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# The role of abduction in proving processes

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**Abstract** This paper offers a typology of forms and uses of abduction that can be exploited to better analyze abduction in proving processes. Based on the work of Peirce and Eco, we describe different kinds of abductions that occur in students' mathematical activity and extend Toulmin's model of an argument as a methodological tool to describe students' reasoning and to classify the different kinds of abduction. We then use this tool to analyze case studies of students' abductions and to identify cognitive difficulties students encounter. We conclude that some types of abduction may present obstacles, both in the argumentation when the abduction occurs and later when the proof is constructed.

**Keywords** Abduction · Argumentation · Proof · Toulmin's model

## 1 Introduction

Abduction is an area of research that has received increasing emphasis in mathematics education, especially in the past decade. However, the term "abduction" is often used in a general way to refer to a number of different processes. Abduction has been considered in relation to mathematical activity in general (see, e.g. Cifarelli & Sáenz-Ludlow, 1996; Krummheuer, 2007; Mason, 1996) and more specifically in relation to mathematical proof (see, e.g. Arzarello, Micheletti, Olivero, Robutti, 1998a, Arzarello, Micheletti, Olivero, Robutti, 1998b; Ferrando, 2006; Knipping, 2003a, b; Pedemonte, 2007, 2008). In these studies, abduction is often described generally as "an inference which allows the construction of a claim starting from an observed fact" (Pedemonte, 2007, p. 29) with reference made to the work of C. S. Peirce (1960). However, Peirce's concept of abduction evolved over time (Fann, 1970), and more recent examinations of abduction have made important contributions to distinguishing types of abductions (e.g. Eco, 1983). In this paper,

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we will explore how more precise descriptions of abduction can allow a more detailed analysis of mathematical activity, especially in relation to mathematical proof.

## 2 Abduction and proof

In mathematics, proof is deductive, but the discovering and conjecturing processes is often characterized by abductive argumentation. When students are engaged in the mathematical practice of proving, they often “come up” with an idea. To analyze what students are doing when this happens, one can refer to abduction. Some studies (e.g. Arzarello et al., 1998b) show that abduction plays an essential role in the dialectic between conjecturing a hypotheses and proving a result: abduction supports the transition to the proving modality, which remains in any case deeply intertwined with it. However, while abduction is crucial in introducing new ideas (Peirce, 1960), it is sometimes an obstacle for students when they have to construct a deductive proof. When solving geometrical problems, some students are not able to construct a deductive proof because they are not able to transform their abductive argumentations into deductive proofs (Pedemonte, 2007). Interestingly, this obstacle is not present if the problem is situated in the algebraic domain (Pedemonte, 2008). In algebra, proof is characterized by a strong deductive structure. Once the solution to a problem has been written as a formula, the proof can consist of the manipulation of the formula to show that it can be derived from other known formulae. For the students, a proof in an algebraic context can be purely mechanical, what Tall (1995) calls a “manipulative proof.” Thus, abductive argumentation is usually not an obstacle for students when constructing a proof in an algebraic context, unlike in geometry. On the contrary, it seems to support it.

Further analysis is necessary to begin to explain these findings. Research is needed that examines whether different types of abduction act as obstacles to or support proving and more generally if the types of abduction used in argumentation can affect the nature of the proving process and if so, how. In this paper, we will offer some analyses of students’ mathematical activity in which different types of abduction occur and which, in some cases, make deductive proof more accessible and in others, more difficult. First, however, it is necessary to describe some types of abduction described by Peirce and Eco.

## 3 Abduction in the work of Peirce

The term “abduction” was introduced by C. S. Peirce to refer to an inference distinct from deduction and induction. In the course of developing the idea of abduction, Peirce used the term in a number of different ways and used a number of terms to refer to it, including Hypothesis and Retroduction. Nevertheless, it is possible to identify the main aspects of Peirce’s thought. In Peirce’s early work on abduction, he emphasizes the logical form of abduction, in which the argument proceeds from a Rule and a Result to a Case. In his later work, he focuses on abduction as a phase in the discovery process. It starts with the observation of a surprising fact, and the goal of the abduction is to explain this fact. In this section, we will review briefly the evolution of Peirce’s thinking and the relationship between his two uses of the term abduction. For a more thorough discussion of this history, see Fann (1970).

Peirce's first discussion of abduction seems to be his presentation in 1867 to the American Academy of Arts and Sciences. At that time, he referred to abduction as "Hypothesis" and characterized it by this syllogism (1960, 2.511[1]):

Hypothesis
Any $M$ is, for instance, $P' \; P'' \; P'''$, etc.
$S$ is $P' \; P'' \; P'''$, etc.;
$\therefore S$ is probably $M$.

Here, $S$ is the subject, a specific case of interest, and $P'$, $P''$, $P'''$ are a number of characteristics of it. For example, one could write:

Any bird has a hard beak, lays eggs, and can walk
A platypus has a hard beak, lays eggs, and can walk
$\therefore$ A platypus is probably a bird

In this case, the stronger conclusion "A platypus is a bird" is false, which indicates a key characteristic of abduction. The argument gives the conclusion plausibility, but not certainty. The only exception to this occurs when the list $P'$, $P''$, $P'''$ of characteristics is exhaustive, in which case Peirce calls the argument "formal hypothesis" or "reasoning from definition" (2.508).

In an article published in Popular Science Monthly in 1878, Peirce first noted the importance of explaining a surprising fact in connection with abduction. Hypothesis now becomes the means to find a general rule to explain a surprising observation.

> Hypothesis is where we find some very curious circumstance, which would be explained by the supposition that it was a case of a certain general rule, and thereupon adopt that supposition. Or, where we find that in certain respects two objects have a strong resemblance, and infer that they resemble one another strongly in other respects. (Peirce, 1960, 2.624)

Note that the second sentence of the quote suggests that analogies are, for Peirce at this time, a special kind of abduction. He gives an example involving white beans, which is often cited by mathematics educators (e.g. Arzarello et al., 1998a, b; Mason, 1996).

> Suppose I enter a room and there find a number of bags, containing different kinds of beans. On the table there is a handful of white beans; and, after some searching, I find one of the bags contains white beans only. I at once infer as a probability, or as a fair guess, that this handful was taken out of that bag. This sort of inference is called *making an hypothesis.* It is the inference of a *case* from a *rule* and a *result.* (Peirce, 1960, 2.623)

Note that in this example, the list of characteristics $P'$, $P''$, $P'''$ has been reduced to one "white." This suggests that Peirce saw abduction as possible on very limited evidence, perhaps because in examining instances of abduction in scientific discovery, he encountered such situations.

