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A mathematical experience involving defining processes: in-action definitions and zero-definitions

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Abstract In this paper, a focus is made on defining processes at stake in an unfamiliar situation coming from discrete mathematics which brings surprising mathematical results. The epistemological framework of Lakatos is questioned and used for the design and the analysis of the situation. The cognitive background of Vergnaud's approach enriches the study of freshmen's processes at university. The mathematical analysis and the results specifically underscore the in-action definitions and the zero-definitions and highlight the need of similar mathematical experiences in education, particularly focused on defining processes and the exploration of a research problem.

Keywords Zero-definitions · In-action definitions · Proof-generated definitions · Concept image · Displacements · Regular grid · Discrete mathematics

Defining processes represent a specific constant of language and of human thought. A definition can be a statement given in order to know what one talks about (such as declarative Euclidian definitions). Besides, defining is often perceived as a meta-process or as formal thinking (Tall, 2008). However, it is also a heuristic (in Schoenfeld's sense, Schoenfeld, 1985) of the mathematical practice. Indeed, Lakatos (1961, 1976) has exemplified how a search of a proof can make room for a new concept and, thus, for defining processes. In this article, I propose to focus on a mathematically centered perspective, questioning more specifically the defining processes in a pre-formal axiomatic activity, from an epistemological point of view. In order to achieve this goal, I will follow Pimm, Beisiegel, and Meglis (2008) who "(…) encourage a new look at and a recognition of Lakatos and his intentions as opposed to what they have become" (p. 478).

In the first section, I will explore and propose the broad outlines of a framework inspired by the one of Lakatos which can be useful for the analysis of defining processes. Section 2 is devoted to a didactical situation implemented with freshmen at university. The designed

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framework is used for the analysis of this situation which deals with discrete mathematics. The experimental results of the intervention are presented. The work expounded in Section 2 states the potentialities and the limits of the Lakatosian tools and situations. It also extends results of a previous research study (Ouvrier-Buffet, 2006). The concluding section highlights new perspectives for the situation and for the defining processes presented in this article.

1 A framework for the conceptualization of defining processes

1.1 A specific light on defining processes

Some researchers have been interested in characterizing the heuristics and behaviors of mathematicians (e.g., Burton, 2004; Carlson & Bloom, 2005; Schoenfeld, 1985). But little can be found about the characterization and the importance of defining processes, except when these are connected to proof. As commonly admitted, a proof may demonstrate a need for better definitions, one of the functions of a proof being the exploration of the meaning of a definition or the consequences of an assumption (Hanna, 2000). This is surely one of the ways to work with definitions and mathematical understanding in the classroom, as well as enriching the students' concept images (Tall, 1991; Vinner, 1991). I shall propose another way. I consider that definitions are temporary statements giving us information about the stages of concept formation at the dialectic interplay with proofs. Dealing with the co-emergence of definitions and proofs (such as the Lakatosian continual process of conceptual revision) is therefore tempting. My goal is to throw a specific light on the students' concept formation through the investigation of defining processes involving unfamiliar concepts. I will now focus on the Lakatosian definitional procedure as a framework.

1.2 What can we learn from the Lakatosian view?

1.2.1 (A part of) an epistemological model of the defining processes

Lakatos attempts to present patterns of mathematical reasoning where each character of his dialogue represents a particular epistemological and philosophical tendency. Inspired by Pólya and Popper, Lakatos deals with the context of discovery (with the construction of conjectures) on the one hand and the context of justification (with the evaluation of conjectures) on the other hand, emphasizing the social dimension. When concepts progressively develop so that they can be used to help solve problems, they do so in stages. Initially, they take the form of tools, followed by one of objects, then of foundations finally becoming formal theories. I refer to Lakatos' thesis (Lakatos, 1961) because it highlights the construction of definitions and mapping, then the formation of concepts with three kinds of definitions—the *naive definitions*, the *zero-definitions*, and the *proof-generated definitions*—whose respective functions are to denominate, to communicate a result, and to prove. A naive definition can be stated first, but it cannot evolve, contrary to a "zero"-definition that marks the beginning of the research process. A zero-definition can be modified in order to protect the primitive conjecture from a "monster" (i.e., a new kind of object) or because the concept is altered by the presentation of a proof. This is the place for growth of a system of concepts, and then a zero-definition turns into a proof-generated definition: it is impossible to get to the proof-generated definition stage without the proof idea.

I insist on the fact that the processes described by Lakatos depend on a restrictive starting situation consisting of: a situation of classification (which delimitates what the class of polyedra is), an initial conjecture (Euler formula), and a proof (that of Cauchy, where a change of thinking is required because it implies a topological view), where non-naive protagonists already have representations of concepts at stake (polyedra). The lack of counterexamples seems to be the only end criteria of the defining process. In fact, Lakatos leaves the first encounter with a mathematical object and the construction of a global axiomatic system aside. For educational issues, it is therefore necessary to continue the study of what comes before and after Lakatos' definitional procedure. One has to explore the pragmatic defining process first. The concept of zero-definition can be useful for didactical issues (I shall exemplify this in Section 2) because it attests to a commitment in a defining process by avoiding the formal, logical, and axiomatic rules that the canonical definitions should follow.

