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Using the onto-semiotic approach to identify and analyze mathematical meaning when transiting between different coordinate systems in a multivariate context

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Abstract The main objective of this paper is to apply the onto-semiotic approach to analyze the mathematical concept of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems. The identification of objects that emerge from the mathematical activity and a first intent to describe an epistemic network that relates to this activity were carried out. Multivariate calculus students' responses to questions involving single and multivariate functions in polar, cylindrical, and spherical coordinates were used to classify semiotic functions that relate the different mathematical objects.

Keywords Double and triple integration · Multivariate functions · Spherical and cylindrical coordinates · Semiotic registers · Onto-semiotic approach · Personal-institutional duality

1 Different coordinate systems

The mathematical notion of different coordinate systems is introduced formally at a precalculus level, with the polar system as the first topological and algebraic example. The emphasis is placed on the geometrical (topological) representation and transformations between systems are introduced as formulas, under the notion of equality ($x = r\cos\theta$, $r = \sqrt{x^2 + y^2}$, etc.). The polar system is usually revisited as part of the calculus sequence; in single variable calculus, the formula for integration in the polar context is covered, as a means to calculate area. In multivariate calculus, work with polar coordinates, and transformations in general, is performed in the context of multivariable functions. It is in calculus applications that the different systems become more than geometrical representations of curves, some familiar (the circle) and some exotic (the rose of 'so many' petals, the

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lemniscate). The different systems, which are related to each other by transformations, are meant to be dealt with through the algebraic and analytic theory of functions, although the geometric representation will still play a large role in the didactic process. As has been established (Montiel, Vidakovic, & Kabael, 2008), the geometric representations need to be dealt with very carefully. For example, it was reported that techniques such as the vertical line test, used to determine if a relation is a function in the rectangular context, were transferred automatically to the polar context. Hence, the circle in the single variable polar context, whose algebraic formula $r=a$ certainly represents a function of the angle θ (the constant function), when θ is defined as the independent variable and r as the dependent variable, was often not identified as a function because, in the Cartesian system, it does not pass the vertical line test. “The graphs are symbolic representations of the process with their own grammar and their own semantics. It is for this reason that their interpretation is not unproblematic” (Noss, Bakker, Hoyles, & Kent, 2007, p. 381).

When multivariate functions are introduced in the rectangular context, in particular functions with domain some subset of R^2 or R^3 and range some subset of R, the *institutional expectation*[1] is that the student will “generalize” the definition of function. The assumption is that students have *flexible mathematical thinking*, that is, that they are capable of transiting in a routine manner between the different meanings of a mathematical notion, accepting the restrictions and possibilities in different contexts (Wilhelmi, Godino, & Lacasta, 2007b).

When dealing with functions in rectangular coordinates that can be graphed as surfaces in R^3, the vertical line test still applies. In a similar fashion, when dealing with functions whose domain is some subset of the polar plane and range some subset of $R(f(r,\theta)=z)$, the vertical line test also applies. The polar plane (or some subset of it) now forms the domain and r is not necessarily a function of θ, but z does depend on the multivariable domain and, geometrically, there is only one value for each pair. On top of all these new elements that are usually introduced in a one semester course, this last example is presented in the context of *cylindrical coordinates*. The cylindrical coordinates refer to a three dimensional domain and a function whose graph can only be imagined in hyperspace.[2] However, in some of the standard exercises that students must confront, they are asked to find the three dimensional volume of that domain when given a triple integral, with an ‘r’ inside that does not represent the function $f(r, \theta, z)$ but is the determinant of the Jacobian matrix. The Jacobian itself (the determinant) is not usually introduced until after the specific cases of polar, cylindrical, and spherical coordinates are studied.

Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent and it is for this reason that no real literature review is given on the subject. It is a ‘new territory’ that is being charted in this respect. Nonetheless, it is in the multivariate calculus course where students, many for the first time, are expected to deal

[1] The duality personal-institutional will be defined in the Section 2. Informally, we are referring to the shared criteria within an institution (university, mathematics department, classroom, business) of what should be previously known, or what should be taught, learned, and understood in some didactical situation or evaluation process.

[2] In single variable calculus, we treat real-variable and real-valued functions ($f: R \to R$), whose graph is in R^2 and we look at the “exponents” in terms of R^n, in this case “1”, and point out that $1+1=2$. When moving to multivariate calculus, we introduce R^n. When our domain is two dimensional, we work with functions $f: R^2 \to R$, whose graph can be represented in R^3 (2+1=3). Now, in terms of spatial dimensions, a function $f: R^3 \to R$ would have to be graphed in $R^4 (3+1=4)$, which is impossible. Triple integrals deal with functions of the type $f(x,y,z)=w$. That is the reason for the relation with hyperspace.

with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace (triple integrals), find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems. As was mentioned above, certain criteria (such as the vertical line test for the graphical representation of functions in the Cartesian plane), which were meant to be context-specific examples but were taken as types, are shown to be wrong when applied in the new curvilinear context. Students must learn operations that are dimension-specific (such as the cross product) and make generalizations which require flexible mathematical thinking. These are just some of the aspects which make multivariate calculus a rich subject for many of the research questions that arise when trying to analyze the epistemology, as well as the didactical processes, in the transition to higher mathematics.