In the last decade of the 1800s, Peirce began to focus more on the role of abduction in scientific thinking and less on its logical form. In 1896, he introduced the term "retroduction" to mean "the provisional adoption of a hypothesis" (Peirce, 1960, 1.68)

[1] Peirce's *Collected Works* are organized into volumes within which each paragraph is numbered. In references to them, we indicate the volume by the digit before the dot and the paragraph by the number following the dot.

using "hypothesis" differently to mean "something, which looks as if it might be true and were true and which is capable of verification or refutation by comparison with facts" (Peirce, 1960, 1.120). An important characteristic suggested by this formulation is that the conclusion of an abduction should be "capable of verification or refutation by comparison with facts," so accounting for a surprising occurrence by making reference to a mysterious force which may never again have any effect, does not count as an abduction.

About 1901, Peirce began to use the term "abduction" instead of "retroduction."

> The first starting of a hypothesis and the entertaining of it, whether as a simple interrogation or with any degree of confidence, is an inferential step which I propose to call abduction. This will include a preference for any one hypothesis over others which would equally explain the facts, so long as this preference is not based upon any previous knowledge bearing upon the truth of the hypotheses, nor on any testing of any of the hypotheses, after having admitted them on probation. I call all such inference by the peculiar name, abduction, because its legitimacy depends upon altogether different principles from those of other kinds of inference. (6.525)

Note the reference here to choosing among several hypotheses as well as considering a single original hypothesis. We will discuss these two possibilities further in the next section.

At about the same time, Peirce began to see abduction, deduction, and induction not only as three distinct logical forms but also as three steps in scientific reasoning, with abduction being the first (see 6.469, 7.202–206). In 1903, Peirce again described abduction using a syllogism, but a much less formal one:

> The surprising fact C is observed,
> But if A were true, C would be a matter of course;
> *Hence, there is reason to suspect that A is true.* (Peirce, 1960, 5.189)

From 1905 on, Peirce returned to using the term "retroduction" to refer to abduction (though Fann, 1970, claims Peirce preferred "abduction" as the best designation). He used "retroduction" to refer to both the logical step of reasoning from consequent to antecedent (6.469) and to hypothesizing an explanation for a surprising state of things (8.229) as the first step in a process of scientific reasoning. He stated explicitly that by retroduction he meant the same thing as he had by "hypothesis" in his earlier work (8.228).

The key characteristics of abduction in Peirce's thinking can be identified in this brief history. The conclusion of an abduction is plausible, not certain. Abduction proceeds backwards, from a result or consequent to a case or antecedent. Abduction explains a surprising fact. The conclusion of an abduction is capable of verification or refutation by comparison with facts. It is the first stage in scientific reasoning, followed by deduction (of further consequences) and induction (testing those consequences).

Peirce's study of logic was unusual in its time because it did not focus primarily on deductive logic. Instead he viewed logic as "the art of devising methods of research—the method of methods" (7.59). Because he took this broad view of logic he was able to discuss it in a way that included all aspects of the process of scientific reasoning. Later, we will consider the work of Toulmin (1958), who also set out take a broad view of argument, examining the arguments used in different fields, and asking "What things about the forms and merits of our arguments are field-invariant and what things about them are field-dependent?" (p. 15, emphasis in original). We shall apply Toulmin's work to our examination of abduction below.

## 4 Abduction in the work of Eco

Eco (1983) makes some useful distinctions based on Peirce's 1878 formulation of abduction: the inference of a case from a rule and a result (Peirce, 1960, 2.623). Eco points outs that the rule needed is not always evident. If it is, then Peirce's formulation applies, but if it is not then other kinds of abduction arise. Eco identifies three kinds of abduction: overcoded, undercoded and creative (see also Bonfantini & Proni, 1983; Magnani, 2001, for related classifications). Overcoded abduction occurs when the arguer is aware of only one rule from which that case would follow (p. 206). It is the same as Peirce's 1878 formulation.

If there is more or less than one rule known to the arguer, then the situation becomes more complex. Before the case can be inferred, a rule must be found and the conclusion of the abduction will depend on what that rule is. As Eco points out, "the real problem is... how to figure out both the Rule and the Case at the same time, since they are inversely related, tied together by a sort of chiasmus" (p. 203).

If there are multiple general rules to be selected from, Eco calls it "undercoded abduction" (p. 206) he gives as an example Kepler's search for an explanation for the surprising positions of Mars at different times. The number of closed curves that are possible paths for a moving object is not infinite, Kepler was selecting from among several possibilities (circle, ellipse, cardioid, etc.). Eco claims that the rules from which the selection is made are "equiprobable" which echoes Peirce's 1901 description of abduction as "a preference for any one hypothesis over others which would equally explain the facts, so long as this preference is not based upon any previous knowledge bearing upon the truth of the hypotheses" (1960, 6.525). However, Eco's example suggests that there might be criteria for selecting among the possible rules, meaning that while all of them "would equally explain the facts" they are not in practice treated as "equiprobable." Astronomers first tried to explain the orbits of the planets by hypothesizing that they moved in circles, the simplest closed curve. Kepler's hypothesis of ellipses arose only after the circle hypothesis became untenable. Other curves (e.g., cardioids, squares) were not considered first because they are less simple (in an everyday sense) than ellipses, especially when considered as paths of moving objects. Peirce introduced a principle of "economy" which includes the application of Ockham's razor: "Try the theory of fewest elements first; and only complicate it as such complication proves indispensable for the ascertainment of truth" (1960, 4.35).

Magnani (2001) links overcoded and undercoded abductions together as selective abductions. Selective abduction is defined as the process of finding the right explanatory hypothesis from a given set of possible explanations. In this case, the arguer should find the most appropriate rule to construct the conclusion from among the set of rules he has access to. However, it can happen that there is no general rule known to the arguer that would imply the given case. Thus, the arguer must invent a new rule. Eco (1983) calls an abduction that involves the invention of a new rule a "creative abduction."

Physicists' attempts to account for the anomalies in the orbit of Mercury provide example of both undercoded and creative abductions. It is possible to account for the anomalies by making use of the rules already available concerning the motion of planets. One hypothesis of this kind that was proposed was the existence of an unknown planet close to the sun that was perturbing Mercury's orbit. In this case, the argument is an undercoded abduction. Many such hypotheses were proposed, but in the end, it was a creative abduction, the creation of a new general rule (Einstein's theory of relativity) that successfully accounted for the anomalies.

## 5 Research focus

We are interested in discovering whether there are kinds of abduction, among those described above, that make deductive proof more accessible and others that, on the contrary, are more difficult to transform into deduction.

