Educ Stud Math (2010) 74:223–240
DOI 10.1007/s10649-010-9235-8

Drawing space: mathematicians' kinetic conceptions of eigenvectors

Nathalie Sinclair · Shiva Gol Tabaghi

Published online: 30 March 2010

Abstract This paper explores how mathematicians build meaning through communicative activity involving talk, gesture and diagram. In the course of describing mathematical concepts, mathematicians use these semiotic resources in ways that blur the distinction between the mathematical and physical world. We shall argue that mathematical meaning of eigenvectors depends strongly on both time and motion—hence, on physical interpretations of mathematical abstractions—which are dimensions of thinking that are typically deliberately absent from formal, written definition of the concept. We shall also show how gesture and talk contribute differently and uniquely to mathematical conceptualisation and further elaborate the claim that diagrams provide an essential mediating role between the two.

Keywords Language · Metaphor · Gesture · Diagram · Motion · Time · Conceptual mathematics · Linear algebra

1 Introduction

The goal of our research has been to extend Lakoff and Núñez's (2000) work on understanding the embodied roots of how abstract mathematical concepts are created and understood by mathematicians. Like these authors, we adopt the perspective in which all human thinking is ultimately rooted in embodied activities. However, unlike these authors, we focus on the conceptual roots of individual mathematicians—ideational mathematics—and less on the conceptual roots of the formal, written mathematics of textbooks and research articles—conceptual mathematics (see Schiralli & Sinclair, 2003)[1]. In particular, we wish to explore the way in which time, motion and gesture are involved in ideational understandings of mathematicians.

[1]The distinction between *ideational* and *conceptual mathematics* resembles Tall and Vinner's (1981) distinction between concept image and concept definition, respectively.

N. Sinclair (✉) · S. Gol Tabaghi
Simon Fraser University, Burnaby, BC, Canada
e-mail: nathsinc@sfu.ca

A motivating interest relates to the proliferation of research in mathematics education on the role of gesture in student thinking and learning. The recent special issue of Educational Studies in Mathematics (2009) draws attention to the importance of gestures in understanding human thinking as well as to the intimate connection between gesture and language. We would thus like to bring gesture to bear on furthering our understanding of how mathematicians conceptualise objects. Moreover, as Sfard (2009), in her commentary on the special issue, suggests doing, we would like to examine the differential roles that gesture and language have on communicating mathematical ideas.

A further motivating interest concerns the recent advent of computer-based technology that offers rich and diverse dynamic representations of mathematical ideas, whether in dynamic geometry environments or dynamic statistics ones, not to mention the web-based smaller visualisation tools. Based on recent arguments in cognitive science, as well as historical sources, we hypothesise that even before such environments were available, ideational mathematics drew extensively on understandings that involved both motion and time. Given the very atemporal nature of conceptual mathematics (see Pimm, 2006), such origins may seem surprising. Indeed, one can read the historical development of mathematics as an attempt to get rid of both motion and time, whether it is the arithmetisation of analysis in the nineteenth century, the refusal to admit "mechanical" curves produced through moving objects as being geometry (Mancosu, 1996), the attack on Cavalieri's principle of infinitesimals (Palmieri, 2009), the constant criticism of Euclid's (and also Cavalieri's) method of superposition, or the very distinction proposed by Aristotle between the nature of mathematics and physics—one based largely on the immobility of the former and the mobility of the latter. Indeed, following on Aristotle's convictions, Russell (1903) protests the description of objects in motion as lending them unwanted material agency:

> To speak of motion implies that our triangles are not spatial, but material. For a point of space *is* a position, and can no more change its position than a leopard can change its spots. The motion of a point of space is a phantom directly contrary to the law of identity: is it the supposition that a given point can be now one point and now another? (p. 405)

Despite Russell's admonition and the atemporal, static nature of contemporary mathematical discourse, Núñez (2006) argues that mathematicians do seem to communicate in ways that suggest they think of mathematical objects in motion. Further, although the activity of mathematical proof might entail detemporalisation, decontextualisation and depersonalition (Balacheff, 1988), one could argue that many new revolutionary ideas in mathematics arise from a recourse to motion[2]. Many have written about the disjunction between formal, written mathematics and mathematics communication in other contexts, including the classroom, but also informal writing between mathematicians (see Solomon & O'Neill, 1998). In this paper, we are concerned with understanding the way mathematicians might be seen as communicating with themselves.

Recent research in mathematics education that studies the way in which bodily actions can foster conceptual development seems to focus more strongly on the motion of the body (Nemirovsky & Borba, 2003; Robutti, 2006; Wright, 2001). Even before this contemporary work drawing on theories of embodied cognition, we point to Papert's (1980) notion of "body syntonicity," which is about using the turtle as a means of developing bodily

[2] This observation, made by Mariolina Bartolini-Bussi (personal communication), can be seen in Archimedes' and Cavalieri's development of infinitesimals, as well as Desargues's founding of projective geometry.

awareness that evokes mathematical ideas. Our research focuses less on studying the way in which carefully designed kinesthetic experiences (kinesthetic in the sense of being mediated by bodily movements and tensions) that support students' learning of concepts such as slope (see Wright, 2001) and more on how people might use kinesthetic abilities to develop mathematical conceptualisations—in contexts not necessarily designed to make them move their bodies in mathematical ways. To do this, we focus on the understanding of mathematicians: we are interested in the motion and time basis of ideational mathematical that arises in contexts that are not as pedagogically intentional. We strongly believe in the value of intentionally kinesthetic activities for learning, but we wish to understand better the mechanisms under which mathematicians use or impose motion to make sense of mathematical objects and ideas.

