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Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects

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Abstract The semiotic approach to mathematics education introduces the notion of "semiotic system" as a tool to describe mathematical activity. The semiotic system is formed by the set of signs, the production rules of signs and the underlying meaning structures. In this paper, we present the notions of system of practices and configuration of objects and processes that complement the notion of semiotic system and help to understand the complex nature of mathematical objects. We also show in what sense these notions facilitate the description and comprehension of building and communicating mathematical knowledge, by applying them to analyze semiotic systems involved in the teaching and learning of some elementary arithmetic concepts.

Keywords Onto-semiotic approach · Object · Meaning · Mathematics · Natural number · Learning decimal numeration · Semiotic system

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1 Natural numbers as equivalence classes of sets

The understanding of the nature of mathematical concepts is a complex question as is revealed in the following class episode. This extract of the interaction between a lecturer and a group of future primary education teachers shows the predominance in the lecturer of a formalist understanding of natural numbers which contrasts with the informal use of these numbers. As we shall see in the following section, from an educative point of view, it is necessary to assume a wider perspective of the nature of the numbers to that shown by this lecturer with this group of students.

Does anyone know what a number is?[1]

The lecturer begins the class on "natural numbers" by saying:
First we will work on the concept of number, the idea and then we will think about the language in which we are going to write it. What are numbers?; for example, What is number five? We are posed with a problem, we have been using numbers from a very early age. However, when we are asked what a number is, we have difficulty in answering.
He asks the students:
Does anyone know what a number is?
One student replies,
A sign that refers to a quantity.
The lecturer asks again:
What is number four?
The students do not reply.
The lecturer writes the symbol 4 on the board and says:
This is no more than a sign. What would the idea behind this be, how could we define it?
The lecturer answers the question himself:
If I want to communicate what number four means we put examples of groups of four, like for example: four pieces of chalk, four fingers, four people, four chairs, etc. What these sets have in common is what we call the idea of being four.
How do we work in Preschool and Primary Education? The numbers are first shown as tools; however as future teachers we are going to take them as an object of study.

The lecturer continues the class by explaining the Logicist construction of the natural numbers as the elements of the quotient set determined on the set of finite sets by the relationship of equivalence or coordinability between sets.

Two aspects are reported from this episode:

- The lecturer's teaching strategy: The lecturer's questions are rhetorical since he is assuming the discourse weight. The initial reply given by the student ("a sign that refers to a quantity") is neither considered nor discussed nor valued. Therefore, the teaching strategy does not take into account the student's role in the teaching/learning process.
- The meaning of number highlighted by the lecturer. The reply pointed out by the student encloses the meaning of the number as cardinal of a set, like the reply to the generic question, how many objects are there here? This meaning differs from the

[1] Didactic incident taken from Arrieche's doctoral thesis (2002).

purpose of teaching, namely, to back up the concept of number on the possibility to establish a one-to-one correspondence between sets.

The lecturer's strategy is crucial from a semiotic perspective in that it avoids a true social interaction in the interpretation of the signs and the construction of the meanings. "From a semiotic perspective neither the cognitive activity of the individual nor his social interaction is primary; both co-exist and co-act in a synergistic manner to support the evolving process of sign interpretation and meaning-making. Thought and communication (taken in its broadest sense as social interaction) both appear to be parallel and interrelated at the same time" (Sáenz-Ludlow, 2006, p. 185).

On the other hand, how do we assess the lecturer's decision to favor a particular meaning of natural numbers? Is it appropriate from an epistemological point of view? Is it admissible taking into the account the conditions and restrictions of the institution? Does it have any implication for the operative (problem solving) and discursive (justification and communication of the activity) mathematical practices?

In the following section, we present a perspective about the natural numbers by implicitly using the notion of system of practice and the institutional and contextual relativity of these practices in addition to the objects that intervene in the same (Godino & Batanero, 1998; Godino, Batanero, & Font, 2007). This approximation clearly shows the plurality of the meanings of natural numbers; a plurality that teachers should consider in order to avoid a formalist emphasis in mathematics teaching.

2 Plurality of number meanings

The nature of the whole numbers, and in particular, their relation with set theory, is a question that is just as interesting to mathematics as to philosophy of mathematics. But numbers are also essential tools in our daily and professional life, and so this is the reason why they constitute a subject of essential study in school from the first levels.

We consider it necessary to distinguish between two uses of numbers:

– The practical and "informal" use: "How many items are there?", "What place does an object occupy in an ordered sequence of items?", and so on.
– The theoretical and "formal" use: "What are the numbers?", "How are the number systems constructed?", and so on.

Within these two broad contexts of use (or institutional frameworks), it is possible to distinguish diverse historical moments at which the questions are tackled with diverse resources and from different approaches, putting into effect specific operative and discursive practices.

From a retrospective viewpoint, we can identify certain constant features that allow us to speak of the "natural number", in singular, but from a local point of view it seems necessary to distinguish between the diverse natural numbers that the primitive peoples and old cultures (Egyptian, Roman, Chinese,…)[2] "handled," and also to distinguish between the numerical practices that are carried out at the moment in primary school, and those that the logicist mathematicians of nineteenth century engaged in or the Hilbertians axiomatic formulations.

[2] Rotman (1988) reached a similar conclusion in his semiotic analysis of mathematical activity, when he considered the numbers studied by the Babylonians, Greeks, Romans, and present-day mathematicians are different.