We suspect that argumentation involving an overcoded abduction makes it easier to construct a proof, compared to the other kinds of abduction, at least if the theorem (the rule) used in the argument is sufficient to solve the problem. In fact, if the rule exists as a theorem, all the elements needed to construct a proof are present. According to the principle of cognitive unity (Boero, Garuti, Mariotti, 1996; Pedemonte, 2005) prior argumentation can be used by students in the construction of proofs if they can organize some of the previously produced arguments into a logical chain. In the case of overcoded abduction, the structural distance between abductive argumentation and deductive proof is shorter because students only have to look for data to justify the claim; the rule and the claim are already present.

However, if the theorem is not sufficient to produce a proof, the student is obliged to change strategy in order to solve the problem. This may be difficult as no other plausible rule is known to the student. In this case, an overcoded abduction is not an aid and may be an obstacle to the construction of the proof.

In the case of undercoded abduction where several plausible rules are known, it is important to select a useful and correct rule in order to produce a proof. As in the case of the overcoded abduction, it is important that the selected rule is sufficient to solve the problem; otherwise, the student is obliged to change strategy to solve the problem However, this may be less of an obstacle in the case of undercoded abduction because other plausible rules are available to be tried.

Creative abduction is probably the most difficult kind of abduction to use as a basis for a deductive proof because a lot of additional checking is necessary to ensure that the created rule is an effective and correct theorem. It could happen that the rule created is incorrect. In this case, the student could construct an incorrect "proof" (as occurred in one of the examples we provide below).

## 6 Toulmin's model

To analyze the relationships between different types of abduction and deductive proving in students' mathematical activity, we shall model their arguments using Toulmin's model (1958, 1993). Through this model, argumentation and proof can be analyzed and compared from a structural point of view (Pedemonte, 2007). Toulmin's aim was to provide a model that could be used to analyze arguments in general, not just deductive arguments, and so it allows us to compare the structure of argumentation with the structure of proof. Here, we use it to differentiate the structure of the kinds of abduction under consideration and then to compare them with the structure of proof.

In Toulmin's model, an argument comprises three elements (Toulmin, 1958, 1993):

C (*claim*) the statement of the speaker
D (*data*) data justifying the claim C
W (*warrant*) the inference rule that allows data to be connected to the claim.

In any argument, the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology, the standpoint is called the claim. The second step consists of the

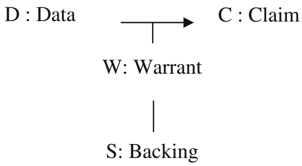


**Fig. 1** Toulmin's model of argumentation

production of data supporting the claim. The warrant provides the justification for using the data; it provides support for the data–claim relationship. The warrant, which can be expressed by a principle or a rule, acts as a bridge between the data and the claim.

Auxiliary elements may be necessary to describe an argument (Toulmin, 1958, 1993): qualifier, rebuttal, and backing. The qualifier expresses the strength of the argument, the rebuttal introduces a counter-argument, and the backing provides additional support for the warrant. We do not discuss rebuttal and qualifier in this paper, but this does not mean that they are unimportant in analyzing arguments, including proofs. As has been highlighted by other researchers (e.g., Inglis, Mejia-Ramos, Simpson, 2007) these elements are very important. We have chosen to not include them here because our focus is on describing kinds of abduction from a structural point of view. For such a structural analysis, the basic form of Toulmin's model is sufficient (Pedemonte, 2007).

We are, however, interested in the backing (S[2]) which supports the warrant.

Toulmin's model of argumentation can be represented as shown in Fig. 1.

In the analysis presented in this paper, the backing is important because it legitimizes the warrant of the argument. It is always present but often in implicit way. In a proof, the backing is a mathematical theory (Pedemonte, 2005), but this is not always the case in an argumentation process. Comparison between the backings used in argumentations and proofs is important because it could give insights into the nature of the rule of abduction versus the theorem used in a proof.

Toulmin's model can be used to represent abductive and deductive structures (Pedemonte, 2007). Deduction can be represented as in Fig. 1 (Toulmin intended his model to represent deductive arguments as well as other kinds; for him, what distinguishes deduction is the warrant and the nature of the backing) while Pedemonte (2007) represents abduction as shown in Fig. 2:

The question mark means that data are to be sought in order to apply the inference rule justifying the claim.

However, this representation of abduction is too limited to show differences between types of abduction. We shall now analyze five case studies of students' mathematical activity related to proving in which abductions occur, in order to further apply Toulmin's model to different types of abduction. Cases 1 and 2 involve undercoded abduction and Cases 3–5involve creative abductions. There are multiple examples of undercoded and

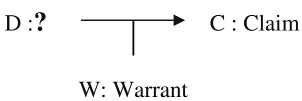


**Fig. 2** Abduction in Toulmin's model

[2] We abbreviate "Backing" as "S" (for Support) rather than B to avoid confusion with the conclusion of the rule A→B

creative abductions because we have observed some interesting differences. Overcoded abduction is also included in Case 2.

## 7 Background for the first four cases

Our first four case studies are derived from a teaching experiment carried out in traditional 12th and 13th grade (15/17 years old) classes in France and Italy, when students are beginning to learn proof in a systematic way. In this context, the didactical contract explicitly required the construction of deductive proofs. The students had some prior experience with proof and knew the theorems necessary to solve the proposed problems.

For the problem considered here, the mathematical activity of 33 student pairs working with Cabri Geometry were analyzed. This software was used on the basis of the hypothesis that the drag function could help students to observe some properties that might elude them when using pencil and paper. In addition, it was expected that two students working together on one screen would be more likely to talk together in order to find a common solution.

The problem posed is the following:

> ABC is a triangle. Three exterior squares are constructed along the triangle's sides. The free points of the squares are connected, defining three more triangles. Compare the areas of these triangles with the area of triangle ABC (Fig. 3).

The problem asks the students to compare the area of each individual exterior triangle with the area of triangle ABC. The areas of four triangles are equal, but this is not obvious to students. In order to find a solution, an abductive argumentation is often constructed. The drag function makes it possible to see the congruence between the base of triangle ABC and that of one of the exterior triangles (see Fig. 4). This provides a context for the students to suppose the congruence of the heights of the two triangles, and so to construct them (see the heights AL and IM in Fig. 5). This fact can become a claim requiring an argumentation to justify it. The claim is that the areas of the two triangles (ABC and ICD in Fig. 5) are equal. It is necessary to look for data and warrant to justify this fact (because the bases are equal, the heights should be equal if the areas are congruent). Once this is done, this abductive structure has to be changed to produce a deductive proof.