On the other hand, multivariate calculus in itself, with its applications, is an important subject for science (physics, chemistry, and biology), engineering, computer science, actuarial sciences, and economics students. For this reason, it is important to analyze the contexts and metaphors used in its introduction and development because generally there are no evident translations between college and workplace mathematics (Williams & Wake, 2007).

2 Conceptual framework

2.1 Mathematical objects

A mathematical object, in this study, will be considered within the onto-semiotic approach of Godino and his co-authors (Godino, Batanero, & Roa, 2005; Font, Godino, & D'Amore, 2007) as anything that can be used, suggested, or pointed to when doing, communicating, or learning mathematics. The onto-semiotic approach (Godino et al., 2005; Font et al. 2007) considers six primary entities, which are as follows:

1. *Language* (terms, expressions, notations, graphics)
2. *Situations* (problems, extra or intra-mathematical applications, exercises, etc.)
3. *Definitions* or descriptions of mathematical notions (number, point, straight line, mean, function, etc.)
4. *Propositions*, properties or attributes, which usually are given as statements
5. *Procedures* or subjects' actions when solving mathematical tasks (operations, algorithms, techniques, procedures)
6. *Arguments* used to validate and explain the propositions or to contrast (justify or refute) subjects' actions

The 'ecology' of the primary entities is not uniquely determined nor objectively established in the cognitive and instructional processes. Mathematical knowledge 'lives' in institutions and within a social context and it 'manifests' itself through concrete practices. In other words, the proposed ontology of mathematical meaning is based on *anthropological* (Chevallard, 1985) and *socio-cultural* (Radford, 1997) principals, as well as *cognitive* aspects (Tall, 1991).

Other aspects, as important as the mathematical objects, are as follows: (1) the agents that move them and the meaning (straightforward or not) that is assigned to them; (2) the concrete appearance of these objects and the reference to ideal entities; and (3) their contextual and relational function with other mathematical objects. For these reasons, in the onto-semiotic framework, the following dual dimensions are also considered when analyzing mathematical objects (Godino et al., 2005, p.5):

1. Personal/institutional
2. Ostensive/non-ostensive
3. Intensive/extensive
4. Unitary/systemic
5. Expression/content

These dual dimensions demonstrate how the primary entities must not be understood in an isolated manner, but according to their function and their relation in a contextualized mathematical activity. Furthermore, the primary entities and the dualities offer a 'photographical' way of seeing the didactical systems, that is, they permit the elaboration of models that capture a changing and dynamic reality. In fact, they are indicators for the identification of the basic processes of mathematical activity. For example, generalization is a mathematical process in which an intensive (type) is determined from a class of extensives (examples) that share a common structure or function. On the other hand, particularization is a process by which an intensive, or general case, is deemed pertinent in justifying or solving a concrete case, determining an extensive that explains or solves the problem. This flexible view of objects and their dualities explains why the onto-semiotic approach does not contradict other theoretical perspectives, such as APOS theory. The fundamental premise of this theory is that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemas* to make sense of the situations and solve the problems (Dubinsky & MacDonald, 2001). The 'personal-institutional' duality shows that these individual tendencies are regulated and normed by the institutional structuring of mathematical knowledge. Indeed, to postulate that the subjects' actions are a primary entity does not invalidate the process/object duality, basic in the process of objectification-encapsulation in APOS theory (Weller et al. 2003). On the contrary, it emphasizes the fact that it is necessary to accept that a mathematical ontology cannot be reduced to ideal or objective elements in a previously established formal structure.

2.2 Systems of practices, emerging objects, and epistemic networks

According to the onto-semiotic approach (Godino & Batanero, 1994; Wilhelmi, Godino, & Lacasta, 2007a), it is necessary to determine the meanings (plural) associated with mathematical objects in different contexts and organize them (the meanings) as a complex and coherent whole. The *operative and discursive systems of practices*, and their subsystems, understood as depending on the institutional and personal contents that are associated with a mathematical object, and the objects that emerge within these systems, form epistemic and cognitive networks. This means that if the systems of practices are institutional, the emerging mathematical objects are considered to be institutional objects, and if the systems correspond to an individual, then the objects are personal, according to the duality specified above. Also, following this duality, the objects that emerge can be ostensive (such as symbols and graphs) or non-ostensive, that is, conceptual or mental. The contextualized and functional use of these objects as elemental entities cannot be divorced from their essentially relational nature that, at the end, justifies their adaptation, whether in particular (extensive) or general (intensive) processes.

Because of this, it can be assured that there is a correspondence between the systems of practices and the expression of the mathematical objects (in a contextual and functional way). Based on Godino et al. (2005) and Font et al. (2007), we summarize this onto-semiotics of mathematical knowledge in Fig. 1.

Whereas the meaning of a mathematical notion represents the structured complex of a system of practices in a context, the holistic meaning of a mathematical notion represents the expression of the different (partial) meanings associated with the notion as one system. The holistic meaning comes from the coordination of the partial meanings associated with a mathematical notion (Wilhelmi et al. 2007b). *Flexible mathematical thinking* is what permits the passage between different partial meanings and the coordination or partitioning of the different meanings when necessary.