2 Theoretical perspectives

The notion of ideational mathematics (developed in Schiralli & Sinclair, 2003) is based on the tenets of embodied cognition, as described by Lakoff and Núñez, (2000), wherein abstract mathematical understanding emerges out of concrete sensory motor experiences. In their book *Where Mathematics Comes From*, Lakoff and Núñez fail to differentiate the term 'mathematics', nor do they indicate whether metaphor might function differently depending on whether one is *learning*, *doing* or *using* mathematics. The interpretation of 'mathematics' implicit in their work has been called *conceptual mathematics* (CM): this is mathematics as a subject matter or discipline. As Schiralli and Sinclair (2003) note:

> CM concepts are not necessarily the same as the mathematical ideas that individual mathematicians (experienced or novice) may form of them. CM concepts are public representations; they exist outside in a public space of shared meanings. As such they are best kept distinct from the internal representations that given people will form of them. (p. 81)

Ideational mathematics (IM) refers to the way in which an individual represents these concepts to herself; IM will be influenced by many experiential and genetic factors, including the particular tools and discourses available. In order to study the IM of mathematicians, we follow in the tradition of Núñez (2006) in which gesture and speech analyses are used to uncover the dynamic quality of mathematicians' thinking. Since we are particularly interested in the kinetic aspects of ideational mathematics, we begin by considering some of the ways in which motion has been studied in the context of mathematical thinking.

Historically, in mathematics education, the notion of visual intelligence has been much more prominent than that of motor intelligence. For example, Presmeg has done extensive work on visual ways of thinking in mathematics which she distinguishes from non-visual ways of thinking. Within the visual, she proposes five categories: (1) concrete pictorial, (2) pattern, (3) memory images of formulae, (4) kinesthetic and (5) dynamic imagery (Presmeg, 1986). In her last two categories, we see a strong link with motion. Including the kinesthetic within visual intelligence may seem quite appropriate, especially since motion is often perceived through visual means[3]. However, there are two reasons for distinguishing the two

[3] Indeed, based on her interviews with over 70 mathematicians, Burton (2004) identifies three styles of thinking of mathematicians, including one that she calls visual, and to which she adds (parenthetically), sometimes dynamic. Her analyses focus entirely on verbal utterances made by mathematicians, and unfortunately, she provides no examples of what she counts as a visual, dynamic style.

abilities: (1) there are some motions that are not perceived visually, some of which may be mathematically relevant, and (2) the motor cortex and the visual cortex are two distinct (though related) areas of the brain.[4] Indeed, Seitz's (2000) emphasis on the motor activity as the basis of cognition draws in part on the fact that brain areas for motor and cognitive functions overlap and are interdependent in developing understanding. It also draws on the presence of "mirror neurons," which imply a close association between one's own intended movements and one's observation of movements performed by another—in other words, seeing someone else do something (walk, gesture, smile) fires the same neurons as doing the thing oneself.

As mentioned above, in their study of "where mathematics comes from," Lakoff and Núñez (2000) focus on the sensorimotor roots of conceptual mathematics and on the conceptual metaphors that are used to move from everyday human activity to abstract mathematics. Some of the metaphors they identify have evident kinetic roots, such as the "motion along a path" grounding metaphor for arithmetic, whilst others, such as "object collection" are described in static terms despite the fact that the grounding metaphor surely involved, in this case, actually collecting objects with one's hands and moving them into piles. Because of their focus on conceptual mathematics, it is not surprising that many of the grounding and conceptual metaphors are stated in static and timeless form (for example, sets are containers, continuity is gaplessness).

In a later work, Núñez (2006) studies the origins of both ideational and conceptual mathematics by analysing written descriptions of mathematical objects as well as real-time explanations of these objects by a mathematics lecturer. By attending to the gestures used by this mathematician, Núñez argues that his explanation of the notion of limit involves the use of motion (the gesture of his hands) and moving objects—despite the fact that the formal concept of limit involves no motion. Núñez's conclusions are especially interesting in light of Seitz's (2000) claim that motor activity is the basis of human intelligence and also in light of evidence offered by mathematicians such as Thurston (1994) who includes both "process and time" and "vision, spatial sense, kinesthetic (motion) sense" as two of the six major facilities that are important for mathematical thinking (pp. 4–5).

Whilst not specifically focused on mathematics, as was the case of Lakoff and Núñez (2000), Seitz (2005) proposes four different types of body-based metaphors, including (1) *perceptual metaphors* (spaghetti as a "bunch of worms"), (2) *perceptual–affective mappings*, (3) *enactive metaphors* (also called movement–movement metaphors, which involve metaphoric associations between moving things: the to-and-fro motion of a long string and a swing) and (4) *cross-modal metaphor* (in which associations are made from one domain—say, motion—to another—say, perception: two pencils akimbo are leaning over to tell each other a secret). Seitz would thus classify Lakoff and Núñez's "motion along a path" grounding metaphor as an enactive metaphor. We note that the last two types of metaphor in Seitz's scheme specifically involve motion and can be used to investigate the way in which ideational mathematics draws on kinesthetic experiences.

The importance of motion in human intelligence has also been suggested by the cognitive linguist Talmy (1996) who, drawing on analyses of written and spoken words, argues that humans often impose motion on otherwise static objects. For example, when

[4] Furthermore, there are several examples of blind people doing mathematics who seem to draw on kinetic ways of thinking. For example, Helen Keller (1969), who was both deaf and blind, described straight lines as "I feel as if I were going forward in a straight line, bound to arrive somewhere, or go on forever without swerving to the right or to the left". Also, Healy (2009) reports a blind student conceptualising a pyramid in terms of a gesture that involves wrapping his fingers around the base and tracing up the sides until his fingers meet at a point.

describing a road as "going along the coast", the use of the active verb "going" suggests that the road is actually moving somewhere. Núñez has used Talmy's notion of "fictive motion" to analyse mathematical speech and found several instances in which mathematical objects were described as being in motion despite begin defined as static entities[5]. However, Núñez points out that "no entities are actually moving or approaching anything" and that the statement is accompanied by a Cartesian graph, which is entirely static with no insinuation of motion. Of course, this does not mean that the entities are static to the mathematician (*pace* Aristotle); that is, in terms of ideational mathematics, it may well be that the entities have mobility. When Núñez says that the entities are entirely static, he means that their formal mathematical definition casts them as fully formed, idealised and immobile.

Given the close connections between gestural, temporal and kinetic aspects (we use the word *kinetic* to describe the motion of material bodies, but not necessarily human bodies) of human thinking, one goal of our work is to examine how they all function in ideational mathematics, if at all, and how they interact with the historical and material development of mathematical concepts. Understanding the gestural, temporal and kinetic aspects of thinking seems particularly interesting in the discipline of mathematics given the tension that time and motion have caused historically and the modern practice of removing both from the mathematical discourse.