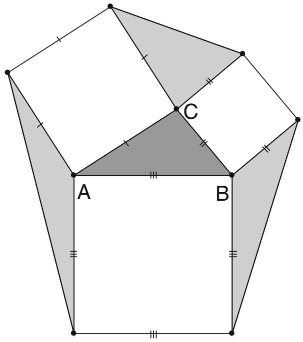


**Fig. 3** Diagram for problem posed in Cases 1–4

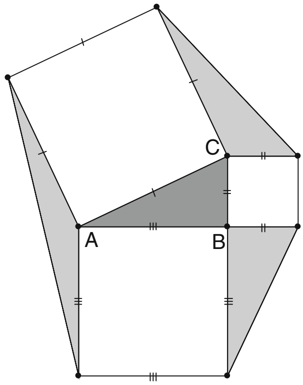


**Fig. 4** Problem figure dragged to suggest congruent bases and heights

Transcriptions of audio recordings and the students' written productions were used to produce solution protocols. The assertions produced by students were identified and the structure of the argumentative step reconstructed, specifying the claim C, data D, and warrant W. The subscripts on the letters D, C, W, identify each argumentative step. The transcript is in the left column, and comments and analyses are reported in the right column. The transcripts have been translated from Italian and French into English.

## 8 Case 1: undercoded abduction

The analysis starts when the students are comparing the area of triangle ABC and the area of triangle ICD. Previously, the students have constructed the heights AL and IM to compare the areas of the triangles ABC and ICD (students have seen bases CB and DC are congruent). Students see that the two triangles ALC and IMC are congruent. This statement is a "fact," where the epistemic value is joined to perception of the figure in Cabri Geometry.

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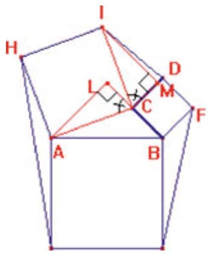
| ….<br>Students together: hey, these are two congruent triangles!<br>L: It's true, ALC and ICM, these are two congruent triangles…what do they have? | <br>**Fig. 5** The figure as represented by the students using Cabri Geometry.<br><br>**$C_1$: The triangles ALC and IMC are congruent** |

Students look for data and warrants justifying the claim $C_1$. The step is an undercoded abduction: it is not clear which of the theorems that establish the congruence of triangles applies. The students look for suitable data in order to apply one of these theorems. They immediately see that AC is congruent to IC because they are side of a square.

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| 44. G: We realized… then AC is congruent to IC because they are sides of the same square.<br>45. L: Wait!<br>46. G: AC is congruent to IC, after…<br>47. L: LC…<br>48. G: It's congruent to CM, why?<br>49. L: Then… Because it's congruent to CM… in my opinion… no wait, this angle is right and this angle is right too. | **$D_1$: AC=IC ?** → **$C_1$: the triangles ALC and ICM are congruent**<br>**W: ?AAS, SSA, SSS congruence**<br>**S: congruence theorems**<br>*Students can select one of the congruence criteria observing that in the two triangles ALC and ICM, there are two equal angles and an equal side. So they select the AAS congruence criterion.* |

Note that in lines 47–49, they are looking at LC and CM, but they are not able to justify why these sides are equal. So they look for other sides or other angles. Once they have AC=IC and the two right angles, they can use the AAS congruence criterion. Note that the backing is fixed: it is the backing that guides students to find data to apply one of the congruence theorems. The choice of the theorem strictly depends on what data are found by the students.

Once the students have found the appropriate data they can construct the proof.

In the proof, students make data $D_1$ explicit to affirm that triangles ALC and IMC are congruent. The abductive structure of the argumentation is transformed into a deductive structure in the proof. Once obtained, claim $C_1$ is used to deduce that the heights of the triangles ABC and ICD are congruent and consequently that their areas are equal.

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| I consider the triangle ABC and the triangle ICD.<br><br>At once I consider the triangles ALC and IMC and I prove that they are congruent triangles by the AAS congruence criterion because we have:<br>• AC = IC because they are two sides of the same square;<br>• ALC = IMC because they are right angles (angles constructed as intersections between the sides and the heights)<br>• ACL = ICM because they are complementary to the same angle (LCI)<br>…. | **$D_1$: AC =IC ALC = IMC ACL = ICM** → **$C_1$: the triangles ALC and ICM are congruent**<br>**W: AAS congruence criterion**<br>*Once obtained, claim $C_1$ is used to deduce that the heights of the triangles ABC and ICD are congruent and consequently that their areas are equal.* |

It seems that in this case, the students have not encountered difficulty in the passage from abduction to proof. It should be noted, however, that in Italy, congruence theorems are the basis of 12th grade curriculum and so the students are very familiar with using theorems like the AAS congruence criterion when they have to prove congruence between two triangles.

## 9 Case 2: undercoded abduction and overcoded abduction

The analysis starts when the two students find that the triangles' areas are equal using the measure function provided by the software. As in the previous example, the students construct the two triangles' heights and see that they are equal (as in Fig. 5). However, they are not able to justify this fact.

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| 27. Y: So… do you see? These two triangles have the same bases<br>28. G: What?<br>29. Y: yes, this is a square…<br>30. G: ah, yes… this square…the bases are the sides of this square<br>31. Y: but the problem is to find out why they have equal heights<br>32. …. | **$D_1$: BC=CD** → **$C_1$: the triangles' areas are equal**<br>**? AL=IM**<br><br>**W: Area Formula** |

The students state and justify that the bases BC and CD are equal. They know that they have to state that the two heights AL and IM are equal, but they are not able to say why. They try to use different theorems they know, but without any result.

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| 47. Y: the height is the median?<br>48. G: No, it depends if you drag this point the figure changes<br>49. Y: yes, but...you can perhaps rotate the triangle but...it is difficult<br>…..<br>64. G: Wait, if you drag… the heights are always parallel to the other side<br>65. Y: No, no we have to find a theorem to state that the two heights are equal...I don't know... | **$D_2$: ?** → **$C_2$: the heights are equal**<br><br>**W: ?$W_1$, $W_2$, $W_3$**<br><br>**S: ?$S_1$, $S_2$, $S_3$**<br><br>*Students are looking for a theorem among those they know to justify the claim $C_3$. Nevertheless, they are not able to select a theorem. In this case the backing is not evident. It could be:*<br><br>$S_1$: Triangle properties (the warrant $W_1$concerns the relationship between medians and heights)<br>$S_2$: Geometric transformations (the warrant $W_2$concerns the rotation of the triangle ACL)<br>$S_3$:Parallelism properties (the warrant $W_3$concerns the visualization that AL//CD and IM//CB) |

The students are not able to continue. After 10 min in which the students do not speak about the problem, the teacher tries to help them. The teacher's suggestion establishes the backing $S_2$: geometric transformations, and guides the students to the warrant: rotation maintains the lengths. In a sense, the students' reasoning can be considered a failed undercoded abduction. The students do not choose a rule until the teacher tells them what to choose. The teacher's intervention transforms the undercoded abduction into an overcoded one.