1.2.2 An operational framework?

Freudenthal (1973) has proposed a distinction of defining activities which has impacted several studies: *descriptive (a posteriori) defining* (systematization of existing knowledge) and *constructive (a priori) defining* (production of new knowledge). In these studies (Borasi, 1992; De Villiers, 1998, 2000; Larsen and Zandieh, 2005, 2008; Ouvrier-Buffet, 2006), two main theoretical tools are used, depending on the mathematical concepts involved: Van Hiele levels for geometrical concepts and Lakatosian tools for others.

Larsen and Zandieh (2008) focus their work on proofs and refutations in constructive defining situations designed and managed like that of Lakatos where the students have to (re)define concepts in a new system of constraints (triangle on the sphere for instance). They present proof as a motivation for defining, as a guide for defining, and as a way to assess defining, in the same way previously quoted by Hanna. They underscore that "the formulation of a definition in a classroom community can evolve through processes like those described by Lakatos" (p. 206). They also show that the Lakatosian stages of proofs and refutations provide a useful framework for the analysis of students' research activity of reinvention, but they do not focus on the defining process itself as Lakatos (1961) does.

I have previously presented an epistemological framework taking into account several conceptions: the Aristotelian one, the Popperian one, and the Lakatosian one (Ouvrier-Buffet, 2003, 2006). I have shown the ability of students to make a zero-definition evolve with the lack of counterexamples and with the reinvestment in a proof. I have also outlined some features of the teacher's guidance in situations of classification involving unfamiliar but graspable concepts for students.

The following questions have not yet been explored: Is it possible to design less restrictive defining situations than the Lakatosian one? To what extent is the Lakatosian framework operational and sufficient to analyze defining processes from both a mathematical and a didactical point of view?

I shall now propose a mathematical problem, still partially open in the ongoing professional research, which has a twofold interest:

- The Lakatosian kinds of definitions can be used as epistemological tools for characterizing the construction of new concepts. They impact the design of the didactical situation, which is also analyzed with the theory of the didactical situations (Brousseau, 1997).
- It was proposed to freshmen at university: the concepts at stake (those to be defined) in this situation are partially unfamiliar for these students and the results (those to be conjectured and proved) are unfamiliar and mathematically surprising.

2 Displacements on a regular grid

2.1 Presentation of the discrete problem

Discrete mathematics arouses interest because it offers a new field for the learning and teaching of proofs (Grenier & Payan, 1999; Heinze, Anderson, & Reiss, 2004). Besides, Goldin (2004) emphasizes “how experiences in discrete mathematics may provide a basis for developing powerful heuristic processes and powerful affect” (p. 58). The situation of displacements on a grid described below illustrates concepts which belong to several branches of mathematics. These concepts are particularly meaningful in the discrete case. Moreover, this discrete situation leads to Uhlig’s (2002) essential exploration of intuitive questions: “What happens if? Why does it happen? How do different cases occur? What is true here?” (p. 338), avoiding the definition–theorem–proof model of mathematics (Thurston, 1994).

Let G be a discrete regular grid (a regular tessellation). In this article, G is a squared one. A *point* of G is a point at the intersection of two lines. A *displacement* on G is defined with two positive integers and two directions (up, down, left, and right). For instance, “2 squares right and 3 squares down” is a displacement. It can be represented with a vector, even if this representation is already considered a modeling of the problem.

> The general “formal” problem (P) is the following: let $E_k = \{d_1, ..., d_k\}$ be a set of k displacements d_i, $k \geq 1$ and $1 \leq i \leq k$. Starting from a point of G, any one, which points of the grid can one reach using nonnegative integer combinations of displacements of E_k?

Comments The following mathematical analysis is still valid for displacements and their opposite displacements. Note that the order of the displacements does not interfere because combinations of displacements are commutative (this is not obvious a priori but it can be easily proved).

I will now deal with the Lakatosian way of focusing on definitions to develop the mathematical analysis of this problem, which allows the mapping of concepts formation with zero-definitions. The didactical variables, and then the didactical situation, will emerge from this analysis.

2.2 Study of problem (P) through defining processes

There are several tracks for studying problem (P). Each of them brings forth zero-definitions and underlying questions. The concepts to be defined and the linked queries are in bold type.

2.2.1 Studying small cases

Let me begin by studying small cases, where $k \geq 2$ (it is obvious that any E_1 does not generate G).

With a set E_2: no set of two displacements allows access to all the points of G. With any E_2, either all the points of a **sector** of the grid (i.e., a set of points bounded by two rays) are reached (Fig. 1), or some points of a **sector** are reached (Fig. 2).