3 Studying mathematicians' conceptions of mathematical objects

3.1 Participants and setting

We designed our interviews using a set of questions aimed at eliciting mathematicians' conceptions of a variety of mathematical ideas (spanning K-12 and undergraduate mathematics). We began the interviews by explaining that we were interested in finding out more about how mathematicians actually thought about various mathematical concepts and that we conjectured that their ways of thinking about these concepts would differ from the ways these concepts were presented in textbooks. We then asked general questions about their field of interest and their current work. Finally, we asked them the following question for each of a set of mathematical concepts X: "What do you think of when I say the word X."

We interviewed a total of six mathematicians whose interests were in both pure and applied mathematics and who were all members of a medium-sized mathematics department in Canada. For one of the participants, only the audiotape recording was available. Each interview lasted between 1 and 1.5 h. Interviews were videotaped and transcribed. We provide the area of expertise for each of the six mathematicians below:

- LG, combinatorics, graph theory, optimisation
- PT, model analysis and simulation
- NN, applied numerical analysis
- KM, number theory and approximation theory

[5] For example, Courant and Robbins (1978) write "The hyperbola approaches more and more nearly the two straight lines $qx \pm py = 0$ as we go to farther and farther from the origin, but it never actually reaches these lines. They are called the asymptotes of the parabola" (p. 76). In using the active verbs in the following phrases "approaching more and more" and "go out farther and farther" and "reaches", the authors seem to be describing moving lines and changing distances.

- JJ, applied discrete mathematics
- JS, fluid dynamics, industrial mathematics and scientific computing

We will present the range of descriptions given by the mathematicians, focusing on the one specific concept of eigenvector. We have two main tools for analysing the responses of the mathematicians, both geared to providing insight into ideational mathematics. The first one, drawing on discursive methods, involves looking for evidence of temporal or kinetic conceptualisations through speech, which may involve the use of fictive motion and/or the use of action verbs to describe mathematical objects. As mentioned above, we stress that the motion is only fictive from the perspective of the formal definition of the object and not from the perspective of the mathematician. The second draws on gesture theory and, in particular, involved identifying gestures that accompany the description of mathematical objects. Of course, many gestures are performed in time and may thus seem kinetic in nature—though this is certainly not always the case.

3.2 Conceptualising eigenvectors

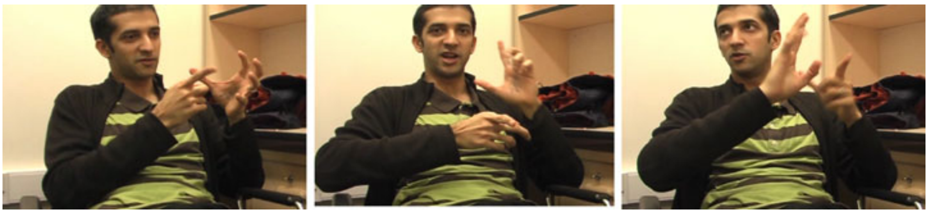
Each mathematician was asked, “What do you think of when I say the word ‘eigenvector.’ We recall here the formal definition of an eigenvector, found in many introductory linear algebra textbooks: “An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ” (Lay, 2006, p. 303).

3.2.1 KM: arrows, clocks, transformations

We begin with KM.

> Oh, things that do not move when you try to transform them. So in my mind I have a picture of bunch of arrows and they are sitting like imagine like in a unit circle so there is a unit circle and bunch of vectors pointing like a clock to different directions [*left hand cupped in a circular shape as he extends his right index finger and moves it in a clockwise direction*, see Fig. 1a] and you just do some transformation to all of them [*moves his left hand from right to left*] and they all bend and twist and do something [*moves his hands, fingers extended, across each other*, see Fig. 1b], but some of them will not change the direction they are pointing so those are the ones [*moves his right hand, fingers extended, in a linear path*, see Fig. 1c].

In his first sentence, KM glosses eigenvectors as “things that do not move when you try to transform them.” This suggests a temporal aspect to his thinking since some transformation has to occur before one can identify the eigenvectors. Although he describes



a. right hand showing unit circle b. vectors bending and twisting c. vector not changing direction

Fig. 1 Gestures associated with KM’s description of eigenvectors

eigenvectors as the things that do not move, we anticipate a kinetic component to his thinking as well since other vectors will be moving. KM follows up with a much more visual image of arrows in a unit circle and the visual metaphor of a clock. He then describes how the (other) vectors will "bend and twist" under some transformation.

KM goes on to explain "I remember learning it, the definition was, you know, matrix times vector is some lambda times vector"—note the lack of time or motion—"and it was okay, whatever, but then I remember our teacher actually put on a picture of just a bunch of vectors [inaudible] going out from the origin and he said let's just apply this 2 by 2 matrix to all of these and what you can see when you apply it most of the vectors just move […]." Since these images were not actually moving on the blackboard, KM has had to produce and coordinate the movements, which he describes as "they turn a little, shrink and turn you know, they change but the change is becoming less and less as you come toward one of the vectors," of the vectors under transformation.

Turning to his gestures now, we first see him using his hands to communicate the image of the clock (the cupped hand) and the vector (the left index finger), the latter moving around the clock. Likewise, his fingers overlap when he describes the bend and twist of the vectors and then extend straight, moving along a linear path to describe the ones that do not change. These gestures offer further evidence of a strongly time and motion-based conceptualisation. The fingers first move around, then bend and twist, and then extend straight, thereby describing a process occurring over time. Furthermore, the circling, extending and crossing depict the movement of the vectors themselves.

In analysing KM's data and trying to be more precise about the source of the ideational mathematics being described, we find that KM draws on Seitz's notion of an enactive metaphor where the unit circle is the clock and the vectors going around the unit circle are the hands of the clock. The transformation distorts the hands of the clock (and even the shape of the clock, as we will see in JS's description). KM's kinetic thinking thus derives from an association between everyday mobility and mathematical mobility.