2.3 Semiotic functions and representation

From the pioneer work of Janvier (1987) and Douady (1987) to the more current proposals (Goldin, 1998; Duval, 2002) it has been clear (theoretically based and contrasted experimentally) that the notion of representation is central to mathematics education. The onto-semiotic approach places great value on the relation between mathematical objects by means of the semiotic function (Godino & Batanero, 1997), as a relation between an expression and a content established by 'someone', according to certain rules of correspondence. Not only language, but the other types of objects such as concepts, situations, actions, properties, or arguments, can be expressions or content of semiotic functions. Font et al. (2007) pointed out that to understand representation in terms of semiotic functions has the advantage of not segregating the object from its representation. Indeed, given an object and a representation, in general it is not possible to identify a unique semiotic function between them, and even the representation can constitute the content in another context. For example, Alson (1989, 1991) shows how a Cartesian graph

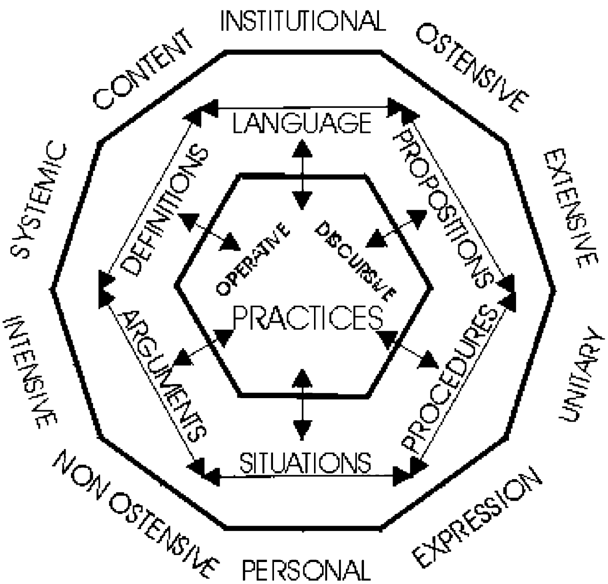


Fig. 1 Onto-semiotics of mathematical knowledge

can be given an algebraic structure before the introduction and analytic development of the theory of functions. This fact determines an objectification process of a representation (in its broadest sense) that is prototypical in mathematics.

The dependence relations can be either representational (one object is put in the place of another), instrumental (an object is used as an instrument by another), or structural (two or more objects conform to a system out of which new objects emerge) (Godino, Batanero, & Font, 2007). An example of a representative semiotic function (as opposed to structural or instrumental) could be, for the purposes of this study, a solid presented geometrically as the expression and the formulation of a double integral 'set up' as the content. An instrumental semiotic function could have as expression the double integral, and as content the numerical answer, while the structural (or 'cooperative') semiotic function could take some region together with a double integral in terms of 'x' and 'y' as the expression, and the set up of a double integral in terms of 'u' and 'v' over a simpler region (using the Jacobian) as the content. It should also be clear that the expression in one semiotic function could be the content in another.

2.4 Five levels of analysis

In different studies carried out within the framework of the onto-semiotic approach to knowledge (Font & Contreras, 2008; Font, Godino, & Contreras, 2008; Font, Godino, & D'Amore, 2007; Font & Godino, 2006; Godino, Bencomo, Font, & Wilhelmi, 2006; Godino, Contreras, & Font, 2006; Godino, Font, & Wilhemi, 2006), five levels of analysis have been proposed:

1. Analysis of types of problems and systems of practices
2. Elaboration of configurations of mathematical objects and processes
3. Analysis of didactical trajectories and interactions
4. Identification of systems of norms and metanorms
5. Evaluation of the didactical suitability of study processes

The present study concentrates on the first level, while touching on the second as well. The same empirical basis, with the same notions, processes, and mathematical meanings will be used in future studies to develop the second and third aspects.

3 Context, methodology and instrument

The context of the present study is multivariate calculus (calculus III) as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Six students were interviewed once, in groups of three, and the interviews were video-recorded. Each interview was approximately an hour and a half long. The students were first given an instrument, which is included in this text, on which they wrote down their responses, and they were then asked to explain them. For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some way, the other two. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. It should be noted that these students participated after taking their final exam, so they had completed the course. Two of the authors of this article were present, as interviewers, with each group. As final grades for the course had still not been submitted, another of the authors, who was the professor of the course, did not participate in the interviews, so that the students would not feel under any pressure in terms of their grades. The students were assured that their professor would not have access to the video-recordings until after the final grades had been submitted.

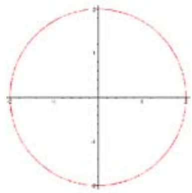
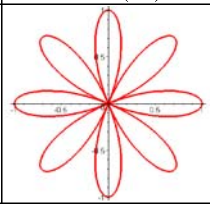
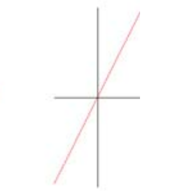
Figure 2 contains the questions presented to the students, the expected (institutional) answers, and a sample of some of the actual student answers are found in the Appendix.

The first question was in three parts and was identical to the question presented to second course calculus students (calculus II of a single variable) and reported upon in Montiel et al. (2008). The objective was to determine if the students could distinguish when a relation between r and θ was a function or not, taking θ as the independent variable and r as the dependent variable. This is not a trivial question, as the geometric representation of the constant function in polar coordinates, $r=a$, is a circle, which is not a function in rectangular coordinates. As was reported in the previous study,

> The generic definition of function, which we can paraphrase as 'a transformation in which to every input there corresponds only one output', seems to often be lost amongst the different representations students are exposed to, without recognizing any implicit hierarchy. (p. 18)

Question 1. Are the given graphs functions in the single variable set up of polar coordinates, when r is considered a function of θ ($r = \rho(\theta)$)?