With a set E_3: if we try to build a generating set with three displacements, taking into account what happens for the sets E_2, we can use Fig. 1, left. These two “unit” displacements allow us to go everywhere within a sector (I call this property **with full density**). And we had to add at least

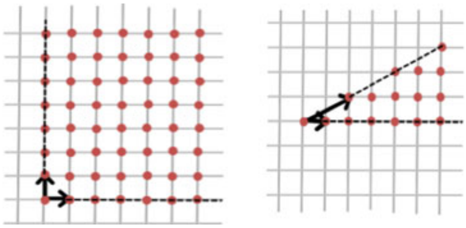

Fig. 1 All the points of a sector are reached

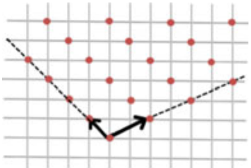

Fig. 2 Some points of a sector are reached

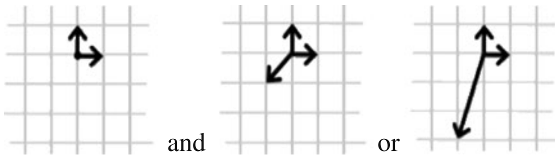

Fig. 3 Construction of a generating set with three displacements

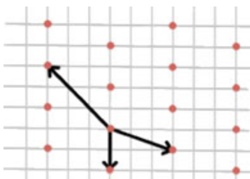

Fig. 4 Set of three displacements with the ALBE property, without the FD property

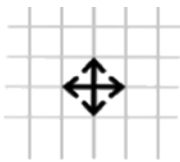

Fig. 5 Minimal generating set with four displacements

one displacement in order to reach other sectors of the grid. A geometrical argument (translation of the sector reached with the two unit displacements) brings us a third displacement (Fig. 3).

Any 3-uplet of displacements does not allow us to go everywhere on G: on Fig. 4, we can go **a little bit everywhere** (**ALBE**[1]), but without **full density (FD)**. Therefore, two properties break away from these examples. We can "zero-define" them as follows:

- **FD**: all the points of a sector of the grid are reached;
- **ALBE**: a set of displacements has the ALBE property if there exists a positive number *M* such that for any point X in the grid there exists a reachable point, "close to X", i.e., whose distance from X does not exceed *M* (it is possible to avoid a metric here altogether).

We can reach all the points of the grid when these two properties are satisfied simultaneously: they imply a zero-definition of a **generating set**.

With a set E_4: there exist obvious sets of displacements that allow all the points of G to be reached (Fig. 5); that is the reason why the study of E_4 can precede that of E_3.

Now consider the grid with the two directions horizontal and vertical. In order to reach all the points of a horizontal line, we have to reach the points 1 square to the left and 1 square to the right (of the starting point, any one). The same idea applied to the vertical line leads us to conclude that if we can reach the four cardinal points, it is possible to reach all the points of G. Moreover, such a set is **minimal** (see Section 2.2.2). **How can one build other generating sets? Do such sets exist? Do they have other mathematical properties?**

2.2.2 Reciprocal problem and minimal aspect

A new question emerges: is it possible to remove a displacement of E_k without changing the set of reachable points? We can say that E_k is ***minimal*** when removing any of its displacements modifies the set of reachable points (note that it is sufficient to check if the **FD** and the **ALBE** properties are kept to prove the minimal aspect). With this definition, the question arises of **how to characterize a minimal generating set of displacements? Do the minimal generating sets of displacements always have the same number of elements (i.e., the same cardinality)?** The answer is obviously no (see Figs. 3 and 5). Let me now go further in the direction of the cardinality.

2.2.3 Discussion on the minimal generating sets and their cardinalities

The exploration of E_3 and E_4 showed us that minimal generating sets could have different cardinalities. We can bring problem (P) back to a problem on $\mathbb{Z}$ (as done with E_4) to use known results about number theory.

Building minimal generating sets on $\mathbb{Z}$ We can choose coprime numbers (i.e., their greatest common divisor (GCD) is equal to 1). Thus, the **FD** property is true for integers with the possible exception of a finite number of points[2] (cf. Bezout's lemma). Some of these

[1] Allusion to "almost everywhere". One can also link this property to the discrete mathematical field of "covers" and raise the question of the minimal cover.

[2] To reach 1, some a_i must be negative. This is supposed not to be allowed by the discrete problem. On the other hand, if negative a_i's are allowed, then there is no need to have a negative number in E to reach −1.

coprime numbers should be negative in order to go **ALBE**.[3] We can therefore build several minimal generating sets of displacements with **different cardinalities** on $\mathbb{Z}$. It leads us to this surprising result:

Proposition 1 for any integer *k*, there exist, in $\mathbb{Z}$, minimal generating sets of displacements with *k* elements.

The previous analysis gives the elements to write a constructive proof. Therefore, the cardinality of minimal generating sets of displacements of $\mathbb{Z}$ is not an invariant feature. The reader can consult the wider and more complex NP-hard Frobenius problem (Ramirez Alfonsin, 2006), also called the "coin problem", which is close to (P).

Building minimal generating sets E_k on G And now, if we want to build a minimal generating set E_k, we can use the previous result and keep in mind the horizontal/vertical representation. We have to choose displacements carefully in order to keep the minimal aspect.

Therefore, one has the following proposition:

Proposition 2 for any integer *k*, there exists, on G, minimal generating sets of displacements with *k* elements.