3.2.2 PT: stretching vectors with his hands

In the next example, PT focuses much more on the behaviour of the eigenvector itself than on the effect of the transformation on all the vectors of the unit circle.

> I think of a matrix I think of applying a vector to, no applying a matrix [*lifts left hand*] to the vector [*moves left hand toward right hand and then back again*] and then what you get out is another vector [*moves his right hand, bended fingers*], that is in the same direction but either stretched [*clenches his hands and extends his arms*, see Fig. 2a] or shrunk [*brings his clenched hands and extended arms together*], but you also have to be aware of like [*moves both hands up and down in an alternate fashion*] flip direction [*hands moving toward each other and index fingers pointed to each other*, see Fig. 2b] or it could just sent to zero [*moves left index finger down*, see Fig. 2c] and that will still count as an eigenvector.

With both his words and his gestures, PT's description has a temporal quality: the words "*applying* a matrix to a vector" and "what you get out" and the gesturing of the left hand moving toward the right hand—which accompanies the word "applying"—the right hand then bending at the fingers, as he describes the vector that you get out. Thus, thinking about the eigenvector also involves a process over time of taking one thing (the matrix) and having it act on another (the vector) and then producing a third thing (another vector).

a. vectors stretched. b. vector flips direction. c. vector sent to zero.

Fig. 2 Gestures associated with PT's description of eigenvectors

PT then uses several active verbs (stretch, shrink, flip, sent to zero) to describe what happens to the vector that "is in the same direction" (the eigenvector), suggesting that he conceives of the eigenvector as actually moving or morphing. The subject of all these action verbs is the vector itself—it is not him doing the stretching and shrinking. It is not clear whether PT has ever seen this motion before (on a computer, for example) or whether he has imposed himself the stretching and shrinking to the phenomena of a longer or shorter eigenvector. In the first case, PT would be drawing on a memory of motion. In the second, he would be producing a continuous motion between two static states. This motion is communicated both through language and through gesture as he begins using both of his hands to represent the vector stretching and shrinking, as if he is actually holding the vector (with the clenched hands) and stretching it apart or shrinking it together—the vector itself is actually physically transforming under the effect of the matrix (his hands). When he describes the vector "just sent to zero," he only uses one hand, with an extended index finger pointing down.

Finally, in contrast then to KM, who uses his fingers to represent the vectors, PT uses his body to act on the vectors (stretch or shrink them). PT's gesture might suggest another enactive metaphor in which he sees eigenvectors as elastic bands that one can stretch or shrink by pulling or pushing on the ends. Thus, the gestures provide some insight—that the words do not—into the nature of motion involved in mathematical thinking.

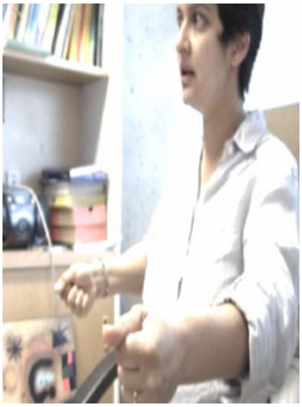
3.2.3 NN: eigenvectors as stresses

We now consider NN whose description of eigenvectors does not mention vectors at all:

> Stresses, so if, so I am thinking about a plate being pulled out [*clenches her hands with palms up and moves her arms back and forth, as if holding a horizontal steering wheel*, see Fig. 3] so it's gonna move along its principal stress directions.

NN describes "stresses" on plates that are "being pulled out" and then motion along principles "stretches out." In this example, the temporality is mostly associated with the motion of the plates and not so much with the process of applying a transformation and then getting some kind of output. Indeed, the eigenvectors are the stresses. Her gesture seems to refer to the back and forth motion of the plates and, possibly, to the general direction in which the stretching occurs. Thus, the motion seems quite primary, with the temporal being a consequence of this motion.

Fig. 3 Gesture associated with NN's description of eigenvectors



We see another example of an enactive metaphor in which stresses on the geographical plate of rock being pulled out is associated with the behaviour of an eigenvector—though NN describes mostly the source domain. This enactive metaphor suggests that NN uses kinesthetic memory to describe the motion of the plates, that is, that she has actually seen this motion and is associating it with the way that eigenvectors should behave.

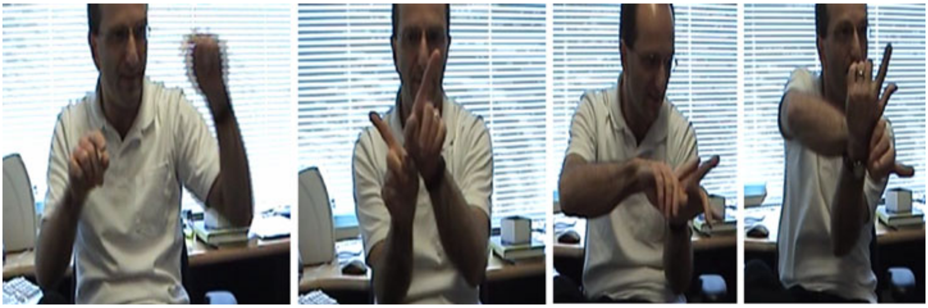
3.2.4 JJ: what a matrix does for its living

Whilst we hypothesised about the origins of PT's dynamic imagery, JJ's conceptualisation is definitely affected by his use of computer visualisation tools.

> […] you take a vector [*lifts and stretches his left hand*] and you map it to something else and there is a nice tool which I can probably pull out where you can see that you can track input vector [*moves his right index finger*] you move the output vector [*moves his left hand*, see Fig. 4a]. You go up here and sort of move around [*rotates his hands, extends right index fingers*, see Fig. 4b] as you play with this. If you set the matrix up by some inputs [*right and left index fingers coming towards each other*, see Fig. 4c] they're gonna come inside and then obviously you say what is the important direction when the two line up [*right and left index fingers come together*]. And some matrices where they just chase each other around [*brings his wrists together then rotates his hands as index fingers are extended*, see Fig. 4d] so you know eigenvector that is a special kind of matrix […]. So the eigenvector is the hidden structure of the matrix that tells you what the matrix does for its living.

