teaching and learning. This is one thing my reasoning was meant to demonstrate. In Fig. 4, I have given a tentative sketch of the three threads I have been following: (1) the thread of perceiving—interpreting—meaning-making—acting; (2) the thread of imitating—acting; and (3) the thread of the embedding interpersonal practice. This figure might not be completely plausible—but it has to be kept in mind that its primary purpose is to illustrate the appalling complexity of the situation of meaning making.

Another principal point of my reasoning has been that meaning making in the forms of empathy, shared intentionality, reciprocity, and cooperation is part and parcel of human

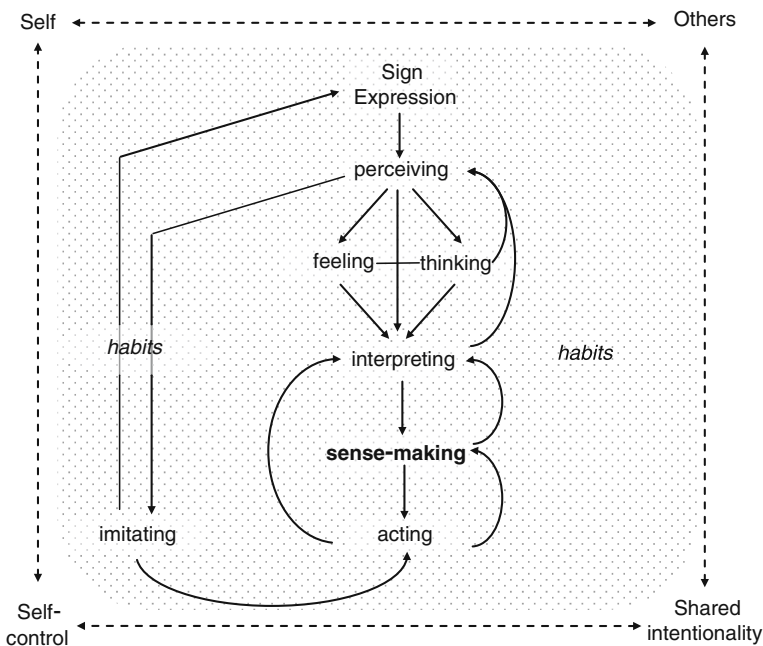


Fig. 4 Meaning-making as network and context

development from birth on. This means that as mathematics educators we have to acknowledge that everything that is done in the classroom is producing meaning—even when we wish to refrain from it. We also have to acknowledge that everything that is done in the classroom arouses feelings and emotions—and that these feelings and emotions are integral part of the process of meaning- and sense-making. They are not some kind of accessory—sometimes blocking the way of exact logical reasoning. They are essential and necessary for self-controlled and independent reasoning—and living.

Emotional attachment is crucially important for a development of an explorative habit—because emotional attachment reduces the fear that is a necessary companion of entering the new territory. This point can be illustrated by referring to attachment theory initiated by John Bowlby (1969, 1973, 1980). He has laid the ground for a very fruitful exploration of the reasons why infant development sometimes fails. It seems that being closely and securely attached to a family is the decisive factor for growing up good. More recent studies are increasingly asking how children and adults perceive this attachment relation. The “secure base” that children can attach to and the attachment relations themselves have to be represented by a corresponding “internal working model” (see, e. g., Bretherton & Mulholland, 1999). It is obvious that neither is independent from the other: Good attachment relations co-occur with a good internal working model and vice versa.

In his work, Michael Otte has often pointed out that the mathematics teacher has to be, as Gramsci coined it, an *exemplary intellectual* (see, e. g., Otte, 1994). This imperative is another expression of the necessity for a “secure base”: for the relations of students to their teachers as well. I do not want to say that there has to be an attachment to teachers like the attachment to parents and family—although in the first grades, attachment-like relations to the teacher tend to be the rule rather than the exception. The relation of the teacher to the students should be one in which the teacher takes over responsibility for providing orientation in the endless weaving by importing authentic “collateral experience” as Peirce would say. It seems a far cry from a “secure base” in learning, if students are taught to construct their own meaning all the time. As the case of Thomas, the custodian of the Zero has shown, students are constructing their own meaning all the time. They need a “secure base” and a deep play context to bring their thoughts and feelings to the fore.

Our contribution to care and precaution for the new generation would be to provide a “secure base” through teaching not only the mind of children and youth, but to teach them as complete human beings, as being empathic, social, and emotional.

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