| | |
|---|---|
| 111. Teacher: What are you doing?<br>112. Y: It is difficult… we have seen that the bases of these two triangles are equal and their heights too but why are the heights equal? The triangle has rotated around this point<br>113. Teacher. Has it rotated?<br>114. Y: yes it has rotated 90 degrees, there is a rotation but I'm not sure that this fact is useful<br>115. Teacher: What does a rotation maintain?<br>116. Y: the angles and the lengths…<br>117. G: yes but…if rotation maintains lengths why are equal the heights?<br>118. Y: ummm… we know that rotation maintains lengths…<br>119. G: yes but we have to say that the heights are equal…<br>120. Y: Ah yes… we can say that….Because there is a rotation… ICM is the rotation of the triangle ACL<br>121. G: ah ok, the triangles are rotated and the heights that are parts of the triangles are equal | *Teacher's intervention allows students to select the backing (geometric transformation) and the theorem (rotation maintains the lengths).After that, student G looks for data to apply the theorem.*<br>**$D_{2G}$: ?** → **$C_{2G}$: the heights are equal**<br>**$W_2$: Rotation maintains lengths**<br>**$S_2$: Geometric transformations**<br>*The student Y helps the student G to find data to apply the theorem. The argument is constructed.*<br>**$D_2$: Triangle rotation in C** → **$C_2$: the heights are equal**<br>**$W_2$: Rotation maintains lengths**<br>**$S_2$: Geometric transformations** |

We observe that the teacher's role is very important to the argumentation because he helps the students select the rule that is useful for solving the problem. Once the teacher supplies the students with the rule they need, then the abduction becomes an overcoded abduction at least for one of the student, student G. Students do not consider any other rules from that point, only what data they can now use.

The subsequent construction of proof does not seem to be difficult for them. The authority of the teacher was important for two reasons: to select the correct warrant and the correct backing and consequently to transform the undercoded abduction into an overcoded abduction. The overcoded abduction is present because, even if the teacher has selected the useful rule, it is not completely clear for student G which data to consider. In fact, the rule is not directly applied on the heights IM and AL, but it has to be applied on the little triangles ICM and ACL as shown in the proof (the last argument is transformed in the proof into two arguments: 2A and 2B, see below)

The construction of a deductive proof follows from this overcoded abduction.

| | |
|---|---|
| *Argument 2A:*<br>The two triangles ACL and ICM are equal because the triangle ACL can be rotated over the triangle ICD through a rotation in C of 90°.<br><br>Because the rotation maintains angles and lengths, the two triangles are equal.<br>*Argument 2B:*<br>Thus the triangle ABC and ICD have equal area, because they have equal bases and equal heights. | **$D_{2A}$:ICM is the rotation of ACL by 90°** → **$C_{2A}$: ACL and ICM are equal**<br>**$W_2$: Rotation maintains lengths and angles**<br>**$S_2$: Geometric transformations**<br><br>*By the congruence of the triangles ACL and ICM, the congruence between the heights AC and IM derives consequently in implicit way. Students can conclude the proof*<br>**$D_{2B}$: $C_{2A}$** → **$C_{2B}$: the heights are equal**<br>**W: inheritance of equality**<br><br>**$D_3$= $D_1$: BC=CD, $C_{2B}$** → **$C_3$=$C_1$: the triangles' areas are equal**<br>**W: Area Formula** |

The overcoded abduction is transformed in two deductive arguments (even if the second argument is implicit). It seems that the construction of the proof was not difficult for students.

## 10 Case 3: incorrect creative abduction

The two students consider particular cases of the triangle ABC, and they observe the effect on the exterior triangles. They consider ABC an equilateral triangle, and they see that the three exterior triangles are equal and so they conclude that the exterior triangles have equal areas. Then they consider ABC an isosceles triangle and they conclude that two exterior triangles are equal and so they have the same area. Finally, they consider ABC generic triangle. In this case, there are no equal exterior triangles. Thus, students state that the triangles have different areas.

Consider the transcript in this last case.

| | |
|---|---|
| 133. S: If the triangle ABC is generic, the exterior triangles are not equal because they don't have equal sides or equal angles…<br>134. C: Yes, but we have to consider areas…<br>135. S: We have to say if the areas are equal or not<br>136. C: yes, but we have to find something to say if they are equal or not<br>137. S: I think they are not equal…they seem not to be equal the areas… see the figure…how can they be equal?<br>138. C: I don't know…<br>139. S: If ABC is equilateral, we have seen that the areas of the exterior triangles are equal because they are equal triangles, if the triangle is isosceles only two exterior triangles have the same area because they are equal,… so in the case ABC generic triangle, there are not equal triangles … the areas are different<br>140. C: But why? How you can say that the areas are not equal?<br>141. S: There is a theorem that states that equal triangles have the same area… then if the triangles are not equal the area are different, aren't it?<br>142. C:… I'm not so sure… but I have no other idea… | The argument can be modelled as follows<br>**$D_1$: ABC generic triangle** → **$C_1$: Exterior triangles are not equal**<br>**W: Angles and sides are different**<br>**S: congruence theorems**<br>*The backing of the argument is the congruence theorems for triangles. The students explicitly look for equal sides or equal angles (133) to state that the triangles are equal. Because there are no equal sides or angles they conclude that the triangles are not equal.*<br>*From the figure Silvia thinks that the areas of the exterior triangles are different. Now students have to justify this fact. The argument is abductive:*<br>**$D_2$: ?** → **$C_2$: Areas of exterior triangles are not equal**<br>**W: ?**<br>**S: ?**<br>*Silvia creates a new false "theorem" (if two triangles are different they areas are different) probably deriving it from the theorem:*<br>*"If two triangles are equal, their areas are equal".*<br>**$D_2$: Exterior triangles are not equal** → **$C_2$: Areas of exterior triangles are not equal**<br>**W: If two triangles are not equal, their areas are not equal**<br>**S: If two triangles are equal, their areas are equal**<br>*The backing is correct but not the warrant.* |

In this example, we can observe how a creative abduction can be constructed. Students are not able to select a theorem from among a set of theorems they know, but they construct a “false” theorem to justify the statement they think to be true “the exterior triangles have different areas.” From this argument, students construct a “false proof.” Observe that the backing is correct. Notwithstanding this, the rule is not a correct rule.

| | |
|---|---|
| In the proof students also consider the cases in which ABC is equilateral and isosceles. Then they consider ABC generic triangle<br><br>…..<br><br>If ABC is a generic triangle… the exterior triangles have different areas because the exterior triangles are not equal. | The argument can be modelled as follows:<br>**$D_2$: Exterior triangles are not equal** → **$C_2$: Areas of exterior triangles are not equal**<br>**W: If two triangles are not equal, their areas are not equal**<br>\|<br>**S: If two triangles are equal, their areas are equal** |

In this example, it seems that creative abduction is more difficult to manage than the abductions in the other examples. In this case, students have to be able to construct a correct warrant to justify the statement. They fail in this task.