JJ first describes how the tool allows one to manipulate the input vector and see what happens to the output vector, and this is done in temporal terms. Unlike the previous mathematicians, he also describes how you can change the matrix as well and see what happens to the system. Here, the "mov[ing]" of the output vector relates to the movement of a concrete vector on the screen, as does the "com[ing]' inside," the lin[ing] up, and the "chas[ing] each other around." JJ thus uses kinesthetic memory to describe eigenvectors.

In both the "two line up" and the "chase each other around," we see two more examples of enactive metaphors. In the case of the second one, JJ associates the movement of the output vectors on the screen as the input vector changes with things (people, animals?) chasing each other around.



a. move output vector b. move vector around c. vectors line up d. chasing vectors

Fig. 4 Gestures accompanying JJ's description of eigenvectors

In terms of gestures, PT first uses his finger to represent a vector. He then rotates his hand as he describes moving the vector, perhaps in a circular fashion around the origin. He again uses his index fingers to show two vectors lining up and then his whole hand rotating around the wrist, which may represent the origin, the fingers extended. Whilst PT is describing what he might see on the screen, he is using his body (fingers and hands), and thus motor logic, to communicate as well. PT might have moved his body in the same way prior to using the computer-based tool, but it may also be the case that the tool has given him a way of moving his body and, in essence, taught his body how to think of eigenvectors.

It is worth pointing out that each of the mathematicians discussed thus far has employed gestures that involve the coordination between their right and left hands. Except for NN, the gestures have also involved each hand engaged in a different motion so that the overall meaning of the gesture depends on the coordination of two motions. This strikes us as interesting given the sharp divide made in mathematics between constructions involving one single motion (the circle, the parabola, etc.) and those involving two (the quadratrix, the spiral, etc.), with the former being deemed 'merely' mechanical. Descartes, for example, argued that double motions lack precision. However, as Netz (2009) points out, they also squarely depend on time in a way that single motions, which can be effectuated almost instantaneously, or over a long period of time, or even in stages, do not. Indeed, when Archimedes describes the spiral, Netz sees him as doing physics, not mathematics (echoing Aristotle's distinction). With their hands, our participants are, in a sense, describing mechanical systems in which one hand becomes the moving matrix whilst the other takes on the role of moving vector.

The final sentence of JJ's transcript describes a matrix as having not only motion, but agency, and even personality, that is, as something that *does* things "for its living." The eigenvector is now described as a hidden structure which no longer moves or turns, but is reified into an object.

3.2.5 JS: arrows and ellipses

JS's description shares features with both KM and PT.

> So at least in 2D well, eigenvector vectors [*moves back and forth both index fingers*] the first thing that comes to my mind is a little arrow, right. But in two dimensions [...] I get a picture of an ellipse in my head because there is a nice way of visualizing

> eigenvectors of 2 by 2 matrices where essentially if we look at vectors of length one [*index and thumb held up showing space between them*, see Fig. 5a] and you multiply those vectors by a matrix sort of if you look at any vector [*rotates his left index finger clockwise*] sort of around just imagine the circle [*traces a circular path with his left index*, see Fig. 5b] and you look at any vector [*extends left index finger*] that goes from the centre of the circle to the outside [*makes a hole using his index and thumb*, see Fig. 5c] and you apply a matrix to it what it does it rotates [*turns his left index finger*], stretches [*moves his hands away from his body*] so basically what happens, say, if I pick this vector [*extends his left index*] that will get rotated [*rotates his index finger*] and stretched [*pulls both hands back, index fingers extended*, see Fig. 5d] so what happens if you rotate all the vectors around the circle, if you multiply all the vectors around the circle by that matrix you end up getting vectors but its an ellipse so some of them get shrunk [*brings his arms and extended hands close to each other*] some of them get stretched [*moves his right hand and arm up and moves his left hand and arm down*] but they all get rotated [*rotates his left hand*] so I see an ellipse actually.

JS has a strong image of arrows, which may act as a simple visual metaphor (a segment with a direction) or may also be enactive in the sense that arrows not only move position through space, but also change direction. The ellipse is another strong visual component of his description, especially in terms of how the circle of vectors transforms into an ellipse of vectors by applying the matrix. Like PT, he describes the rotation, stretching and shrinking of the vectors in active terms, suggesting the vectors as being imbued with motion. The application of the matrix to the vector also has a strong temporal component. Interestingly, he does not actually describe the eigenvector and instead develops the image of the transformation of the circle to the ellipse under the mapping of the matrix. It may be that the image of the ellipse itself constitutes his idea of eigenvector in that the long axis will represent the eigenvector. This "way of visualising eigenvectors" seems to be something he has seen before, but not necessarily in a dynamic context, in which case he would be imputing the motion.

In terms of gestures, the index finger is used once again to communicate the idea of vector as well as a vector rotating. Interestingly, as with PT, the gesture that accompanies the stretching and shrinking of the eigenvector involves his whole hand and his body, perhaps because stretching cannot as easily be communicated with an extended finger. The second time he talks about rotation, his gesture involves the turning of his whole hand. We thus see once again the use of motor logic that is being used to describe the effect of the 2 by 2 matrix on a circle of vectors—but this time within a strong visual image of the circle mapping onto an ellipse.

a. vector length 1 b. imagine the circle c. vector from circle centre d. stretched vector

Fig. 5 Gestures associated with JS's description of eigenvectors

3.2.6 LG: resonating and swinging

As with NN, LG seems to have a more physical conceptualisation of eigenvectors:

> I guess eigenvector reminds me of resonance. So if you're in a big tunnel and you start to sing and hit the right note it really starts to go really loud and resonate in your ears. And you say what is special about that frequency. And the same thing happens you know when you're pushing a swing if you just push somebody on the swing you're gonna get knocked over but if you get just the right rhythm you get really high up and there's a resonance. And the same thing with the matrix if you multiply a matrix by a vector you get another vector and it may or may not be related but if you get a resulting vector which is the same direction as the one you started with you're gonna get a resonance there. You can multiply it again and it's just going to get longer again and longer again. It's not just gonna get all over the place.