But, if *k* is given (as big as one wants), we do not know how to construct **all** the minimal generating sets of displacements. This is an open question. A track to explore it is to study how to convert a minimal generating set into another using geometrical transformations on the grid. Other ways to explore problem (P) such as algebraic and algorithmic tracks exist.

I will now summarize the situation and enlighten it through potential defining processes by characterizing didactical variables.

3 The didactical situation

3.1 Context—methodology

This activity was carried out with two classes of students from scientific and non-scientific courses (freshmen year) during the academic term. The methodology used for this activity consisted in putting students into groups (six groups of three or four students in each class) in order to facilitate exchanges and discussion, which were audiotaped. Their written productions were also collected and the students were asked to write a research report after the experiment. The students who took part in this experiment are used to working in groups and writing research reports. The teacher took the position of a manager–observer (MO, in Ouvrier-Buffet's sense; Ouvrier-

[3] If we want to generate $\mathbb{Z}$ with four integers, we build four natural numbers which are coprime as a whole: for instance $2\times3\times7$, $3\times5\times7$, $2\times3\times5$, $2\times5\times7$, i.e., 42, 105, 30, and 70. After 383, all integers are reached (see http://www.math.uu.nl/people/beukers/frobenius/index.html for exploring it). Now, if we take one of these numbers as a negative one, we get a minimal generating set of $\mathbb{Z}$: $\{-30; 42; 70; 105\}$.

Buffet, 2006). The situation consisted of three problems. The intervention lasted 3 h. The general instructions were the following:

> We will take an interest in the displacements on a regular squared grid. For each problem (Table 1), we will use a set of displacements and choose a starting point (called A), any one. The questions[4] are:

1. Starting from point A, which points of the grid can we reach?
2. What are the consequences if we remove one or more displacements?

3.2 A priori analysis

I have tried to design a milieu[5] that allows the devolution[6] of the discrete problem (Brousseau, 1997). Let me explain how I have set the didactical variables and dealt with the devolution of the situation.

3.2.1 Didactical variables and zero-definitions

The design of a didactical situation that focuses on defining processes depends on the potentialities offered by the zero-definitions connected to the topic, which are codependent on the questions that the solver explores. Three main didactical variables emerge through the mathematical analysis:

- The number of displacements: the exploration of small cases leads to several zero-definitions which can evolve into new zero-definitions;
- The queries: the various paths explored highlight different zero-definitions. For the guidance of the situation, design instructions have to be selected (to explicitly question the minimal aspect or not, to explicitly ask for a definition or not, to engage students in a proof process or not, etc.);
- The nature of the tessellation: other modellings can appear and then the question of the generalization of solutions.

I have chosen to avoid a guidance of the situation such as the Lakatosian one. The MO does not ask for definitions or the minimal aspect or proof. I will now show that the zero-definitions focusing on the “generating” aspect have a real potential evolution through the situation itself.

Zero-definition of a generating set A natural definition can be: a generating set is a set of displacements that allows access to all the points of G. It is not operational to build such sets (except in the graphic register, with only two displacements) and it is very demanding if we want to check whether a set of displacements is a generating one. Thus, such a zero-definition should evolve in a problem where the number of displacements is >2. It can

[4] There is also a question relating to the paths (“Let us take another point, called B. Can we reach B from A? If so, are there different paths to do it? What does “different” mean to you?”), but due to its difficulty, I avoid it here.

[5] In the sense of the theory of didactical situations, the *milieu* includes material or symbolic objects that are able to provide feedback to the students’ actions on them.

[6] “The act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem and accepts the consequences of this transfer of this responsibility” (Brousseau, 1997, p. 230).

Table 1 Problems that make up the situation

Problem	Sets of displacements
Problem 1	$\mathbf{d_1}$: 2 squares to the right and 1 square up
	$\mathbf{d_2}$: 3 squares to the left and 3 squares down
Problem 2	$\mathbf{d_1}$: 2 squares to the right and 3 squares up
	$\mathbf{d_2}$: 5 squares to the left and 2 squares down
	$\mathbf{d_3}$: 5 squares to the right and 3 squares down
	$\mathbf{d_4}$: 1 square to the right
Problem 3	$\mathbf{d_1}$: 3 squares to the right and 3 squares up
	$\mathbf{d_2}$: 2 squares up
	$\mathbf{d_3}$: 1 square to the left
	$\mathbf{d_4}$: 1 square to the left and 3 squares down

evolve into an operational definition including two properties, namely, ALBE and FD. Zero-definitions are validated by the exploration of small cases, and their use for proving that a set is a generating one testifies to their local validity.

Zero-definition of a minimal generating set A first zero-definition can be: a minimal generating set of displacements is a set of three displacements allowing access to all the points of the grid. This definition can emerge during a problem E_2 or E_3 or a problem E_4 where two displacements are dependent for instance. Solving a problem E_4 where the four displacements are independent allows the invalidation of this zero-definition.

A second zero-definition can be: a minimal generating set of displacements is a set of non-dependent displacements (this zero-definition has a geometrical root; it can mobilize knowledge on collineation of vectors in the plane and coplanarity in space).

The status of these zero-definitions should evolve to that of proof-generated definitions through the following proof path: from proving the existence of minimal generating sets of displacements, to building such sets, to proving that a set is a generating one, and to proving that a set is a minimal generating one.