Therefore, the understanding of the nature and meaning of numbers requires adopting an anthropological–sociocultural vision of mathematics, like the proposal, among other approaches (Chevallard, 1992; Radford, 2006), from the "onto-semiotic approach to mathematical knowledge and instruction" (Godino & Batanero, 1998; Godino et al., 2007) that we shall describe in section 3.

2.1 Some features of the informal uses of natural numbers

In order to communicate to other people, and as a means to register for ourselves at other moments, the size or amount of elements of a set, it can be done using different resources and procedures:

1) In our present western culture, the use of the "numerical words" is generalized, one, two, three…, and the Arabic numerals, 1, 2, 3,… These limitless collections of words and symbols are those that our students use when we ask, for example, how many students are there in class?, and they respond, "there are 91 students," or, they write, "91." To do this, they have had only to apply a rigorous procedure of counting, putting in bijective mapping of each student of the class with a unique numerical word recited in the established order, and respecting the principles of counting.
2) If we ask students to communicate the counting result without using the "number or symbol words," they might invent other means to express the number of students in the class (or the cardinal of the set formed by all the people in the class). For example:

   – The collection of marks ///..., or other symbols, on the sheet of paper, as many elements as the set has.
   – A combination of symbols for different partial groupings (*, to indicate ten students, / to express the unity)

As we have freedom to invent symbols and objects as a means to express the cardinality of sets, that is to say, to respond to the question, how many are there?, the collection of possible numeral systems is unlimited. In principle, any limitless collection of objects, whatever its nature may be, could be used as a numeral system: diverse cultures have used sets of little stones, or parts of the human body, etc., as numeral systems to solve this problem.

We see, therefore, that the informal semiotic systems in which the natural numbers are used are characterized by a specific and empirical problematic (to describe the cardinality of collections of things), as well as by using particular linguistic resources, procedures, properties, concepts, and justifications to solve these empirical problems.

2.2 Some features of the formal uses of natural numbers

The mathematical entities that intervene in problems of counting and arithmetic calculation are analyzed formally within the internal framework of mathematics, that is, from a structural point of view. Therefore, numbers are not considered as a means to inform the magnitude amounts (numbers of people, or things, the role that it fulfils in a situation, etc.) and are interpreted, either like elements of one structure characterized according to the set theory or according to Peano's axioms.

In this context of mathematical formalization, other questions are posed:

– How should we define numbers?
– How should we define the arithmetic operations starting from Peano's axioms?

- How should we define arithmetic operations when natural numbers are conceived as cardinal of finite sets?
- What type of algebraic structure does the set **N** of natural numbers have with the addition operation?

The answer to these questions requires the elaboration of specific linguistic resources:

- *Operative techniques*: recursion and set operations
- *Concepts*: set definitions of addition and subtraction, recursive definitions and algebraic definition of subtraction
- *Properties*: semi-group structure with null element for the addition and multiplication
- *Argumentations*: deductive.

Really, it is a system of operative and discursive practices with specific features, adapted to the generality and rigor of mathematical work.

In spite of the differences between the informal–empirical and formal meanings of numbers, a fruitful synergy relationship between the same always existed: "Practical requirements have driven notational innovations such as the refinement of place value systems and the introduction of negative number notation. Conceptual developments have underpinned these developments, ensuring that the rules of procedure reflect the underlying meaning structures, as well as developing knowledge of other properties" (Ernest, 2006, p. 80).

2.3 Plurality of numbers and meanings

Figure 1 represents the plurality (without looking for thoroughness) of informal and formal meanings of natural numbers. Counting situations have been solved by diverse cultures using different practices and tools, giving rise to "different numbers." These diverse numerical configurations are articulated in new formal contexts of use giving rise to different numerical constructions.[3]

Here, a configuration is the set of objects that intervene in a mathematics practice, in the action (operative practice) as well as in argumentation and communication (discursive practice) relative to a specific context of use (formal or informal). In section 3, we shall describe what the meaning of "practice" and "configuration" in the "onto-semiotic approach," is: this is the theoretical framework that has served us as a guide to carry out the analysis of the nature of the whole numbers and show the partiality of the whole numbers construction made by the lecturer (section 1). These notions will also permit us to describe the behavior shown by a pupil solving a task relative to the natural numbers (section 4). However, before this, it is necessary to state some aspects about the formal–informal distinction of the practices.

It is important to point out that the informal practices do not merely have an "historical" existence. They coexist in time with the scientific formalization in the usual practices at schools and determine the personal progress of meaning. They are not the "lesser of two evils," but a landmark necessary in the mental development of children and consubstantial to the processes of didactic transposition.

Numbers are the social answers to the problem of communicating the size or cardinality of sets, to ordering a collection of objects, and to analyzing iterative–recurrent processes. But each primitive culture, each form of life gave its own answer to this problem. In

[3] In this case, the "set theory" context refers to the constructions of N based on set coordinability, whereas "axiomatic" refers to Peano's axiomatics (or other equivalent ones).