Circle your choice and explain the reason.

Function	r = 2	r = cos(4θ)	$\theta = \rho / 3$
Graphs			
Answer	YES NO Explanation: …	YES NO Explanation: …	YES NO Explanation: …

Question 2. Shade the region and set up how you would calculate the area enclosed by: outside $r = 2$, but inside $r = 4 \sin(\theta)$; Use DOUBLE integration. [DO NOT CALCULATE THE INTEGRAL.]

Question 3. In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes.

(a) In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

(b) In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$? Sketch.

Question 4. What are the names of the following surfaces that are expressed as the polar functions:

(a) $z = f(r,\theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \leq r \leq 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?

(b) $z = f(r,\theta) = r^2$. Sketch the surface. Find the volume of the solid by triple integration.

Fig. 2 Questionnaire

For this reason, in the previous study the vertical line test, valid for the rectangular system but not for the polar coordinate system, was used as a criterion to say, mistakenly, that $r=a$ was not a function. This same question was now asked of students who had completed a multivariate calculus course and who were expected to know how to identify and 'do calculus' with not only single variable functions, but multivariable functions as well, in rectangular, cylindrical, and spherical systems.

The second question was also identical to the item presented to single variable calculus students in the previous study, but this time, instead of asking them to set up a single variable integral with the information given, they were asked to use double integration. This question corresponds to *argumentation* as a primary entity, and it also corresponds to a *discursive practice*, more than to the subjects' actions (procedures) as an operative practice. This is because the students had already learned how to solve the integral by the single variable method and procedure is not the main aspect that is being considered. What is put into play here is the flexible mathematical thinking that is expected and, fundamentally, their ability to translate the definition of double integral into a setup for area. This is considered as the primary entity 'argumentation' as opposed to 'procedure'.

The students were then prompted to explain why double integration, instead of single integration, is now being used to find area (as integration had been introduced through Riemann sums, and the motivation was to find 'the volume under the surface area', as an extension of single variable Riemann sums and the 'area under the curve'). They were also asked to show the relation between this setup in a double integration context and what they had been taught to do in the single variable context.

The third question was included to detect the students' geometrical transition to 3-space where, in the rectangular context, much emphasis was placed at the beginning of the course on the coordinate planes and the octants. Although we have been unable to discover any literature on the subject, through informal discussions and comparisons it has been noted that the average student has difficulty with associating the algebraic equation, say, $y=a$, with a plane parallel to the xz-plane, or the actual xz-plane if $a=0$. The cylindrical or spherical coordinate systems are named as such because of the equations $r=r_0$ and $\rho=\rho_0$ (where r is the radius of a circle on the polar plane and ρ is the radius of a sphere). However, $r=r_0$ is not a function in the usual setup of cylindrical coordinates ($f(r, \theta)=z$), but $\rho=\rho_0$ is a function within the algebraic representation $f(\theta, \phi)=\rho$ associated with the spherical system (Leathrum, 2002). That is, $r=r_0$ is a relation, but not a function, as r is one of the two independent variables in the domain (like $x=x_0$ in single variable functions). On the other hand, $\rho=\rho_0$ is a function, as ρ is the dependent variable. Students are expected to complete exercises about these geometric representations in all the textbooks that were reviewed (Larson, Hostetler, & Edwards, 2005; Stewart, 2004; Varberg & Purcell, 2006; Salas, Hille, & Etgen, 2007) but simultaneously, in these same sections, they are expected to set up and solve integrals that sometimes will represent volume, and sometimes hypervolume in 4-space, using the concept of function. The textbook used at the university where the study took place is Salas, Hille, and Etgen. It should be mentioned that, as is usual in the calculus textbooks written in the United States, θ represents the *azimuth* or longitude, and ϕ represents the *zenith* or colatitude. An important discussion on the issue of conventions in spherical coordinates and the contradictions found in their presentations between mathematics, on the one hand, and science and engineering on the other, can be found in Dray and Manogue (2002).

The fourth question asks for the names of the quadric surfaces, but expressed algebraically as $f(r, \theta)=z$ instead of as the 'formulas' that are taught when learning to identify quadric surfaces. Then the students are asked to find the volume of the solid by triple integration which, once again, connects volume to a triple—not a double—integral.

The question was raised as to how the process of finding that volume could be done with a double integral, in the context of polar coordinates. As one of the two surfaces is the upper nappe of a cone (the other is a circular paraboloid), it was also asked if the answer by integration coincides with the formula for the volume of that solid. To be able to respond to this question depends, of course, on whether the solid was identified correctly and whether the formula for the volume of a cone is remembered.

4 Analysis using the onto-semiotic approach

While analyzing the case of different coordinate systems within the onto-semiotic approach, it is important to remember what was mentioned in the conceptual framework about the classification of the mathematical objects and the primary entities. There are aspects that characterize each of these entities, but by no means can there be a sharp separation between them. The plan will be to go through the four questions; as there are six students and two groups, S1, S2, and S3 will represent the participants in the first group and S4, S5, and S6 the participants in the second interview session. Usually the two sessions will not be differentiated as emphasis will be placed on the questions themselves and the mathematical content. There are also written answers which will be referred to at times.