3.2.2 Specificities of the three problems

Problem 1 Only some points of a sector of G can be reached with the two given displacements. The ALBE and FD properties are not verified here.[7] This problem allows students to engage in the situation and to assimilate the rules of the displacements and the questions. Students should conjecture that: in order to generate all the points of the grid, more than two displacements are required. Therefore, this first problem has two main interests:

- The situation is different from the continuous case. Problem 1 dismisses possible students' preexistent concept images coming from vector space of dimension 2 where two vectors are enough to build a basis.
- It can allow the emergence of a conjecture and of zero-definitions of the ALBE and FD properties.

[7] The ALBE property is obviously verified for the generated sector (the distance between every point of the sector and a reached point is inferior or equal to 1).

Problem 2 The set of the four given displacements is a generating one: one can prove that the four cardinal points can be reached with the displacements. Is it a minimal one? We can remove one well-picked displacement without changing anything to the generating aspect. The notion of dependency can also emerge here. In fact, if we remove d_3 or d_4, the generating property is preserved because d_3 is a positive integer combination of d_1, d_2, and d_4 and d_4 of d_1, d_2, and d_3. Thus, the sets $\{d_1, d_2, d_3\}$ and $\{d_1, d_2, d_4\}$ are both generating ones. And they are necessarily minimal because two displacements are not enough, as seen in problem 1. This problem leads to the proof that three displacements can be enough to generate all the points of the grid. This does not allow a conclusion that all the generating sets are made up of three elements.

Problem 3 The main goal of the third problem is to demonstrate that three displacements are not always enough. In fact, the set presented here features four displacements and is a minimal generating one. Therefore, we can reach the following result: the corollary of the exchange theorem (i.e., the existence of dimension, true in a vector space) is false in the discrete case.

How can we prove that it is not possible to remove one displacement of this set? We can prove that it is not possible to write a d_i using the three others (algebraic resolution) or we can prove that the removal of one of the d_i implies the loss of the ALBE or of the FD property.

There are four subsets of displacements with a cardinality of 3: $\{d_1, d_2, d_3\}$ $\{d_1, d_2, d_4\}$ $\{d_1, d_3, d_4\}$ and $\{d_2, d_3, d_4\}$. Figure 6 covers the cases to study.

For the subsets in the middle, we have to verify the FD property. These subsets have d_1 and d_4 in common. We can study the points generated by $\{d_1, d_4\}$ and add d_2 or d_3 in order to characterize the points which can be reached using these subsets. We can do this graphically or by making an arithmetical comment: on the vertical, the displacements d_1 and d_4 can generate only multiples of 3. Thus, the points with coordinates (x; 1 mod 3) and (x; 2 mod 3) (x is an integer) cannot be reached with d_1 and d_4.

- If we add d_2 to $\{d_1, d_4\}$, d_2 brings an even vertical component. Therefore, more points are reached. But horizontally, there are some non-reached points. In order to prove that this subset $\{d_1, d_2, d_4\}$ does not allow us to go everywhere with FD, it is sufficient to find a non-reached point. Using arithmetic means, we can prove that the point (0; 1) cannot be reached. Therefore, $\{d_1, d_2, d_4\}$ is not a generating set for the whole grid.

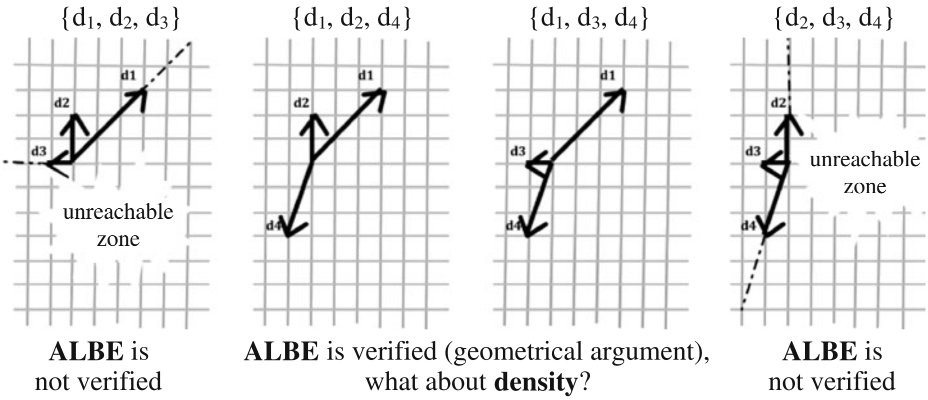

Fig. 6 Subsets composed of three of the four displacements

- If we now add d_3 to $\{d_1, d_4\}$, we can use the same kind of arguments as in the previous case in order to prove that $\{d_1, d_3, d_4\}$ is not a generating set.

In conclusion, it is not possible to remove a displacement from the given set $\{d_1, d_2, d_3, d_4\}$; otherwise, we lose the generating property, either by losing the ALBE property or by losing the FD property. The set $\{d_1, d_2, d_3, d_4\}$ is therefore a minimal generating set.