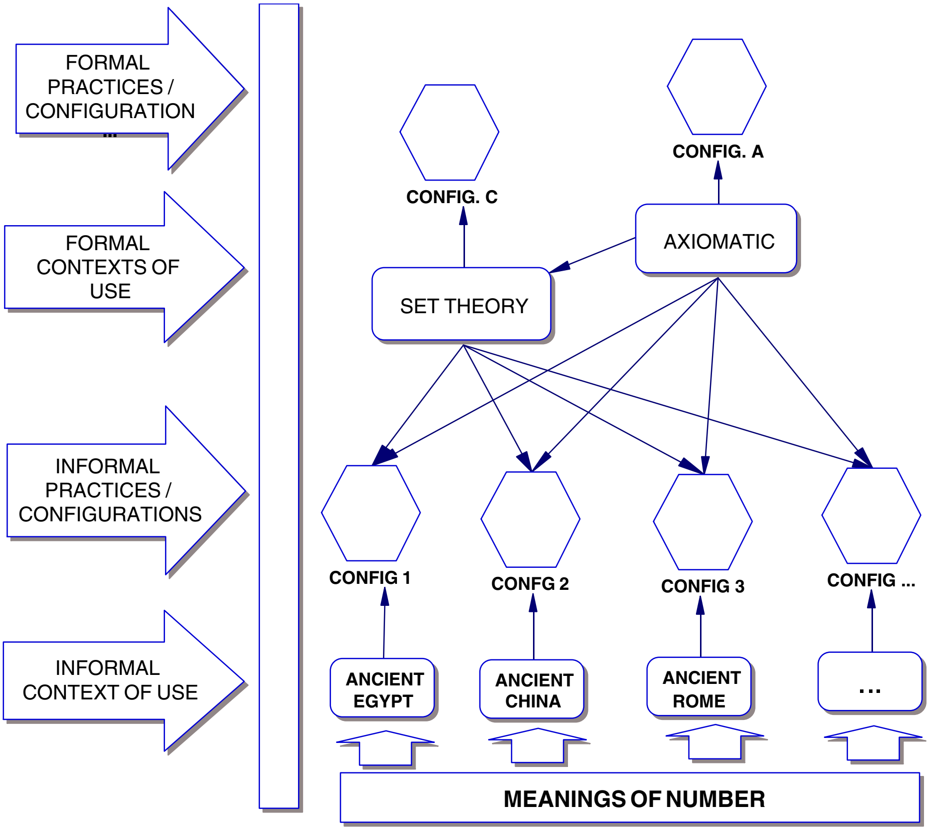


Fig. 1 Plurality of number meanings

principle, each society, culture, and historical stage, has its own numbers, and their own arithmetics, which are distinguishable according to the configuration of objects and processes that characterize them. In each configuration, there are recursively organized objects, with a first element, and a unique following for each element. These organizations permit the solving of the generic problems of quantification, ordering, iteration, and codification.

3 Some onto-semiotic tools for understanding the nature of mathematical objects

The vision that we have presented of the nature of numbers enables us to foresee and understand potential conflicts that the future teachers who receive a partial and excessively formal vision of the numbers, will have, just as we have seen in the sequence presented by the lecturer (section 1). This way of seeing the numbers does not take into account the different contexts of use of numbers and their relation with the linguistic elements and artifacts that intervene in the school practice with numbers.

The presentation of the number in section 2 is based on the onto-semiotic approach to the knowledge and mathematics instruction (OSA; Godino et al., 2007; Font, Godino, & Contreras, 2008; Font & Contreras, 2008). This approach sustains a pragmatist-anthropological conception of mathematics concepts, not limited to the merely formal development of the same.

In this section, we shall show the notions of systems of practices and configuration of objects and processes, included in the OSA, which make up the tools that help to understand the complex nature of mathematics concepts and to explain the learning difficulties of the same. Likewise, we shall see that the interpretation of the meaning in terms of the content of semiotic functions, as well as the types of objects that can intervene in these functions as antecedent or consequent, permit a reformulation of the notion of semiotic system, which emphasizes the role of the semiotic perspectives in mathematics education.

3.1 Systems of practices and pragmatic meanings

All kinds of performances or expressions (verbal, graphic, gestural, etc.), carried out by someone in order to solve mathematics problems, communicate the solution obtained to others, validate it, or generalize it to other contexts and problems, are considered to be mathematical practice (Godino & Batanero, 1998). These practices might be idiosyncratic of a person or be shared within an institution. In the study of mathematics, rather than a particular practice to solve a specific problem, it is interesting to consider the systems of practices[4] (operative and discursive) carried out by people when faced with problematic types of situations.

The meaning of a mathematical object is conceived in terms of the system of practices in which that object intervenes, playing a relevant role. This implies assuming a pragmatist postulate about the meaning. Since the systems of practices are relative to the contexts of uses and institutional frameworks, the need to recognize the plurality of meanings for concepts, is derived, as has been illustrated for the numbers (Fig. 1).

The systems of practices (and hence, also the meanings) have been categorized in the OSA, taking into account diverse points of view. First is the distinction between the personal, or idiosyncratic character of practices (personal practices), and the institutional one (social or shared practices). Learning processes involve the progressive fitting of personal and institutional meanings, as well as the student's appropriation of these institutional meanings; teaching requires the student's participation in the communities of practices that hold institutional meanings.

3.2 Configuration of objects and processes

The notion of "system of practices" is useful for certain macro-didactical analysis, particularly, when we try to compare the particular form adopted by mathematical knowledge in different institutional frameworks, contexts of use, or language games. Getting a finer analysis of mathematical activity requires introducing a typology of mathematical objects. In the description of the informal and formal practices relative to the natural numbers, we have seen previously that procedures, linguistic elements, properties, and different ways of justifying intervene.