4.1 Question 1

In the presentation of the first question, the theories of 'Interplays between settings' (*Jeux de cadres*) (Douady, 1987) or of 'the change of register' (Duval, 2002) contribute a key idea: the framework or register of the representation does not only refer to the set of ostensives (basically, the formulas and graphs of the functions), but to a set of reference situations and operational invariants, as well as rules for action.

The essence of the first question is the fact that the exact same geometrical representation, a circle, which is not considered a function in rectangular coordinates, is in fact a function in the polar coordinate system. Language seen as a mathematical object, one of the primary entities, and understood as terms, expressions, notations and graphics, and semiotic functions that map language (expression) to content (meaning), play an important role here. For example, S2 specifically mentioned that the vertical line test could not be used, making it understood that the 'definition of a function by the vertical line test' was not valid in polar coordinates, because in polar coordinates "anything goes". What is inside the quotations, of course, are personal objects in a very colloquial language, although from the institutional point of view the answer is correct, given that she circled "yes" for 'a' and 'b', and "no" for 'c. However, as can be seen in the answers, her explanation differs from the usual institutional expression. It can be appreciated that S3 gave as his explanation "for every θ there is only one r", using the concept (definition) and properties of function in its underlying, structural meaning, which does not rely on a particular coordinate system, as well as employing impeccable institutional expression. S4 related the two systems by saying that "in the rectangular system there is one y for each x, so here there is one r for each θ", while S1 used the radial line test to justify the equation as representing a function; the radial line test had been briefly mentioned in class.

The onto-semiotic approach contributes the mathematical ontology and the dualities to this analysis, according to a pragmatic, anthropological, and socio-cultural view of mathematics. It is necessary to interpret the mathematical activity of the students in relation to the particular institutional meaning. The concept (definition) of function, as seen from the onto-semiotic approach, can be understood in different mathematical contexts (Wilhelmi et al., 2007a), such as topological, algebraic, or analytical. Furthermore, when the concept of function is first introduced, usually at the secondary algebra level, it is not possible to embrace all the systems of practices, so even when the underlying structural definition is given (‘for every element in the domain, there corresponds one and only one element in the codomain’, or, ‘for every input there is only one output’), what often remains in students’ minds (Montiel et al., 2008) is the geometric language with the vertical line test, as different coordinate systems are not included. Even though polar coordinates are introduced at the precalculus level, their geometric representations are usually presented in textbooks as exotic curves (lemnicate, limaçon), not as functions.

This sequence of developments is delimited by the institutional requirements of: (1) assuring efficiency in the solution of problems, (2) minimizing cost in carrying out an assignment, and (3) controlling mathematical activity. This way, the calculations with polar coordinates are preferred to the setup in Cartesian coordinates as only one integral is used (it is not necessary to divide the region in three parts) which minimizes the cost in the procedure and looks to maximize the success rate. The control is exercised by means of an approximate calculation by procedures of classical plane geometry (Fig. 3).

However, this pragmatic justification (whose basis is epistemological) has a cognitive cost that must be identified. The unitary-systemic dichotomy also is applicable here, because all the different coordinate systems, including the general ‘curvilinear’ coordinates, and the transformations between them together with the determinant of the Jacobian matrix, form a compound object, that is, a system. The actual curve in a particular system, as graphical language, would be an example of an elementary—or unitary—object. At the same time, the ostensive/non-ostensive duality is also relevant, as the graphical representations and the setup of double and triple integrals in different systems lead up to the mathematical concept of changing variables in multiple integration.

Many standard calculus textbooks do not help in clarifying the concept of function in polar coordinates. As was mentioned in Montiel et al. (2008), in the standard precalculus and calculus courses, the function concept in polar coordinates is typically not introduced until absolutely necessary for setting up Riemann integrals to find the area between two polar curves. In the textbook by Varberg and Purcell (2006, p. 572), the authors state that

> ...there is a phenomenon in the polar system that did not occur in the Cartesian system. Each point has many sets of polar coordinates due to the fact that the angles

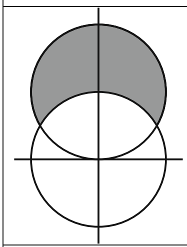
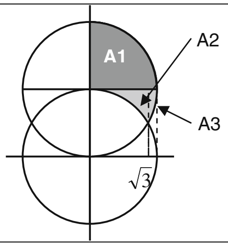
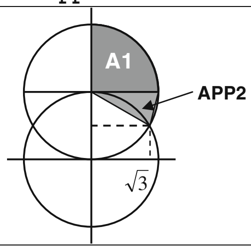
Polar method	**Cartesian method**	**Geometric upper approximation**
		
	$A = A1 + A2 + A3$	$A1 = \pi\ \ 2^2/4;\ APP2 = 2\ \ 1/2$ $UpperApprox = 2(A1 + APP2)$

Fig. 3 Analytical methods and geometric upper approximation for question 2

$\theta + 2\pi n, n = 0, \pm 1, \pm 2, \ldots,$ have the same terminal sides. For example, the point with polar coordinates $\left(4, \frac{\pi}{2}\right)$ also has coordinates $\left(4, \frac{5\pi}{2}\right), \left(4, \frac{9\pi}{2}\right), \left(4, \frac{-3\pi}{2}\right)$ and so on.

However, we ask, if there is a switch from Cartesian to polar coordinates, in what way is the element $\left(4, \frac{\pi}{2}\right)$ equivalent to $\left(4, \frac{9\pi}{2}\right)$?