3.3 Results

I have chosen groups of students (from scientific courses) who have solved the three problems and/or whose strategies involve a variety of ways of reasoning. I give an account of the main results regarding the concept formation process that appear to be the most significant with respect to the defining process developed above. Three groups of students are taken into account for the analysis: the blue one (students A/B/C), the yellow one (students D/P/J/F), and the pink one (students S/R/T/Z).

3.3.1 Summary of students' productions

All the groups have dealt with the three problems, except for the pink group. The representations of the problem used by students were very diverse. They focused on the paths (Fig. 7, left), on the notion of sector and the reached points (Fig. 7, right), or on algebraic modeling.

The blue group gave up the Cartesian equations they used in problem 1 because of the time-consuming and inefficient aspects of this method. Then, they used some knowledge from linear algebra and very quickly investigated the four cardinal points and the ALBE property with linear systems whose resolution was not an obstacle for them.

Conversely, the yellow and the pink groups considered the distance between the discrete problem and linear algebra and geometry (especially in problem 1) and excluded their attempt at using standard basis and algebraic modeling. Their work was mainly a graphical one and evolved toward the search of four unit displacements with a trial and error approach. They took into account directions and parities as well as the ALBE property. The pink group approached the notion of reached points in a sector, but no definitions were stated, in any of the groups.

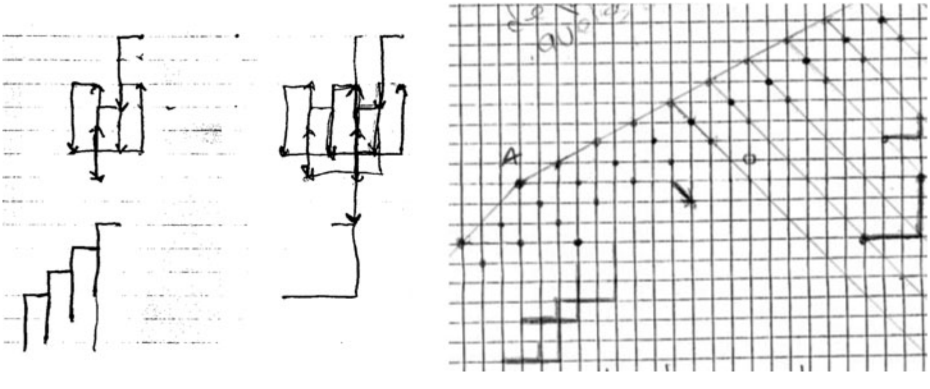

Fig. 7 Some representations of the problem used by students

3.3.2 The discrete problem: a problem in itself

Problem 1 presents the situation as different from one in a vector space, as announced in the a priori analysis. Students are aware that: "this problem is different from linear algebra" because the "rules are different" (the nonnegative integer combinations). They also said:

> P: It is obvious that you have points that you can't reach because we have these displacements. They are not vectors in fact. If you take vectors, it works. Otherwise… (Yellow group—problem 1)

After problem 1, all students conclude that "two displacements are not enough to reach all the points of the grid." However, the yellow group continues to think that "two displacements can be enough" even if these students had concluded in the first problem that it was not possible to reach all the points of the grid with the two given displacements:

> P: We have four displacements. We only need two vectors as a basis.

These turns show the persistence of concept images from analytical geometry and the need to engage students in refining their concept images in the discrete situation. The generalization (from studying small cases) is also an obstacle that students encounter.

3.3.3 The ALBE and FD properties

Students have talked about "reached sectors" in their written productions without talking about density. Even if the generating aspect is understood in a natural sense, the ALBE property is predominant for all the students:

> P: We can't reach the points of this sector. (…)
>
> D: If we use d_1, we can't go higher. And if we use d_2, we can't go more to the left. So, we can reach all the points of these parallel lines. And if we start from one of these points, and we use d_2 or d_1, we go back to one of these points. We can't go outside this set of points. (Yellow group—problem 1; Fig. 7, right)

The notion of sector and the problem of the density in a reached sector appear in one group. In the following excerpt, there is a potentiality for the emergence of the FD aspect, but the students established no definition:

> R: We noticed that if we remove d_4, it is not like removing d_2 where a whole sector disappears. If we remove d_4, a lot of points are removed, but we still have points in a whole zone. (Pink group—problem 2)

Moreover, a natural definition of "generating set" (to reach all the points of the grid) has been produced by all the students, but it has not been connected to the ALBE and FD properties. It has been transformed into an operational property (to generate four points or unit displacements—see Section 3.3.5).

3.3.4 Lakatos' categories of definitions are not enough for the analysis

The previous Lakatosian analysis focuses on mathematical concepts and problematics as more basic than theorems. It provides markers (such as the FD and ALBE properties and the questions on minimal aspects) to characterize students' pre-formal defining processes through zero-definitions. However, such an analysis does not take in charge the real beginning of the creative process. As seen in the previous paragraph, students' actions include some conceived properties, but no zero-definitions. Now, how may we characterize their process?