[4] In the OSA, we use a weak notion of system as an organized or structured set of elements, which is common in cognitive and social sciences. This allows speaking of "mathematical object" as an entity emerging from the subjects' system of practices to solve a class of problem situation, mediated by linguistic and material artifacts.

The definition of an object as emergent from the systems of practices and the typology of primary objects introduced in the OSA intend to respond to the necessity of describing the systems of practices in order to compare them and take decisions about the design, development, and assessment of mathematics teaching and learning processes. The types of objects that we are going to describe permit the identification of epistemic and cognitive configurations, configurations that can be used as reference for the description of semiotic processes, that is, processes of construction and communication of meanings involved in mathematics activity, as we shall illustrate in section 4.

3.2.1 Configuration of intervening and emergent objects

For the accomplishment of a mathematical practice and for the interpretation of its results as satisfactory, it is necessary to put certain knowledge into practice. If we consider, for example, the knowledge required to find the number of objects in a set, it is necessary to use some verbal or symbolic tools, procedures, counting principles, etc. Consequently, when an agent carries out and evaluates a mathematical practice, it activates a configuration of objects formed by problems, languages, concepts, propositions, procedures, and arguments. The six types of primary entities postulated extend the traditional distinction between conceptual and procedural knowledge when considering them insufficient to describe the intervening and emergent object in mathematical activity. The problems are the origin or reason of being of the activity; the language represents the remaining entities and serves as an instrument for the action; the arguments justify the procedures and propositions that relate the concepts to each other.

The primary objects are related to each other forming configurations, defined as the networks of intervening and emergent objects from the systems of practices. These configurations can be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects).

3.2.2 Processes

The entities described might be analyzed from the *process–product* perspective. The emergence of primary objects (languages, problems, definitions, propositions, procedures, and arguments) takes place by means of the respective mathematical processes of communication, problem posing, definition, enunciation, elaboration of procedures (algorithmization), and argumentation.

These processes, which are essential in mathematics activity, are not unique. It is necessary to consider other processes that are essential in this activity, such as, "generalisation–particularisation," "representation–interpretation," splitting–reification (processes which are intimately related to the *mathematics connections*), and materialization–idealization (linked to the *language game*[5]). Likewise, the institutional and personal nature of the meanings[6] of mathematics objects make up other essential processes in the

[5] Mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated *ostensive* objects (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the *language game* (Wittgenstein, 1953) in which they take part.

[6] Institutional objects emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person. "Personal cognition" is the result of individual thinking and activity when solving a given class of problems, while "institutional cognition" is the result of dialogue, agreement, and regulation within the group of subjects belonging to a community of practices.

aforementioned activity, namely, institutionalization–personalization processes, which are intimately related.

3.3 Meaning, conflicts, and semiotic functions

One way of conceiving a word's meaning is to consider it as the content that is associated with the said expression. Meaning is the content of any *semiotic function* (Eco, 1978; Hjelmslev, 1943), that is to say, the content of the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified, or meaning), established by a subject (person or institution) according to distinct criteria or a corresponding code. This is an elemental or "unitary" way of understanding meaning. Given a mathematical object, which is considered as expression, the meaning is the mathematical object considered as content. A prototypical example would be the semiotic function that associates a definition (content) with a term (expression).

Another possible way of tacking the problem of "meaning" is to do so in terms of usage. From this perspective, the meaning of a mathematical object must be understood in terms of what can be done with it. This is a "systemic" perspective, as it considers the meaning of the object to be the set of practices in which the said object plays a determining role (or not).

These two ways of understanding meaning complement one another, since mathematical practices involve the activation of configurations of objects and processes that are related by means of semiotic functions.

The notion of semiotic conflict has been introduced in the OSA as an explanation of students' errors, difficulties and obstacles in the learning of specific mathematical content, and in general, of difficulties arising in classroom communication. The relativity of the systems of practices (and hence, also the meanings) to the institutional frameworks, and the ecological relationships between institutions (dominance, dependence, subordination,...) lead us to consider that the following general definition of *semiotic conflict* is useful: It is any disparity or mismatch between the meanings given to an expression (antecedent of a semiotic function) by two subjects (people or institutions) in an interactive communication.

A student answers the question, does anyone know what a number is?, saying, "A sign that refers to a quantity." The meaning that this student attributes to number enters into conflict with the formal meaning that the lecturer wants to present to the class, but whose solution he does not want to tackle. Likewise, the number meaning that the lecturer wants to present to the class enters into conflict with the holistic meaning we have described in section 2.

3.4 Relationship between configurations and semiotic systems

The configurations of intervening and emergent objects and processes help us to give a definition of "semiotic system," which we consider is operative and well adapted to the analysis of teaching and learning processes: A semiotic system is the system formed by the configuration of intervening and emerging objects in a system of practices, along with the interpretation processes that are established between the same (that is to say, including the network of semiotic functions that relate the constituent objects of the configuration). In the case of the numbers, each partial meaning is a semiotic system, like the holistic meaning described in section 2, which is made up of the articulation of the different subsystems that form the partial meanings.

Since the systems of practices depend on the people who carry them out, and on the institutions (communities, cultures, etc.) where they are shared, the associated semiotic systems will also depend on people and institutions. Practices are linked to the solution of types of problem, which might have a particular local or global character, and hence, the semiotic systems will have these levels of generality too.