It should be pointed out that this 'phenomenon' comes about because a point in polar coordinates is being identified with an equivalence class. That is, a point (r, θ) is equivalent to another point (r, θ') if $\theta' = \theta \pm 2n\pi$. In other words, it is presupposed that the dual dimensions example/type and expression/content should be avoided, as they constitute an unnecessary difficulty. However, this 'simplification' can limit students' access to the overall institutional meaning.

In Salas et al. (2007), on page 479, it is also stated, "Polar coordinates are not unique. Many pairs (r, θ) can represent the same point." On page 492, the problem is avoided by strictly stating the domain of the variable θ as limited to (0, 2π). There is no mention of the radial line test in any of these texts.

When the geometric language, and the system of practices developed around it, are not taken specifically into account, the elementary algebraic entity, in the example above, is a perfectly defined function $r(\theta)=4$, with no restriction on the domain. If the formal structure of the object 'function' must be coherent in all coordinate systems, then the fact that the point is 'apparently' the same does not make for sound mathematics. If 'for every input there is only one output' captures this underlying structure, then the textbooks might need to take this into account.

4.2 Question 2

The second question also repeats one of the items given to single variable calculus students in Montiel et al. (2008), but it is restated for the multivariate context. In this case, students are given a region in the polar plane and asked to set up, but not solve, a double integral that represents this area. All participants, with the exception of S6, shaded the correct region; we can interpret the subjects' actions (one of the primary entities) through a semiotic function with mathematical English and symbolic language (Wells, 2003) as the expression, and topological language as the content. That is, the communication process is by means of written language, through which the institution expects the students to understand a task that should be seen, in itself, as unproblematic. The tension 'expression-content' is resolved in the same way for the student (personal meaning) and the professor (who represents, in this case, the institutional meaning). This fact guarantees that the students' activities can be evaluated according to the same institutional pattern.

The students' correct shading of the region lends itself to an analysis based exclusively on the change of register between natural, formal, and graphical language by the participants. However, it is of interest to describe how these individual tendencies interact with the institutional structuring of mathematical knowledge, that is, what are the tensions of the 'personal-institutional' dualities that result from the initially coherent articulation of meanings?

The students were prompted to answer why they were using double integration to find area, when the motivation of the double integration had been the calculation of volume. In this case, the discrepancy between institutional and personal meaning could be detected in their language, as well as in their expression of concepts and interpretation of properties, as will be shown in the next paragraph. The formulation of the questions makes up the evaluated institutional meaning, while the students' answers show the personal meaning. These meanings are an empirical indicator of the agreement between the designated institutional meanings and the personal meanings that are learned and, ultimately, are what permit the evaluation of mathematical competence in relation to the objectives and learning outcomes.

The question was direct, but the responses did not actually address it, although S4 and S5 did give a geometric explanation after much prompting, as will be shown. Both S1 and S2 responded that the limits of integration were giving the bounds of the region and for this reason the double integral made sense. The object which they used was graphic language, as they pointed to the rays $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, as well as the curves r=2 and $r = 4\sin(\theta)$, with the expression (domain of the semiotic function) being the rays and curves and the content being the shaded region (the meaning expressed in topological language).

S4's use of language as a personal object contrasted seriously with the institutional object, as will be explained. However, his resolution of the dichotomy expression-content by means of a semiotic function that went from the combination of mathematical English with symbols as expression, to the correct setup of the symbolic representation of area as content, would be successful in the institutional context (the multivariate calculus test, if this was the case).

What occurred was that S4 confused the term 'function', as a language object, with the term 'region' when he said "We're not integrating a region, 'r' is compensation, not a region." It should be clarified that the 'r' he is talking about is the determinant of the Jacobian matrix, which was explained in the class as a 'compensation' for the distortion of the original region in the rectangular system, when it was first introduced. This was because the 'r', in all the textbooks consulted, is used many sections before general transformations of regions using the Jacobian are explained. The term 'compensation' itself could be considered as a personal language object of the professor, which would make it an institutional object for the students.

S4 was correct in explaining that when there is no 'apparent' function (in other words, when the constant function $f(r, \theta) = 1$ is implicit), the double integral can be interpreted as area. However, he used the word 'region' for 'function' and then, when pressed to explain this 'compensation', he said that it was the Jacobian, that is used to convert a complicated function to an easier one, in this case using the word function instead of region.[3]

S5, after going through an explanation of Riemann sums of rectangular solids, realized that he was explaining how the double integral serves to calculate the volume under a surface area. The meanings (plural) of integration are expressed by semiotic functions that can take as expression, say, a single integral and as content either area, volume (geometric objects) or centroids, work, etc. If the expression of a semiotic function is a double integral, the content could be area, volume, mass, moments of inertia, and so on; if the expression is a triple integral, geometrically the content could be volume or hypervolume in the fourth dimension. These semiotic functions respond to a classical and formal institutional perspective that prefers the analytic representation over the graphical one. This, in itself, is not problematic; the problem is that this is done without a clear consciousness of the assumptions of this institutional act. For example, it is assumed that the learning of a 'one-way' semiotic function will lead students to master, in a flexible manner, expression and content. Question 2 could be interpreted as the maximum area covered by a solar disk in a

[3] As was mentioned, his actions as well as the properties and arguments used were impeccable, if the words function and region had been exchanged in the two cases. On the other hand, the pragmatic nature of the meanings of the terms 'function' and 'region' can themselves cause the mistake, and can be explained in terms of institutional use (including the professor's explanations). The identification of a function with its graph and, consequently, the area (definite integral) with the region, shows a semantic and syntactic coherence, in spite of the mistaken use of the terms.

partial eclipse. The graphical representation would then be the expression and the integral (in polar or Cartesian coordinates, single or double, see Fig. 3) the content. It is clear that this way of relating the objects 'integral' and 'graph' is not direct. In fact, a modeling process is insinuated. The integral is a model of the region that is covered, making it possible to determine the approximate area covered by the solar disk (the 'darkness') and relate it with the part of the solar diameter that is hidden (the magnitude of the eclipse) and the level of light in some place.