## 11 Case 4: correct creative abduction

The analysis starts after the two students have found that the triangles’ areas are equal using the measure function provided by Cabri. At this point, they try to justify this fact. They see that the bases of the triangles ABC and ALE are equal (Fig. 6). They have to find something to say that the heights are equal. This argument has an abductive structure.

133. L: the problem is different now… we have to say why the heights are equal …

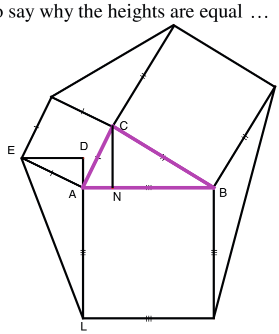



**Fig. 6** Diagram for Case 4.

….

136. L: This is a square …we can take the compass … we put the centre here (*in the point A*) and we can start here (point E) and we arrive exactly here (point C) because it is an arc… if we make a rotation of this point… this point arrives here, does not it?

137. C: Yes.

…

138. L: Then…through this rotation the two heights are overlapping

139. C: yes, but the rotation…. What is its definition?

140. L: We have not studied rotation… How is it defined?

141. C: I don't know…

142. L: however, in this rotation there are equal heights… perhaps rotation maintains lengths…

143. C: No, we have to prove it

*At this point students change strategy and they consider the congruence criterions to prove that the little triangles ANC and ADE are equal.*

*The argument can be modelled as follows:*

**$D_1$: ?** → **$C_1$: the heights are equal**

**W :?**

*This is a creative abduction because the students construct a new rule to justify the claim. The new rule is a correct rule: the lengths in a rotation are maintained. The students consider the rotation of AE over CN.*
*The question marks can be replaced.*

**$D_1$: C is the rotation of E** → **$C_1$: the heights are equal**

**W : rotation maintains lengths**

**S : Empirical geometric rotation**

*The support of this argument is not geometric transformation. The rotation is considered in an empirical sense: the use of a compass allows student to see the rotation. The warrant is explicit by student L who states: perhaps rotation maintains lengths. Note the word perhaps. There is no theoretical background to strengthen the argument. The rotation is given by the centre A but there is no mention of the rotation angle.*

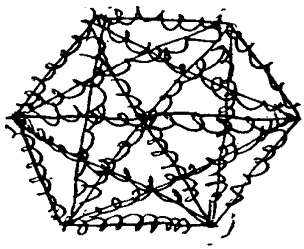
In this example, we can observe how a creative abduction can be constructed but also abandoned. Students explicit the warrant, but they cannot use it because it has not the status of theorem. They do not know this theorem even if they state it. They know about the compass tool and how to use it, but they do not know definition of rotation and theorems concerning it. The students are able to construct a new correct rule but they are not able to use it in the proof.

## 12 Case 5: correct creative abduction

Here, we consider a case from a different context that also shows a creative abduction. We have chosen to include a non-geometric case because it provides a contrast to the cases above, and allows us to see how the models used in them have to be adapted in this new context.

This example was presented in a previous paper by one of the authors of this article (Reid, 2003). Three Canadian grade 8 students (age 13–14) were solving the problem of determining the number of handshakes that occur when $n$ people shake hands. They were first asked to explain why 15 handshakes occur when six people shake hands, which they did

**Fig. 7** Diagram for $n$=6

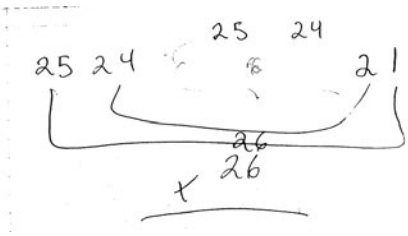


using a diagram (see Fig. 7). They were then asked to determine the number of handshakes for 26 and 300 people. They solved the case of 26 people by adding 25+24+…+2+1. As Jason added the numbers using a calculator, Sofia claimed to "know an easier way." She tried to explain her way using a diagram (see Fig. 8) but Jason protested "How would you write that down so you wouldn't have to draw all those lines? Is there a way?"

The transcript begins when Sofia tried to show how to use her method to solve the case of 300 people, proposing 150 times 300 as the answer. Jason suggested seeing if it worked for 26. Sofia said it should be 26 times 13, which Jason calculated, getting 338, not 325 as he had obtained by adding.

| | |
|---|---|
| Jason: See where you have 300 people like the next question says<br>Sofia: Yes.<br>Jason: You, You…<br>Nicola: That's going to be a<br>Sofia: OK you can just divide it in half, ok?<br>Sofia: Divide 300 in<br>Jason: But still there's got to be an easier way<br>Sofia: half<br>Jason: to do it<br>Sofia: No! You divide<br>Jason: So you divide 300 by so you'd get<br>Sofia: In half<br>Jason: 150<br>Jason: Still that would be one big line<br>Sofia: 150<br>Sofia: Times 300<br>Jason: 300 times 300 times 150?<br>Sofia: 300 times 150<br>Nicola: That one's gone, it leaves just these two.<br>Sofia: Right?<br>Jason: OK, well let's try it with the other one, see if it works.<br><br>Sofia: Try it with the<br>Jason: 20, so…<br>Sofia: 25<br>Jason: 25 times 12.5?<br>Sofia: What?<br>Jason: 26 times,<br>Sofia: 26 it would be 26…No divide it in half,<br>Jason: It, it would give you 13.<br>Sofia: 13 times<br>Jason: 26<br>Sofia: 26<br>Jason: Equals338, and this equals 325 | *We can model the Sofia's argument in the following way.*<br><br>**$D_{1S}$: 300 people** → **$C_{1S}$: 150*300**<br><br>**$W_S$: generalisation of the diagram (fig. 9)**<br><br>*Note that Jason's statement works as a rebuttal in Sofia's argument.*<br><br>**$Re_{1J}$: This is not correct because it doesn't work for 26 people (338 and not 325)**<br><br>**$D_{1S}$: 300 people** → **$C_{1S}$: 150*300**<br><br>**$W_S$: generalisation of the diagram (fig. 8)**<br><br>*This rebuttal seems to guide the students to reconsider the problem for 26 people in order to find a correct rule.*<br>. |

**Fig. 8** Sofia's diagram




Sofia tries to guess other numbers near 13 and 26 but without any result.