The epistemic/cognitive configurations are made up of linguistic elements of different nature (gestures, words, inscriptions…) and also of conceptual, propositional, and argumentative objects, as well as the problems that are their origin and reason for being, and the relations that are established between the aforementioned constituents. So, we are dealing with heterogeneous, multimodal, and dynamic systems that change and are enriched as time goes by. The notion of configuration, conceived in this way, is in accordance with the semiotic system described by Ernest (2006).

The components of a semiotic system that Ernest (2006) considers are some of the elements considered in the configuration of objects and processes activated and emergent in mathematical practices. Indeed, the set of signs (S) is the "language" component of the configurations of intervening and emergent objects and processes, when it is considered from its ostensive nature. The set of rules of production of signs (R) are, in our case, the remaining primary entities (definitions, propositions, procedures, and arguments). The relations between the signs and their meaning, embodied in an underlying structure of meanings (M), are considered in our case, by the system of objects and processes of the configuration, looked at from the point of view of the expression–content duality.

Likewise, our notion of semiotic system is compatible with that described by Radford (2002), and the notions of semiotic set and bundle by Arzarello (2006), as long as we widen our notion of the linguistic constituents of the configurations by incorporating any kind of material medium (artifacts) that participate in carrying out the mathematics practices. As the example of the meanings of the natural numbers (section 2) has shown, the systems of practices which characterize them can be formed by different subsystems, each one providing a partial meaning of the numbers.

Our notion of semiotic system is more general than Duval's (1993) "register of semiotic representation," which basically refers to the type of language used in a specific activity (graphical or algebraic register, natural language,…), whereas semiotic systems include the "structured set" of objects that take part and emerge from this activity. Duval (2006) conceives the semiotic system as a set of signs, depending on each other, according to clearly identifiable principles of organization. For example, this interpretation allows the author to differentiate a representation of numbers using stick collections from another representation structured by the abacus positional principle, where the number value depends on the column where the marks are placed (p. 54). Even if the author describes this essential difference between "sets of signs" and "systems of signs," Duval, nevertheless does not systematize the implications of this difference. Thus, for example, in relation to the representation of numbers, it would be pertinent to deal with questions such as:

- What problems can be posed and solved with stick collections representation and using the positional principle of the abacus?
- What procedures are applicable with each of these two representations and what are their relative effectiveness?
- What arguments are susceptible to be used in solving the problems or justifying of procedures and properties?
- What properties can be formulated in an intelligible way?
- Etc.

3.5 Partial meanings and the phenomenon of compartmentalization

The OSA tries to extend the vision of representations and translate the study of the transformations between representations towards the study of configurations of objects and processes and the articulations between them. These configurations include the expression tools, along with the situations, conceptual, propositional, procedural, and argumentative elements. The meaning of an object is given by the flexible articulation between the diverse configurations linked to that object and the practices that these configurations make possible; each pair, practices–configuration, constitutes a partial meaning of the object.

Replacing the analysis of "representations and its transformations" by the study of "configurations and their articulation" implies the revision of some classic learning phenomena, such as compartmentalization (Vinner & Dreyfus, 1989; Elia, Gagatsis, & Gras, 2005), which refers to the subject's cognitive incapacity to coordinate at least two representation registers for a concept.

Since the learning of an object can be described in the OSA in terms of the partial meanings associated to the object that the students must construct, compartmentalization can be seen as the subjects' incapacity to articulate diverse partial meaning for the object. This inability prevents him or her from solving related problems formulated in diverse contexts or justifies the appropriateness of actions. In other words, besides observing compartmentalization in the conflicts with the translations between representations of an object, it can also be observed in the incapacity to:

– Solve a problem by means of two configurations that correspond to different partial meanings
– Characterize an object with a new definition that provides a new interpretation or a more effective action
– Consider two propositions that correspond to different configurations to be equivalent
– Etc.

Consequently, the teacher's interventions for overcoming the cognitive compartmentalization should take into account not only the students' learning of changes of registers in which can be represented an object (Duval, 1993), but the students' articulation of diverse meanings of the same (including its different linguistic representations).

4 Onto-semiotic complexity of learning the tens

The theoretical tools introduced in the onto-semiotic approach (system of practices and configuration of objects and processes) can be used to describe and understand the semiotic systems formed by the students' answers given to specific mathematical tasks.

Giroux and Lemoyne (1998) provide experimental data of the complexity that the construction of the symbolic knowledge of the representation of the numeric system supposes. The problems posed to the children, which involve the operations of addition and subtraction, show how, for example, the construction of the equality 10+3=13 is not made unless it is after the process that starts from the basic notion of counting: "10+3→11, 12, 13→13" (p. 291). The description of this process "requires additional work to analyze the knowledge and the conditions under which this knowledge is constructed" (p. 300).

Sáenz-Ludlow (2004) presents an analysis of the mathematical activity of a fourth grade class (9–10 years old) that shows the relevance of the semiotics dimension in the evolution of the numerical learning. This work puts the emphasis on the utility of the semiotic approaches to provoke in situations of dialogical interaction the emergence of "own" signs that facilitate strategies of arithmetic calculation (mainly addition and multiplication).

In this section, we illustrate the use of the semiotic tools described in section 3 to improve our understanding of some difficulties of learning the tens by analyzing the answer given by a 6-year-old child to a task of counting and writing numbers greater than ten in the system of decimal numeration (Fig. 2).