It is this plethora of partial meanings that makes this particular question (why the double integral for area) so slippery. S5's action, once he realized that he was asked to specify why the double integral would be used to calculate area, not volume, resorted to a geometric concept. His response was "We can't calculate volume, it's not a three dimensional surface", which is correct, but which does not answer the original question of why a double integral is being used to calculate area.

4.3 Question 3

The third question had a more geometric emphasis than the others. The results of interpreting the instructions, written in a combination of mathematical English and symbols, were sketches, a graphical language object. When looking at the students' sketches, the representation of $\theta=\theta_0$ seemed to be the least precise, in both (a) and (b). They all mentioned a vertical plane, but S2 admitted that "I just did all this by memorization, I have no good explanation". In the Appendix it can be seen that she did not understand how that vertical plane is constructed. As institutional objects these sketches are considered language entities and these subjects' actions in the problem situation would be considered formally correct, although not precise (it was not clear, with the exception of the sketches of S3 and S4, that $\theta=\theta_0$). However, S1 also made the sketches "by the definitions", but was able to give content (meaning) to these expressions by means of a semiotic function in which spoken mathematical English (language) played the role of signified, asserting that "as θ is fixed, r and z can be anything". S5 also was able to give content to his expression, using a non-mathematical metaphor (Pimm, 1987), "θ is stuck at one position, like when you cut a pie". None of the students actually remembered that in spherical coordinates, the equation $\phi=\phi_0$ must be broken down to five different cases, with restrictions on ϕ.

4.4 Question 4

The fourth question is related to the second, as the students were asked to set up a triple integral to find volume. As students were motivated geometrically to understand double integration as a means of calculating volume and as the geometric notion of hypervolume under some hypersurface in 4-space is no longer feasible to graph, the problematic itself was easier to deal with than in the second question.

The semiotic function that describes the mathematical activity is compound. First, the expression is a statement in mathematical English with symbolic language embedded in it and the content is an integral setup which captures the meaning of the expression. Then this content (the triple integral) turns into the expression, whose meaning is a number that represents the volume asked for. However, even when this formal process is carried out, in several cases (S1, S2, S5) personal meaning does not coincide with institutional meaning, to get the volume of the cone in (a), or the paraboloid in (b), the object must be inverted; if not, the volume that is obtained is that which is under and outside of the object, not of the object itself. S4 did realize this dilemma and explained it with words. “It is like when you find the area under the parabola. It’s not the area inside. To get the area inside you would have shift it up and turn it upside down. I don’t remember how to do this”.

In the previous example S4 communicates through mathematical English that he has been able to give meaning to an expression, although he has not been able to formulate it (to operate) in the mathematically correct symbolic manner. How is this correct communication of mathematics as one type of language object (mathematical English) to be evaluated, when he is unable to present it as a conventional institutional object (the actual triple integral and the calculation)? The ability to carry out actions such as operations, procedures, and so on, does not automatically indicate understanding. This fact has been widely researched in the literature on APOS theory, for example (Dubinsky & MacDonald, 2001). However, the opposite phenomenon, described in this last example with S4, has not been researched as much. This is the type of question that the onto-semiotic approach can lead to and is an example of its heuristic power, that is, of its capability to identify and describe phenomena that broaden the issues and that have not been analyzed or identified through other approaches.

5 Synthesis, conclusions, and prospective activity

Different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in science and engineering. Central in the cognitive and instructional processes of this subject, is the generic notion of representation. The focus on changes of registers and on individual processes of objectification, conceptualization, and meaning contributes to a coherent view of mathematical knowledge and the means of its construction and communication. As has been seen in this study and in a previous work (Montiel et al., 2008) in the standard textbooks reviewed, there is no emphasis on different representations once the function concept is introduced. This in itself would not necessarily be problematic, but when there are misconceptions about the function concept itself and confusion between the definition of a function and certain local criteria in the rectangular system that identify particular functions, it becomes problematic. The fact that students have learned to convert from polar, cylindrical, or spherical coordinates to rectangular coordinates seems to make some of them think that whenever θ, ϕ, r or ρ appear they are dealing with polar, cylindrical, or spherical coordinates. Several students, in the previous study, said that the presentation in terms of $r\cos\theta\,(r\sin\theta)$was in polar coordinates. Yet another factor that comes into play is the fact that θ and r have very precise meanings as ‘angle’ and ‘radius’ from geometry and trigonometry, which is different from the abstract x and y.

The overriding definition of function, which we can paraphrase as ‘a transformation in which to every input there corresponds only one output’, seems to often be lost among the different representations students are exposed to, without recognizing any implicit hierarchy.