LG draws less on visual images than does NN: the resonance in the tunnel derives from an aural metaphor, with the eigenvector being associated with loud resonating. The metaphor is thus a cross-domain one. The swinging example is very physical as well and draws on an enactive metaphor, with the eigenvector being associated with the "right rhythm" of the back-and-forth swinging motion. LG speaks of singing and of swinging and describes the eigenvector in both cases as the optimal case along a continuum of resonance and height. Both the resonance and the swinging are temporal in nature, as is the process he describes of multiplying a matrix by a vector. However, his talk about eigenvectors is much more static: the eigenvector is the resonance; it is in the same direction.

Both the physical metaphors are plausibly ones he has undertaken himself in which case he draws on previous kinesthetic awareness of his own body (producing sounds and being pushed on a swing) in order to think about eigenvectors. He later explains how he develops intuitions for eigenvectors by building visualisations in Maple. In this case, he seems to be using the computer to create new kinesthetic (and visual) experiences, which he will be able to draw on in his work: these new motions will form part of his kinesthetic memory. In his response to the quadratic function prompt ('What do you think of when I say the word quadratic function?'), LG talks about creating his own imagined motions using neither the computer nor his own physical experiences: he extends the quadratic function into three dimensions and describes varying the values of its coefficients: "Sometimes I get a sphere or an ellipsoid or a paraboloid or hyperboloid with two sheets or a football shape thing and I try to image them [the coefficients] changing continuously […] I often amuse myself trying to picture these changes." LG's interview thus points to three different sources of kinesthetic thinking: motions physically experienced by the body, motions generated by tools and perceived by the body, in this case through vision, and motions orchestrated in one's imagination.

4 Discussion

Despite its single, atemporal and static formal definition, the mathematicians described the eigenvector—both with their words and with their gestures—in ways that suggest diverse (including temporal and kinetic) ways of thinking. Four of the mathematicians focused specifically on the transformations of a vector under matrix multiplication ('shrink,' 'rotate,' 'bend,' etc.), as well as the characteristic 'personality' of the eigenvector ('in the same direction,' 'no changes in direction,' 'lines up,' 'comes inside'). In contrast to the algebraic statement $Ax=\lambda x$ (and to the usual sequence of manipulations that follows in which the eigenvalues are first determined, and then the associated eigenvectors), none of the mathematicians talked solely about an algebraic equality. Instead, the multiplication of a matrix by a vector was described in terms of A acting on x to transform it, with the eigenvector corresponding to a near identity transformation. For NN and LG, the temporal component related more to the source of the enactive metaphors they used and less to the vectors themselves. Whilst we do not have videotape data for LG, it is interesting to note that NN was the only one to not use her hand or fingers to represent a vector; we suspect the LG's gestures were similarly less concerned with representing actual vectors than with communicating a sense of the dynamics of the situations of singing and swinging.

Of the metaphors we identified, all of them contain a motion component, whether they are movement-to-movement metaphors (enactive) or cross-domain ones. This was not true for all the concepts we used in the interviews. For example, NN responded to the prompt of quadratic function by saying that she thinks of a goblet, thus using a perceptual metaphor. However, even with less spatiotemporal concepts such as commutativity[6], five out of the six mathematicians described it using motion-based metaphors, describing to some kind of switching through gestures involving crossing one finger over another or rotating the hand by a half-turn.

Turning to the question of the different roles language and gesture might play in mathematicians' conceptualisations, we suggest that for these mathematicians, gesture offers more possibility than spoken language for expressing continuity, time and motion. Recall PT whose gesture of stretching and shrinking (as if holding the end of an elastic band) communicates a more physical, causal aspect of the transformation he describes in words[7], saying the vector is "either stretched or shrunk," which, spoken in the present passive tense, no indication of the agent of stretching or shrinking—the agent is of course the matrix—indicates a change in state that occurred instantly and discretely, not continuously over time as communicated by his gesture. For NN also, the language describes the setting of the metaphor (shifting plates) whilst the gesture gives a sense of the dynamical system involved. Of course, there are many mechanisms of spoken language—such as tone of voice, cadence, rhythm and pauses—that communicate the smoothness and continuity that gestures often do. However, perhaps for these mathematicians, for whom mathematical writing—which omits such prosodic elements and which is meticulously detemporalised (see Pimm, 2006)—is such a defining and imposing linguistic form, gesture remains as the less constrained form of communication.

Referring again to NN, we see yet another possible differentiating role of language and gesture that is related to the genesis of the concept of eigenvector. Her gesture seems to communicate an eighteenth century idea of eigenvector, whilst her language of plates and principal stresses describes a twentieth century application. According to Hawkins (1975), the concept first arose in the eighteenth century as mathematicians such as Bernouilli, D'Alembert and Euler were studying discrete mechanical systems. These involved, for example, describing the motion of a string fixed at one end and swinging at the other. In the latter part of the eighteenth century, D'Alembert attacked the string problem by defining a system of differential equations with constant coefficients describing horizontal and vertical

[6] We included commutativity (along with area, distance and symmetry) so as to have examples of concepts that do not point back in time to a spatiotemporal conception—this based on Harris's (2008) critique of Núñez's work which used limit and continuity, both arguably already imbued with temporal conceptions,

[7] Ochs, Gonzales, and Jacobyet (1996)) note a similar phenomenon with physicists who, even in their spoken language, associated themselves with the physical phenomena under consideration, for example: "When I come down I'm in the domain state."

displacement of the string. He characterised the roots of these equations in terms of the stability of the mechanical system under consideration. In a sense, NN's gesture, in which she moves her arms back and forth towards a state of equilibrium, focuses on the underlying behaviour of the eigenvector. Indeed, as an applied mathematician, NN may be more concerned in her work with the characteristic features of mechanical systems, such as their stability or their direction of displacement.

Similarly, whilst LG uses the noun "resonance" to describe eigenvectors, we conjecture that his corresponding gesture would involve some kind of continuous motion in which the singing is getting louder and the swinging higher, that is, where the gesture is used to communicate something about the behaviour of the voice and the swing. It may be argued that the gesture communicates a less reified conception, and since the mathematical discourse involves constant reification (see Sfard, 2008), we should expect initial conceptualisations to be less reified than later ones. However, since reification is usually taken as a sign of increasingly sophisticated mathematical thinking, we would not want to imply that the talk is more mathematical than the gesture. Indeed, there are many cases in which mathematical thinking requires attention to the *behaviour* of mathematical objects (see Sinclair, Healy & Sales, 2009).