The worksheet asks the child to count the number of chocolates represented in several drawings, and it gives a guide to write the answer in the first task (one ten is written, and the 6 of the units is suggested with dots that the child has to write over). We can see that the teacher has crossed out the zero of the ten that the boy has written.

4.1 Institutional and personal mathematics system of practices

The teacher expects the child to carry out the following actions:

- Read and understand the task
- Count the number of chocolates in the picture
- Write the results of counting in three different ways:

  ✓ Ordinary language, one ten and the number of units, written in specific places
  ✓ Sum of the cardinal of two subsets (chocolates inside and outside the box)

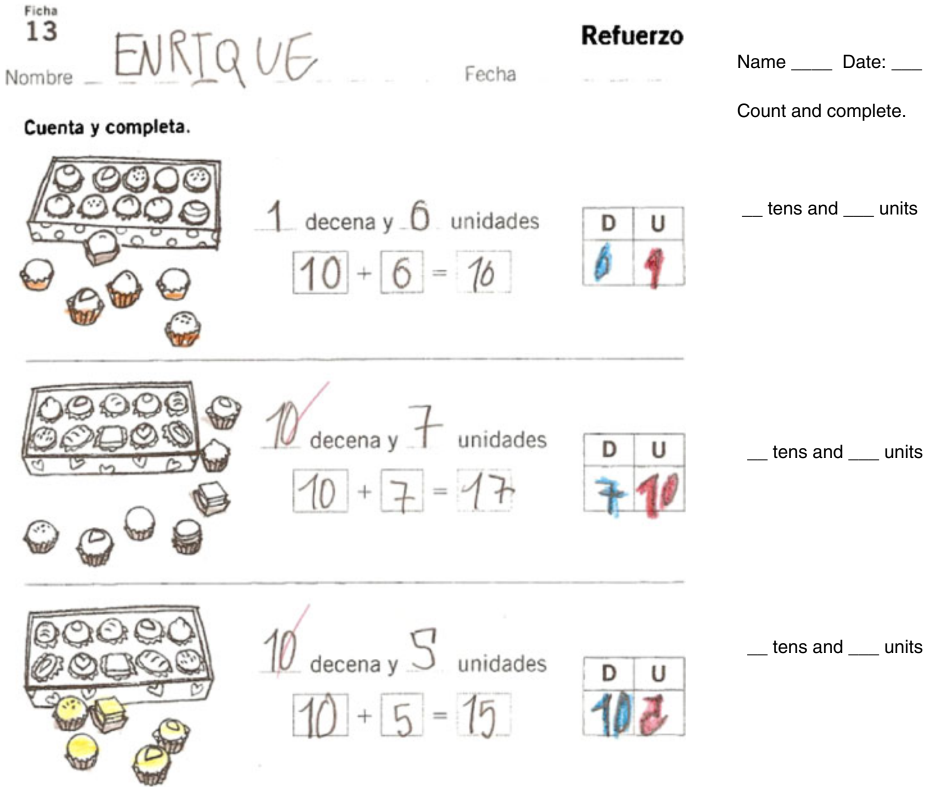


Fig. 2 Counting and writing numbers greater than ten

✓ Identify the tens and the units of each result and write them in the array ad hoc.

In this case, we consider the student's reading and solving the task as a practice. The worksheet shows an incomplete example, since it is a worksheet of "reinforcement" and the child has already done similar activities.

The child counts well, but shows difficulties with the identification of the tens and the units. The personal meaning does not seem to differentiate between digit and number, in spite of distinguishing between the relative value and absolute value of numbers (one ten is ten units). In other words, the operative practice (correct) does not have correlation with the discursive practice. This inconsistency should not be evaluated solely in terms of "the student know—or does not know."

It is supposed that writing in the table D–U (Tens–Units) requires "solely" the identification of numbers as a "symbol aggregation" and to interpret these according the conventions of the tabular language.

4.2 Interpretation and representation processes

Next, we include a detailed analysis of the diverse objects that intervene in the statement and expected solution of the task and the meanings given to these objects. We use the notion of "semiotic function," relation between an antecedent object (expression, signifier) and another consequent object (content, signified), which is established by the subject that carries out the interpretation process when applying a criterion or rule of correspondence[7].

We classify the network of semiotic functions in four groups, according to the types of objects that take part like expression or antecedent: linguistic objects, concepts, procedures, and propositions. The subjects' actions are in the nucleus of the criterion–rule that determines the expression–content relation, that is to say, they are consubstantial to the relational nature of mathematics and to the semiotic functions that allow the description of mathematical activity. The arguments form part of a justification of procedures and proposition. This analysis allows us to understand Enrique's difficulties in terms of the onto-semiotic complexity of the task.

4.2.1 Interpretation of linguistic elements

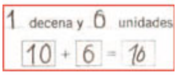
In Table 1, we present a summary of the words, expressions, and other linguistic elements included in Fig. 2, their intended meanings, and the implicit rules that the interpreter uses to connect the expression with the content.

The first vignette establishes the institutional rule to follow: When we have a set with ten objects, we say that it forms a ten, and a one is written; we call the rest of objects that do not reach ten, units, and their number is written, 6, in the position of the right.