I have borrowed from Vergnaud's cognitive point of view to develop another aspect of definitions. I extend his characterization of *concepts-in-action* and *theorems-in-action* (Vergnaud, 1996, p. 225) to *in-action definitions* and *in-action propositions* in order to lead an analysis on the students' invariants in the context. An in-action definition is a statement used as a tool (not an object) that enables students to be operational without explicit definition. It comes before the zero-definitions stage and can be weighed against a zero-definition. In the same way that definitions depend on propositions, in-action definition(s) can be linked to an in-action proposition (see Section 3.3.5).

3.3.5 An operational property: to generate four cardinal points (or unit displacements)

I have identified two in-action definitions in the students' productions. There is one for a "minimal generating set" (all the displacements are used during the search of four unit displacements) and one for "non-minimal set" (when one of them is an integer combination of the others). Students do not define these properties, but they can be operational with these in-action definitions (see the following excerpt). I can connect them to a common in-action proposition: to have the four directions represented is a necessary condition in order to have a generating set of displacements.

> B: In order to go everywhere, you try, and if you find (1; 1), it means that you can reach this point. After that, you go on with this system to see if you have solutions. You process with (1; −1) in order to see if you can reach this point and with (−1; 1) and with (−1; −1).
>
> A: Yes! You can see if you can reach the four points around!
>
> B: Therefore, it means that you can go everywhere. If we can prove that, we have no more questions. And if we can't prove it… it means that there are other conditions, it is complicated, or there is a technique.…
>
> A: It is not these four points. We have to take those.
>
> B: Ah yes! We have to take (1; 0) (−1; 0) (0; 1) and (0; −1). We have a lot of systems to solve! *(they solve systems)*. There are many solutions, but proving that there is one is enough. And this solution should work for the others too.…
>
> A: With that, we have proven that we can go everywhere. (Blue group—problem 2)

At this stage, the students' process did not move to a generalization that would have allowed an evolution of in-action definitions (which could be supported by the MO). It is well known that this distance between manipulation and formalization is too rarely approached in the teaching process. But there is another hypothesis to explain this phenomenon. The students tried to reinvest tools and knowledge from their other mathematical courses.

> A: In vector spaces, you can remove one element without changing anything if it is a linear combination of the others. (…)

B: We have to isolate.… And if we use a matrix?

C: But a matrix, it is to compute the determinant! (…)

A: And what will you do with that?

B: I don't know!

(Blue group—problem 2)

MO: What do you mean by linear combination?

P: It means that we can build it with the other two.

D: If we can build one with the others, it is useless… to go everywhere. (…)

D: Are the vectors like you said? Generating… and independent.

J: Then, there is only one way to go there with three vectors.
(Yellow group—problem 2)

These two excerpts show that for some students, the in-action definitions of "minimal" or "non-minimal" are actually imported from their linear algebra course. This exogenous aspect and the fact that the students usually know formal and definitive definitions do not encourage the evolution of their in-action definitions.

The guidance of this situation could have evolved toward something more focused on the defining process in itself, that is to say explicitly asking for definitions and giving the opportunity to the students to reinvest their results in a proof. I have shown that the role of such interventions can be crucial if one wants to leave the in-action definition stage in some kinds of situations (Ouvrier-Buffet, 2006). The MO has tried to engage students in a formalization of their results, even if it had not been planned a priori, but without results:

MO: You have explored several things… there are perhaps some properties…

D: Something like a conclusion…

B: Well, to generalize things… starting from examples… we might make mistakes!

MO: It would be good if you were able to formalize some things…

A: For instance, if we take three displacements, like this, we can reach all the points… making the sum of all the horizontal displacements… something like that?

B: And with two vectors, we can't go everywhere, it is obvious. (…)
MO: This is problem 3. It would be good if you felt the need to verify some things from what you have said so far. (Blue group—problem 2)

This excerpt underscores the need of a long time format and of efficient MO's requests. To go beyond the in-action definitions stage in this situation seems to be difficult in the conditions of this intervention. Indeed, the fact that students used the in-action proposition about the four cardinal points impeded their investigation of the FD property because in this particular case, the ALBE property implies the FD property. More generally, we have to engage students in defining processes and mathematical explorations more often during their courses at university in order to develop their abilities to question concepts and the relations between them. To conclude this article, I will deal with these questions: What kinds of didactical perspectives are brought by this situation? What about further studies of defining processes in education?

4 Discussion and new perspectives

4.1 Future of the situation

The results of the intervention show that defining is not as natural as the Lakatosian model and the analysis of the situation suggest. It also remains difficult to design didactical situations involving defining processes as a necessity for the resolution of a problem. This is not only because students have not yet encountered such processes but also because "all" the features of the defining processes are not yet elucidated. An important part of learning mathematics is actually to become aware of the importance of defining just like the importance of proving.

The definitions of the involved concepts do not come first in the resolution of the discrete problem: they depend on the ability of the solver to question the equivalence relation between the properties of some sets of displacements as well as to be engaged in a proof process. In this intervention, in order to help students go beyond the in-action definitions stage, the MO has to get involved. It could be done at a denomination stage (by asking definitions) and more deeply at a proof stage (by asking students about existence proofs, construction of generating sets with k displacements in order to leave the four cardinal points, cardinality, etc.). The MO can also question the following equivalence relations to bring new conjectures:

- "Minimal generating set" is equivalent to "independent" (false in the discrete case: $\{2; 3\}$ is a counterexample for $\mathbb{N}$);
- "Maximal independent" corresponds to "generating set."