Jason tries to understand why the result for 26 people is 338 and not 325. He is also searching for a general rule for finding the number of handshakes, one that he can use for the next question where the number of people is 300. He knows the conclusion for 26 people (325) and Sofia has suggested a rule that almost works (which she obtained by partly remembering a deductive argument she had seen before). He has data and conclusion and creates a new rule that goes from one to the other, and also explains the near success of Sofia's rule.

| | |
|---|---|
| Jason: Umm, I know, maybe it's… just a second, just a second..<br>Sofia: Wait, wait, no no, it's 27, it's 27!<br>Jason: Maybe it's the number times half the number, umm, subtract half the number<br>Sofia: You lost me<br>Jason: Because that would work, 325 subtract 13, which is half of 26 is right.<br>Sofia: Try it again. | *Jason tries to understand why the result is 338 and not 325. 338 is obtained from 13 times 26 but the difference between 338 and 325 is 13.*<br>**$D_2$:26 people** → **$C_2$: 325 handshakes**<br>**W: ?**<br>**S: ?**<br>*He constructs a computational process to obtain 325 from 338:*<br>*325=338-13=26*13-13.*<br>**$D_{3J}$: 26 people** → **$C_{3j}$: 325 handshakes**<br>**W: 26*13-13**<br>**S: Computational rules and the fact that 325 is the correct result**<br>*Jason generalizes to the case n*<br>**$D_{4J}$: n people** → **$C_{4J}$: n*n/2 –n/2 handshakes**<br>**W: generalisation of the previous rule** |

Jason has used abductive reasoning to arrive at the general rule:

> The number of handshakes is the number of people times half the number, subtract half the number

which he shows does in fact allow him to go from the given data to the known conclusion:

> Because that would work, the number of handshakes for 26 people is 325 which is 338 subtract 13, which is half of 26, is right.

Reasoning like this, from a specific case to a general rule, is sometimes confused with induction. Peirce refined his own thinking about the difference over time. In his early work (c. 1878), induction is characterized as inference from a case and a result to a rule, which is exactly the logical form here. However, his example in 1878 and the symbolic form he gave for induction in 1867 both indicate that the "case" includes multiple instances of objects of a type, and the result multiple instances of them having a property. If Jason had compiled a table to compare the number of people with the number of handshakes in multiple instances, and had observed a pattern in that table, that the number of handshakes seemed always to be the number of people times half the number, subtract half the number, that would be, according to Peirce, an induction. In this case Jason arrived at his rule on the basis of only one case (he did not seem to be considering the result for six people), which is insufficient for an induction. Peirce's later writings give a second justification for considering this an abduction. Recall that in 1901, when he started to use the term "abduction" it referred to "the first starting of a hypothesis and the entertaining of it" (Peirce, 1960, 6.525). "Abduction … is merely preparatory. It is the first step of scientific reasoning, as induction is the concluding step" (7.218). Abduction occurs when a rule or data is hypothesized. Induction occurs latter when more cases are tested to determine if the rule is correct.

By comparing Case 4 with Case 5 we can observe that when a new rule is constructed by a creative abduction it seems important that the students recognize it as a theorem. In Case 4, the students constructed a new correct rule, but they did not use it in the proof because it was not a theorem for them. In contrast Jason constructed a new rule and went on to use it to solve other problems, showing that it had the status of a theorem for him.

## 13 Discussion

In representing students' mathematical activity in the above cases using Toulmin's model, we have constructed different representations for the different kinds of abduction. In addition to different representations for overcoded and creative abduction, we have also found two different representations for undercoded abduction.

An overcoded abduction can be represented in Toulmin's model as follows (Fig. 9):

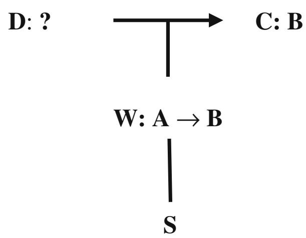



**Fig. 9** Overcoded abduction in Toulmin's model

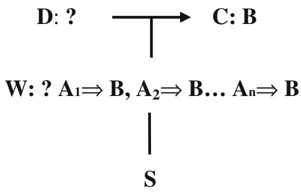



**Fig. 10** Undercoded abduction of the first kind represented in Toulmin's model

We observe that when an overcoded abduction is represented by the model a unique general rule is present as the warrant of the argument. There is a backing (explicit or implicit) that legitimates the use of the rule in the argument. In Case 2, when the teacher establishes the backing as geometric transformations, the student's argumentation becomes an overcoded abduction (argument 2 G) where claim and warrant are fixed (the heights are equal and rotation maintains lengths), but data have to be found (rotation of the triangles).

We have two different representations for undercoded abduction as shown in Cases 1 and 2. In the first kind, the backing is the same for all the rules that the arguer has to choose from. In the second kind the backing can be different for each rule.

The first kind of undercoded abduction, where the backing is the same for all the rules that the arguer has to choose from, can be represented in Toulmin's model as shown in Fig. 10.

This is what happens in the Case 1 where the backing is given by the congruence theorems and a specific theorem has to be chosen inside this domain.

The second kind of undercoded abduction in which the backing can be different for each rule the representation is shown in Fig. 11.

Case 2 shows that each rule or theorem might be supported by a different mathematical theory. Thus, the warrant as a particular rule has to be selected from among a set of rules and from among a set of theories. This selection seems more difficult for the student. In the example above the intervention of the teacher was necessary to support students in this choice.

Both kinds of undercoded abduction could present difficulties in the construction of a proof. Undercoded abduction of the second kind could be more difficult to manage for the student because he has to select not only the correct rule but also a "good theory." Undercoded abduction of the first type seems to be easier to manage (because the choice is limited to the rule, not the theory) but in certain cases the theory that seems to be the evident backing might not be sufficient to solve the problem (because rule needed is not in fact supported by it).

A creative abduction can be represented in Toulmin's model as shown in Fig. 12.

The question mark in the warrant means that a rule has to be sought to justify the claim. Unlike the undercoded cases, this rule has to be created, not selected from among a set of existing rules. For this reason, creative abduction is probably the most difficult to manage,

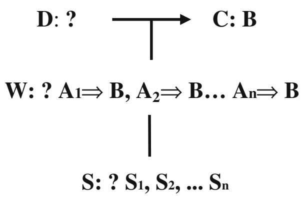



**Fig. 11** Undercoded abduction of the second kind represented in Toulmin's model

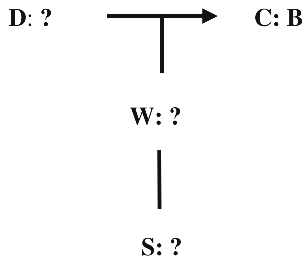



**Fig. 12** Creative abduction in Toulmin's model

as shown in Cases 3 to 5. Students have to be able to not only construct a correct theorem but also to recognize it as a theorem.