The worksheet displays three different ways to express the same mathematical fact: "1 ten and 6 units"; "10+6=16"; and the tabular–symbolic register that remembers the positional value of each digit. These three mathematical expressions are also representing a specific situation of the daily world: the cardinal of set union of two subsets of chocolates shown by means of icons. The result of modeling the situation, 16 chocolates, is implicit; the worksheet only expresses the numerical value of the measurement, 16.

[7] Coherently with the OSA semiotic perspective, we present a triadic characterization of the data, which are analyzed and systematized by a three column array.

Table 1 Implicit rules connecting expressions to contents in Fig. 2

EXPRESSION (Signifier)	CONTENT (Meaning)	CRITERION/RULE
Count	- Problem: How many chocolates are there drawn in the picture? - Counting procedure - Concept of cardinal, number of items in the collections	- To find "how many are there", you have to count; - To count you have to apply the counting technique
	- Set of object to count, split in two subsets (inside and outside the box)	- The cardinality of set union is the sum of the cardinal of disjoint subsets
Fill in	- Write the counting result in the empty spaces, as the sample given, applying the rules of the decimal numeration rules	- Writing rules: divide ten and units; addition in a row; tabular writing
1 ten and 6 units	- Concepts of ten and unit; - 1 ten refers to the amount of chocolates inside the box; 6 units refers to amount of chocolates outside the box.	- Conceptual rules - Decimal grouping principle
	- Addition operation - Concepts of addition, sums and result - Polynomial expression of a number; splitting a number in units and tens	- Procedural rule of sum; - Conceptual rules - Representational rules
	1 ten and 6 units can be written as 10+6; and also as 16	- Representational rule
	- U, refers to Unit; D, to Tens (Decena) - In the right cell each digit has its own value; in the left cell each digit has the value of ten	- Representational rules - Principle of positional value of digits

The student must understand the meaning (meaning process) of each linguistic element of the text and, mainly, he must understand the text globally. The child's accomplishment of the task shows his difficulties in applying the writing rules of the decimal numeration system in the most elementary case, as it is the writing of the ten as a unit of second order. The student does not see a ten, but ten units. The meaning of symbols D (Decena, Tens) and U (units) does not seem obvious to this student.

4.2.2 Concepts interpretation

In Table 2, we present a summary of the concepts that intervene in Fig. 2, their intended meanings, and the implicit rules that the interpreter uses to connect the concepts with their meanings.

Table 2 Implicit rules connecting concepts to meanings in Fig. 2

Expression (Signifier)	Content (Meaning)	Criterion/rule
Number of elements	—The size of the three sets of chocolates grouped in tens and units: 16, 17, 15	Implicit definition
Ten	—Collection of ten chocolates considered as a unit (complete box)	—The ten as a container
	—Second position in the positional decimal writing	—Writing algorithm of two digit numbers
Unit	—Objects not included in the decimal grouping	—The unit as "remaining" elements
	—First position in the positional decimal writing	—Writing algorithm of two digit numbers
Addition	—Grouping the chocolates inside and outside the box; go on counting	—Fix order of the number series
Equality	—Result of the sum operation	—The operation and its result are related with the "=" sign

The child counts the number of objects inside the box well and knows that ten is written 10, but he does not understand the role (meaning) that the 0 and the 1 have in this writing. Enrique's answers to the requested tasks essentially show the complexity of the notion of unit (of first order) and ten (units of second order): a collection of ten units (chocolates) must be seen unitarily like a new unit, and not like ten units. Likewise, the sample of the activity does not allow us to determine what the meaning that Enrique gives to the rule is, "in order to add one digit number to 10, it is enough to replace the 0 (of the 10) by this number." In fact, since when posed with the question "how many tens are there," Enrique puts "10," the previous task represents a mere game of symbols for this child, without reference to the ten like "grouping of units."

4.2.3 Interpretation of procedures

Procedures and propositions suppose a "higher level" of mathematical connection. This fact, within the OSA, is taken into account considering that the criterion–rule of the semiotic functions in which the procedures participate require arguments that justify its use and it makes the sentence, in which they are inserted, coherent.

Table 3 Implicit rules connecting procedures to meanings in Fig. 2

Procedure (Antecedent)	Use (Consequent)	Criterion/rule (Justification)
Counting technique of the number of elements of a collection	It is used to find the size or number of elements inside and outside the box	The condition of application are fulfilled (finite collection of objects)
Writing the numbers in the positional system, separating units, and tens	Writing 16, 17, and 15	They are numbers greater than the numeration base, and there is an algorithm to write these numbers
Addition operation	It is used to find the total number of chocolates, inside and outside the box	The condition "disjoint collections" is fulfilled

In Table 3, we present a summary of the procedures that intervene in Fig. 2, their intended meanings (uses), and the implicit rules that the interpreter applies to connect the procedures with their meanings.

Enrique counts the collections of objects well, but he makes mistakes in the procedure of positional writing of numbers.

4.2.4 Interpretation of propositions

In Table 4, we present a summary of the propositions that intervene in Fig. 2, their intended meanings (uses), and the implicit rules that the interpreter applies to connect the propositions with their meanings.

Enrique applies the counting principles well because he is able to find the number of chocolates.

4.3 Generalization and particularization processes

In this exercise, the chocolates are intended to be used as generic objects, that is to say, we try to make the students generalize (for the first problem) that 10 objects+6 objects are 16 objects. With the sequence of the three exercises, he should know that, if a ten of objects is joined with a number of objects less than ten, the result is a number of objects equal to 1 followed by the number that represents the units. The student should also learn the general rule of the positional numeration systems for numbers of two digits: "Ten simple units, or first order units, form a unit of higher order and it is written to the left like a new unit of higher order." We can observe that the student does not understand this rule.