Given these possible differing roles of language and gesture, it may be fruitful in the context of teaching and learning to identify more explicitly gestures that could underlie reified mathematical ideas, especially if, say, the spoken 'public' language of the lecturer (and even more of the textbook) tends so much to the static and discrete. Indeed, it would seem worth further research to find out whether, in the context of a mathematics classroom, students attend at all to gestures and whether they notice the unwitting mobility of their lecturers' mathematics.

There is another reason why the gesture may be seen as the more natural and frequent communicator of mobility and this is related to the extensive use of the hand in mathematics to draw and sketch. Given the importance of diagrams in mathematics, we explore this idea in more detail in the following section.

4.1 Diagrams and gestures

KM, JS and JJ seem to use their gestures to illustrate the situations they are describing, whether it is the mapping (one hand moving onto another), the direction and motion of vectors around a unit circle or the behaviour of the set of vectors under transformation. One gets the sense of the mathematicians drawing in space much like they might draw diagrams with pencil and paper where the motion of the pencil on the page to exemplify how a mapping or an arrow occurs, unfolds over time. Indeed, KM seemed to be drawing diagrams from memory (the diagram offered by his teacher), with the diagrams of JS being based on dynamic images.

In his study of the practice of mathematics, Châtelet's (1993) sees the gesture as a core mediating link between the body and the mathematical diagram[8]—and, whilst committed to the bodily sources of mathematical knowledge, seems much more interested in the bidirectional interaction between mathematics and the body than are Lakoff and Núñez (2000). Diagrams, for Châtelet, "capture gestures mid-flight" (p. 10). They do not simply illustrate or translate something already known; diagrams come from gestures, they carry the meaning of the gesture—the motion, the visceral, the corporeal—and they convey "all this talking with the hands" (p. 11). Indeed, for Châtelet, the making of a mathematical

[8] He also sees the metaphor as another mediating link between the body and mathematics.

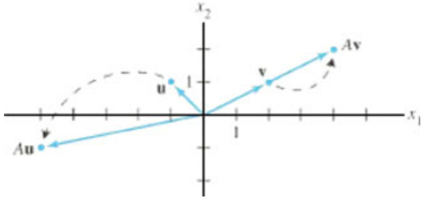


Fig. 6 Textbook diagram accompanying description of eigenvectors

diagram is a symbolic process of condensation and amplification that precedes formalism and that acts as a kind of mid-wife for implicit, intuitive and even irrational thought, and—borrowing the mathematician André Weil's words—for the obscure analogies, murky reflections, furtive caresses and inexplicable tiffs that animate mathematics knowledge[9]. The gesture "cuts out a form of articulation" (p. 8) and gives rise to new gestures, partly by suggesting to oneself new ways of moving in the world.

Thus, in Châtelet's scheme, the gesture precedes the diagram. There is some developmental evidence as well that gesture precedes speech, at least in the case of cognitive mismatches—where the gesture communicates a more sophisticated idea than the spoken word (Goldin-Meadow, 2003). Other researchers have also reported instances in which students communicate their thinking through gestures before they do so through speech (see, for example, Radford, 2009). In watching the mathematicians describe eigenvectors, many of the gestures recalled some of the conventional diagrams of textbooks: for example, JS's rotating hand becomes the curved arrow indicating the turn of a vector (as in Fig. 6). But Châtelet also insists that diagrams create space for mathematical intuition, as we see in KM's case, in which the diagram he was shown by his teacher gives rise to intuitions that evoke new gestures (the ones he makes whilst describing the diagram in our interview). We note that the gestures he uses do not merely depict the diagram since KM's fingers and hands are not static, but instead, dynamically depicting transformation of the vectors.

As a cultural tool then, the diagram may encode gesturing possibilities for learners—perhaps because, as Châtelet says, the diagram carries the motion and bodily meaning of the gesture—and can even evoke new gestures. If Châtelet is right, then the implications for education would be twofold. First, learners would gain more meaning if they were encouraged not only to use gestures in the course of their mathematical activity but also to make diagrams as they work. Teachers often invite students to write reflections or explain their ideas in words, but might diagramming be a fruitful intermediary? Second, if diagrams provide a space for intuitions, then an increased use of diagrams as focal technologies would not only give rise to further gesturing—and ideas about how to move one's body—but students might also learn how to make their own mathematical diagrams.

If the diagram, and its gestural genesis, is as central as Châtelet argues, then their use in textbooks and blackboards is worth serious scrutiny. If good diagrams can create space for mathematical intuition, then bad diagrams may crush or mislead them. Consider for example one of the diagrams (see Fig. 6) found in a popular linear algebra textbook *Linear Algebra and its Applications* (Lay, 2006). It depicts two examples of matrix multiplication, $A\mathbf{u}$ and $A\mathbf{v}$, and is meant to illustrate that $\mathbf{v}$ is an eigenvector, but $\mathbf{u}$ is not. The arrows seem

[9] See also Sinclair (2010) for a discussion of the role of "covert" thinking in mathematics, which includes not only the surface intuitions mentioned by Châtelet, by also the more subconscious (irrational?) ways of knowing hinted at by Weil.

to be used in order to indicate some kind of mapping from **v** to A**v** and from **u** to A**u**. The curved, dashed path of the arrow from **u** to A**u** might be interpreted as a kind of rotation (otherwise the path could have been a straight line), which is how several of the mathematicians above described the mapping from **u** to A**u**. However, the curved dashed path from **v** to A**v** cannot represent rotation since the mapping is actually a dilation. Therefore, the curved, dashed path should be interpreted simply as indicating a mapping. There are few gestural possibilities for thinking about stretching, or even about shrinking. There are also few possibilities for expressing the important "line up" feature described (in words and gestures) by the mathematicians in our interviews. Of course any static diagram will have to flatten time and pause motion. However, absent more dynamic diagrams available using digital technologies, it would seem that much more attention could be paid to creating more and more fruitful diagrams for learners.