The study of the latter is not easy and implies a definition of "maximal".[8] If there is an unreached point, we can add a displacement d allowing us to reach this point. This new displacement is not a combination of the others. But, in this new set of displacements (including d), it is possible that one of them is a combination of some of the others including d. This underscores the complexity of the discrete situation but also the opportunity to lead students to conjectures and other kinds of reasoning.

In short, it seems that the current framework would be useful to analyze and boost defining processes.

4.2 Nature of a framework to study defining processes

Lakatos has definitely had an influence in teaching and learning mathematics. However, his approach has some limits (De Villiers, 2000; Hanna, 2007). Still, in the present analysis, the use of the expounded theoretical elements shows that the Lakatosian frame of thinking

[8] For instance: a set of displacements is a maximal one if and even if when one adds a displacement, one loses the independence (i.e., some of the displacements become dependent).

(especially the zero-definitions) allows a characterization of concept formation. Moreover, such a process enlightens how the concepts at stake are connected and brings a design for a situation as well as a first analysis of the students' tracks. I have shown the need of using another category of definitions to analyze the pre-zero-definitions stage: the in-action definitions characterize the first encounter with a mathematical object where students deal, in action, with an unfamiliar object or property. We can then determine how the students are involved in the defining process through their actions and their discourse. Even if the students mobilized in-action definitions, they were not able to coordinate several properties (see Section 3.3.5).

The framework also brings elements for the MO to make the defining process evolve (if it is his/her goal) such as: to help students in the recognition of pertinent problems, results, and questions; to engage students in a deeper formalization, in constructive proofs, and in the awareness of constructing, justifying, and writing definitions. But the MO needs to have had a specific experience in such guidance. This is a key point to integrate in the teacher training. Then, a more refined theorization of the management of such situations could be done.

Other theoretical frameworks should be used to characterize the levels of conceptualization of in-action definitions. Indeed, a parallel could be drawn between the level of in-action definitions and the interiorization of an operational conception (Sfard, 1991) on the one hand and the APOS theory (Dubinsky, 1991) on the other hand—especially the level taking into account an "interiorized process with conscious control." This will be the object of a later article with further experiments, exploring how an interiorized process becomes a dynamic one, not only in the action but also in a formalizing process. In fact, definitions should capture our intuitive representation of concepts but also a valid pre-formal idea of a concept. Of course, all these theoretical tools stress the need of a shift to a more structural level. This shift should lead students to efficient zero-definitions. In order to achieve this goal, the guidance of the situation should be clearly definitions-oriented and proof-oriented.

There still remains another moment of the mathematical activity to investigate more deeply: the conceptualization with an axiomatic change of perspective. The movement to greater abstraction being very hard to problematize, it should be taken into account too.

4.3 Toward higher mathematical structures—a link between the discrete and the continuous cases

To study the discrete case is more difficult than to study the continuous case, but it is also more graspable and it provides a natural exploration of some results that are not always true while avoiding excessive formalism. Indeed, a minimal generating system or a maximal linearly independent system is not a basis in the general case. The discrete situation allows an access to the reasoning behind the construction of concepts and then contributes to the development of proof abilities through an active mathematical exploration of the problem (construction of relations between properties and proofs of implications). I intend to explore in further interventions how that mathematical "discrete" experience would be suitable and valuable at university. A comparable study like the one by Harel (1998) is obviously required for such a research study.

The expounded discrete situation also brings opportunities to reinvest the constructed concepts and the ways of reasoning in other mathematical fields. It actually questions the axiomatic system of the fundamental concepts of linear algebra for instance while getting around some well-known obstacles (see Dorier, 2000; Uhlig, 2002) and trying to instill a new kind of concept images system. The suggested discrete situation is therefore a way to situate linear algebra's questions in a wider context, but more easily than by considering vector spaces over a field as a subsidiary notion of modules over a ring. If the discrete problems are sometimes (and even often) easier to grasp than the continuous ones, the mathematics behind can be quite advanced. Such a discussion leads us to further studies. There is room to explore new questions about the teaching of continuous structures, with the help of the discrete ones.

4.4 Opening

Are some concepts more conducive to defining situations than others? Any mathematical concept being given (or any set of concepts), one can anticipate the existence of situations involving defining processes. The situation of displacements shows us that abstract concepts can be the core of a defining problematic, even if one did not expect it, precisely because of the abstract nature of these concepts.

Beyond the concepts themselves, we have to question the place and role of discrete mathematics. This comparatively young branch of mathematics brings graspable concepts which are fruitful to study defining processes and their links with proofs. It also provides a mathematical experience. Discrete mathematics is therefore a field of experiments that questions concepts involved in other mathematical branches as well. Furthermore, discrete mathematics represents a mathematical field that takes on growing importance in our society. We should take this into account for educational purposes and investigate this realm more deeply.

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