Recall that cases 1–4 are drawn from a study of 33 student pairs. Of these, only two pairs constructed a deductive argument, arriving at a conjecture by way of a trigonometric process (areas of triangles are calculated using trigonometric formulae). Seventeen pairs solved the problem through an inductive argument considering specific cases for the triangle (equilateral, right, and isosceles triangles). Only 14 pairs produced abductive arguments. Among these five pairs produced a deductive proof. In these cases, the abductive argument was an undercoded abduction of the first kind for three pairs (case 1 is a typical example), an undercoded abduction of the first kind transformed by the teacher into an overcoded one for one pair (case 2) and a correct creative abduction for one pair (case 4).

Among the students who were not able to construct a deductive proof, five pairs had used undercoded abduction of the first kind and four pairs used creative abductions (case 3 is a typical example).

## 14 Conclusion

In this paper, we have considered the ways in which different types of abduction can be involved in proving processes. We have observed three types of abduction described by Eco (overcoded, undercoded, and creative abductions) in students' mathematical activity and have used Toulmin's model to represent each type. This analysis has also revealed two kinds of undercoded abduction that seem to present different challenges to students as they attempt to construct a proof.

Through this analysis, we can distinguish two specific cognitive difficulties students may encounter using abductions: a cognitive difficulty related to the abduction itself and a cognitive difficulty related to use of this abduction when constructing a deductive proof.

Before an abduction can be used in a deductive process, the abduction must occur. Depending on the type of abduction this can involve locating data needed to apply a known rule, selecting from among several known rules backed by a theory, selecting from among several theories that provide backing for several rules, and creating a rule.

The construction of a proof seems to be more accessible to students when the abduction in the argumentation is overcoded or at least an undercoded abduction of first type. In these cases, students have to manage less information and they can select a rule from a limited set of rules. In contrast, undercoded abduction of the second type and creative abduction seem to be more complex to manage because a great deal of irrelevant information may be involved in the argumentation process, confusing, and creating disorder in the student's thought process.

Moreover, in the case of creative abduction, it is important that the rule constructed by students in the abductive argumentation is proved (or at least justified) during the argumentation. If this rule is only supposed to be true, students could not use it in the proof because they do not recognize it as a theorem.

Consequently, it seems that there is not a simple link between the use of abduction in argumentation and constructing a deductive proof. Both the claim that abduction is an obstacle to proof and the claim that abduction is a support, if considered in a general sense, are oversimplifications. Some kinds of abductions, in some context may make the elements required for the deductions used in a proof more accessible. Some are probably less dangerous to use and can make the construction of a proof easier to get to because they could make easier to find and to select the theorem and the theory necessary to produce a proof. However, other kinds of abductions present genuine obstacles to constructing the proof. This suggests that teaching approaches that involve students conjecturing in a problem solving process prior to proving have potential, but great care must be taken that the abductions expected of the students do not become obstacles to their later proving.

## References

Arzarello, F., Micheletti, C., Olivero, F., & Robutti, O. (1998a). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twentieth-second Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 24–31). Stellenbosch, South Africa.

Arzarello, F., Micheletti, C., Olivero, F., & Robutti, O. (1998b). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twentieth-second Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 32–39). Stellenbosch, South Africa.

Boero, P., Garuti, R., Mariotti M. A. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the Twentieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 121–128). Valencia, Spain.

Bonfantini, M., & Proni, G. (1983). To guess or not to guess. In U. Eco & T. Sebeok (Eds.), *The sign of three: Dupin, Holmes, Peirce* (pp. 119–134). Bloomington, IN: Indiana University Press.

Cifarelli, V., & Sáenz-Ludlow, A. (1996). Abductive processes and mathematics learning. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the eighteenth annual meeting of the North American chapter of the international group for the psychology of mathematics education Vol. I* (pp. 161–166). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Eco, U. (1983). Horns, hooves, insteps: Some hypotheses on three types of abduction. In U. Eco & T. Sebeok (Eds.), *The sign of three: Dupin, Holmes, Peirce* (pp. 198–220). Bloomington, IN: Indiana University Press.

Fann, K. T. (1970). *Peirce's theory of abduction*. The Hague: Martinus Nijhoff.

Ferrando, E. (2006). The abductive system. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the thirtieth conference of the international group for the psychology of mathematics education Vol. 3* (pp. 57–64). Czech Republic: Prague.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*, 3–21.

Knipping, C. (2003a). Argumentation structures in classroom proving situations. In M.A. Mariotti (Ed.), *Proceedings of the Third Conference of the European Society in Mathematics Education* (unpaginated). Bellaria, Italy. Retrieved from http://ermeweb.free.fr/CERME3/Groups/TG4/TG4_Knipping_cerme3.pdf

Knipping, C. (2003b). *Beweisprozesse in der Unterrichtspraxis: Vergleichende analysen von mathematikunterricht in Deutschland und Frankreich [Proving processes in teaching practices – Comparative analysis of mathematics teaching in France and Germany]*. Hildesheim: Franzbecker.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal of Mathematical Behavior, 26*(1), 60–82.

Magnani, L. (2001). *Abduction, reason and science: Processes of discovery and explanation*. Dordrecht: Kluwer.

Mason, J. (1996). Abduction at the heart of mathematical being. In E. Gray (Ed.), *Thinking about mathematics & music of the spheres: Papers presented for the inaugural lecture of Professor David Tall* (pp. 34–40). Coventry: Mathematics Education Research Centre.

Pedemonte, B. (2005). Quelques outils pour l'analyse cognitive du rapport entre argumentation et démonstration [Some tools to analyse the cognitive aspects of the relationship between argumentation and proof]. *Recherche en Didactique des Mathématiques, 25*(3), 313–348.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*, 23–41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM – The International Journal on Mathematics Education, 40*(3), 385–400.

Peirce, C. S. (1867). On the natural classification of arguments. Presented 9 April 1867 to the American academy of arts and sciences. *Proceedings of the American Academy of Arts and Sciences, 7*, 261–287. Compiled in Peirce, C. S., 1960, 2.461–516.

Peirce, C. S. (1878). Deduction, induction, and hypothesis. Popular science monthly, 13(August), 470–82. (Compiled in Peirce, C. S., 1960, 2.619–644).

Peirce, C. S. (1960). *Collected papers*. Cambridge, MA: Harvard University Press.

Reid, D. (2003). Forms and uses of abduction. In M. A. Mariotti (Ed.), *Proceedings of the third conference of the European society in mathematics education (unpaginated)*. Italy: Bellaria. Retrieved from http://ermeweb.free.fr/CERME3/Groups/TG4/TG4_Reid_cerme3.pdf.

Tall, D. (1995). *Cognitive development, representations and proof* (Proceedings of justifying and proving in school mathematics, pp. 27–38). London: Institute of Education.

Toulmin, S. E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Toulmin, S. E. (1993). *Les usages de l'argumentation* ([The uses of argument] (P. De Brabanter, Trans.)). Paris: Presses Universitaires de France.