5 Concluding remarks

In this paper, we have reported on the responses that mathematicians gave to prompts about how they think of the concept of eigenvector. By analysing both the verbal language and their gestures, we found that contrary to the formal, written mathematical discourse, mathematicians use both language and gesture to convey a sense of motion and time in their thinking. In particular, the gestures underline the continuity of motion and time in describing the effect of a matrix on a vector. Gestures also played the role of dynamic diagrams in the sense that they were used to sketch objects and their relations, providing information about time, location and motion. We have thus taken the work of Lakoff and Núñez (2000) in a different direction by focusing less on conceptual mathematics than on what we have been calling ideational mathematics and making full use of the communicational means available to the mathematicians.

We have drawn attention to the intimate relationship between gesture and diagram which, in thinking about mathematics education more directly, leads us to some further research questions. First and foremost, how are learners' gestures related to their diagrams and how do each of these forms of communication evolve together? Second, in addition to offering situations for learners that enable mathematically rich sensorimotor activities, might there be some benefit to inviting students to engage much more explicitly in the production (and refinement) of diagrams than is currently customary? Finally, in light of the description provided by JJ (who was clearly drawing on his experience using a computer-based visualisation tool), we wonder to what extent dynamic representations might point back to the original gestures that, in Châtelet's terms, link the body to the diagram and to what extent they might further 'teach the body how to think,' that is, how their use may occasion quite different gesturing for mathematicians and students alike. Indeed, Rotman (2008) has pointed to this possibility in the case of mathematicians, and, in our own current research, we are finding preliminary evidence that engaging with dynamic geometry representations results in the use of much more and qualitatively different forms of gesturing.

Acknowledgements This research has been supported by the Social Sciences and Humanities Research Council of Canada. We would like to thank the mathematicians who participated in this study. We would also like to thank David Pimm for his feedback on previous versions. We extend our acknowledgements as well to three anonymous reviewers and to Norma Presmeg for her guidance in our revisions.

References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics (translated by Pimm D.). In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder and Stoughton.

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Dordrecht: Kluwer.

Châtelet, G. (1993). *Les enjeux du mobile*. Paris: Seuil [English translation by R. Shore & M. Zagha: *Figuring space: Philosophy, mathematics and physics*, Dordrecht: Kluwer, 2000].

Courant, R., & Robbins, H. (1978). *What is mathematics?* New York: Oxford University Press.

Goldin-Meadow, S. (2003). *Gesture: How our hands help us think*. Cambridge: Harvard University Press.

Harris, M. (2008). Why mathematics? You might ask. In T. Gowers, J. Barrow-Green, & I. Leader (Eds.), *The Princeton companion to mathematics*. Princeton: Princeton University Press.

Hawkins, T. (1975). Cauchy and the spectral theory of matrices. *Historia Mathematica, 2*, 1–29.

Healy, L. (2009). Relationships between sensory activity, cultural artefacts and mathematical cognition. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*. Thessaloniki: PME.

Keller, H. (1969). *The story of my life*. New York: Collier-Macmillan.

Lakoff, G., & Núñez, R. (2000). *Where mathematics come from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lay, D. C. (2006). *Linear algebra and its application*. USA: Pearson Addison Wesley.

Mancosu, P. (1996). *Philosophy of mathematics and mathematical practice in the seventeenth century.* NewYork: Oxford University Press.

Nemirovsky, R., & Borba, M. (2003). Perceptuo-motor activity and imagination in mathematics learning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education, 1* (pp. 103–135). Manoa: University of Hawaii.

Netz, R. (2009). *Ludic mathematics: Greek mathematics and the Alexandrian aesthetic*. Cambridge: Cambridge University Press.

Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), *18 Unconventional essays on the nature of mathematics* (pp. 160–181). New York: Springer.

Ochs, E., Gonzales, P., & Jacobyet, S. (1996). "When I come down I'm in the domain state": Grammar and graphic representation in the interpretive activity of physicists. In E. Ochs, E. A. Schegloff, & S. A. Thompson (Eds.), *Interaction and grammar* (pp. 328–369). Cambridge: Cambridge University Press.

Palmieri, P. (2009). Superposition: On Cavalieri's practice of mathematics. *Archive for History of Exact Sciences, 63*, 471–495.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas*. New York: Basic Books.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 160–189). New York: Springer.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.

Radford, L. (2009). No! He starts walking backward!: Interpreting motion graphs and the question of space, place and distance. *ZDM the International Journal on Mathematics Education, 41*, 467–480.

Robutti, O. (2006). Motion, technology, gesture in interpreting graphs. *The International Journal for Technology in Mathematics Education, 13*(30), 117–126.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Russell, B. (1903). *The principles of mathematics*. Cambridge: Cambridge University Press.

Schiralli, M., & Sinclair, N. (2003). A constructive response to where mathematics comes from. *Educational Studies in Mathematics, 52*(1), 79–91.

Seitz, J. A. (2000). The bodily basis of thought. *New Ideas in Psychology, 18*, 23–40.

Seitz, J. A. (2005). The neural, evolutionary, developmental, and bodily basis of metaphor. *New Ideas in Psychology, 23*, 74–95.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Sfard, A. (2009). What's all the fuss about gestures? A commentary. *Educational Studies in Mathematics, 70* (2), 191–200.

Sinclair, N. (2010). Knowing more than we can tell. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers, advances in mathematics education* (pp. 591–608). New York: Springer.

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM, 41*, 441–452.

Solomon, Y., & O'Neill, J. (1998). Mathematics and narrative. *Language and Education, 12*(3), 210–221.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

Talmy, L. (1996). Fictive motion in language and "caption". In P. Bloom, M. Peterson, L. Nadel, & M. Garrett (Eds.), *Language and space* (pp. 212–273). Cambridge: MIT Press.

Thurston, W. P. (1994). On proof and progress in mathematics. *The American Mathematical Society, 30*(2), 161–177.

Wright, T. (2001). Karen in motion: The role of physical enactment in developing an understanding of distance, time, and speed. *Journal of Mathematical Behavior, 20*, 145–162.