4.4 Idealization and materialization processes

In this task, there is an implicit process of idealization since, after counting, we have only the empirical evidence that 10 chocolates+6 chocolates are 16 chocolates. This is the physical operation of grouping objects. However, when writing the numbers, the chocolates disappear and it is concluded that 10+6 are 16, that is to say, we have changed from an operation with physical objects to a mathematical operation with numbers. This process of idealization is combined with the generalization process previously described,

$$\text{Chocolates} \rightarrow \text{Any objects} \rightarrow \text{Numbers}$$

Table 4 Implicit rules connecting propositions to meanings in Fig. 2

Proposition (Antecedent)	Use (Consequent)	Criterion/rule (Justification)
Counting principle: the cardinal number is the ordinal number of the last counted element, …	It is used in the procedure of counting the collections	The conditions of application are fulfilled (finite collection of objects)
The cardinal of the union of two disjoint sets is the sum of the cardinal of each set	It is used to calculate the total (16, 17, 15)	The condition "disjoint collections" is fulfilled
There are 16 (17, 15) chocolates	They are the answers to the questions posed	Empirical checking (there are 10 objects in the box and 7 outside, hence I write 10 and 7 according to the model)

The concept of ten is materialized first with a box that contains ten chocolates and later by means of three different notations: one ten, 10, "second position to the left" in the writing of numbers. The unit idea is also materialized first by one chocolate and the number of units like a set of chocolates "without container," this number is also materialized with the notation "6" and by the right cell in the table that divides the number into tens and units. The sum is materialized with the fact that there are two disjoint collections (the chocolates inside and outside the box).

4.5 Reification and splitting processes

The reification process of the ten should be achieved first by the presentation of the collection of objects to count divided in two subsets: the box of ten chocolates and the rest outside the box. While for the ten, the container-contained scheme should be applied so that the students understand the ten chocolates as a unit of higher order, this scheme is excluded explicitly in the case of the units. This reification is later reinforced with the writing "1 ten…" and by the use of the table that divides the number into tens and units.

The main conflict observed in Enrique's answer is that he does not conceive the ten objects as a unit (of higher order), he writes 10 instead of a 1 in the cell on the left that separates the number in tens and units. Enrique does not interpret the writing 10 assigned to a collection of ten objects in terms of 0 units of first order and 1 of second order. This fact allows us to affirm that the child has not reified the ten chocolates in one ten of chocolates.

5 Final reflections

Different works of research have tackled the conceptual analysis of the numeration and the stages of development that the subjects go through in their learning (Bednarz & Janvier, 1982; Steffe & von Glaserfeld, 1985; DeBlois, 1996). Our aim has not been to characterize the notion of numeration and establish learning levels of the same thoroughly, but to unfold the network of semiotic functions implied when carrying out a specific task done by a pupil, revealing the system of rules that the child should progressively learn.

In this paper, we have shown that the notions of system of practices, configuration of objects and processes, semiotic function, and semiotic conflict permit the implementation of several levels of detailed analyses for mathematical activity, and consequently, get new explanations of phenomena regarding teaching and learning.

In the case of numbers and arithmetic, which we have analyzed, the onto-semiotic approach permits the description of the diverse elements that characterize the institutional meaning of numbers (understood as pairs of practices and configurations of objects and processes) and explain the conflicts of learning in terms of the complexity of objects and meaning involved.

The analysis of the child's answer to the school task of writing numbers greater than ten has allowed the systematic and structured study of the rules involved in the use of numbers: rules of linguistic, conceptual, procedural, propositional, and argumentative nature, as well as the associated processes of generalization, idealization, and reification. The personal–institutional duality focuses on the analysis from the point of view of teaching and learning, identifying the conflicts between the meaning that the student constructs and the intended institutional meaning. This type of analysis helps us to be aware of the complexity of knowledge called on and the difficulty of developing the operative and discursive competences on natural numbers

The emphasis on the ontological aspects that the OSA proposes is compatible with the socio-constructivist and anthropological assumptions taken as starting postulates. The object, considered as coming from a system of practices, can be considered as unique and with a holistic meaning (Wilhelmi, Godino, & Lacasta, 2007). However, in each sub-system of practices, the configuration of objects and processes in which the object at issue "appears" is different, and, therefore, different practices are made possible. The systems of practices can be divided up into different classes of more specific practices, made possible by a certain configuration of objects and processes, allowing the distinction between meaning and sense: the senses can be interpreted as partial meanings. This point of view for mathematical objects is closely related to Ernest's position (1998, p. 261); he considers that the social constructivism adopts an approach to mathematical objects that can be described as nominalist, when considering them as objects of conceptual/linguistic nature.

In the OSA, contrary to a traditional realistic position on the nature and ontological status of mathematical objects—that locates it in the abstract and intangible world of the Forms (Plato) or the World 3 (Popper) or even directly in the empirical world (Maddy, 1990)—we locate it in language games and cultural space of mathematics. The configurations of intervening and emergent objects and processes and the analysis of how they become apparent in professional and school mathematical languages allow us to explain how the language games lead to conferring a certain type of existence on mathematical objects.

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