Based on the onto-semiotic approach, it can be added to this general conclusion that it is also important to emphasize the anthropological and socio-cultural character of this knowledge, indicating the tensions between the personal and institutional meanings. The primary entities and their dualities, together with the semiotic functions, allow the description of this personal-institutional tension, related to the notion of meaning and mathematical objects in different coordinate systems.

The relation between the theories of change of register and the onto-semiotic approach supposes:

- A mutual external validation of these approaches, given that they allow the identification and description of issues related to the didactic system that other theories also identify and describe.
- An example of the heuristic power of the onto-semiotic approach, whose specificities are shown in some of the concrete aspects of the process that was observed and analyzed.

The onto-semiotic complexity that was identified is an empirical indicator that should guide the search for ways to improve and control the didactic systems related to the learning and teaching of notions, methods, and meanings associated with integration in the multivariate context.

As previous research, within any framework, on this mathematical concept, and on multivariate functions in analysis in general, is practically non-existent, a much more sophisticated description of an epistemic network for this subject is a goal that we hope to reach in the near future. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, could be accomplished because of the rich layering and complexity of the mathematical concept at hand.

"The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research" (Godino et al., 2005). It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. However, the onto-semiotic approach gives us a framework to analyze, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics. It is hoped that this attempt to apply this ontology and these instruments to a mathematical concept that involves so many subsystems, provides an example of the kinds of studies that can and should be undertaken. Further studies on this particular mathematical concept can only clarify aspects of the knowledge needed in the communication and understanding of it.

Appendix

Expected answer

(a) Answer: **YES** NO

Explanation: For every element θ in the domain, there corresponds one, and only one, element in the codomain.

For every input θ, there corresponds one, and only one, output.

(b) Answer: **YES** NO

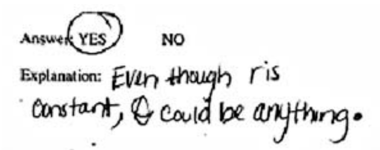
Explanation: Same as in part (a).

(c) Answer: YES **NO**

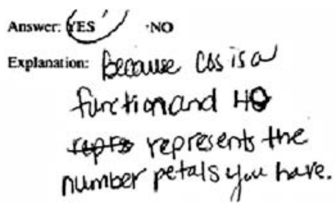
Explanation: For π/3 there are infinite values (more than one) of *r*.

Answer from S2.

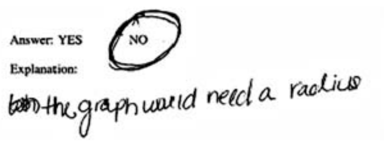
(a)


Answer: YES NO
Explanation: Even though r is constant, θ could be anything.

(b)

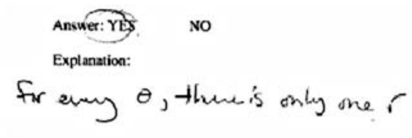

Answer: YES NO
Explanation: Because cos is a function and 4θ ~~reps~~ represents the number petals you have.

(c)

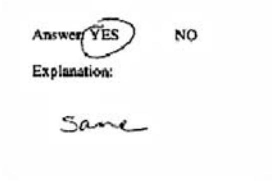

Answer: YES NO
Explanation:
~~both~~ the graph would need a radius

Answer from S3.

(a)


Answer: YES NO
Explanation:
For every θ, there's only one r

(b)


Answer: YES NO
Explanation:
Same

(c)

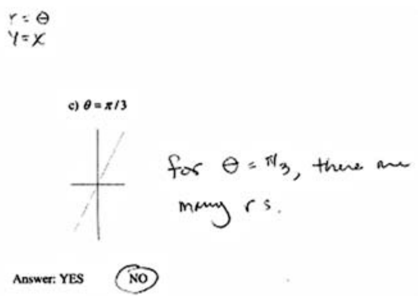
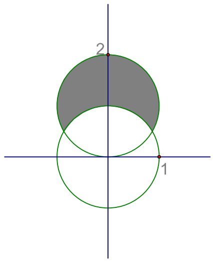


Fig. 4 Expected answers and actual student answers of question 1

Expected answer



They intersect when $2 = 4\sin(\theta)$; $\frac{1}{2} = \sin(\theta)$; $\theta = \pi/6, 5\pi/6$

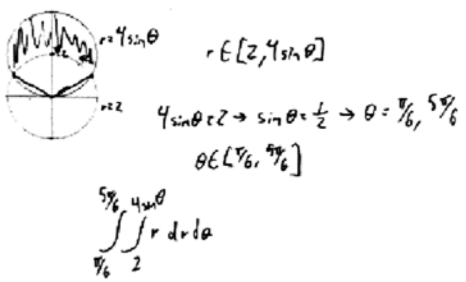
In single variable calculus this integral (this type of integration) was presented as

$$\frac{1}{2}\int_{\theta_1}^{\theta_2}([\rho_2(\theta)]^2 - [\rho_1(\theta)]^2)\,d\theta$$

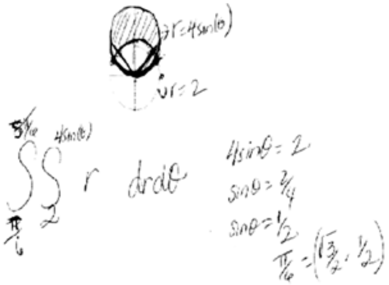
However, we asked for <u>*double integration*</u>*:*

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{2}^{4\sin(\theta)} r\,dr\,d\theta.$$

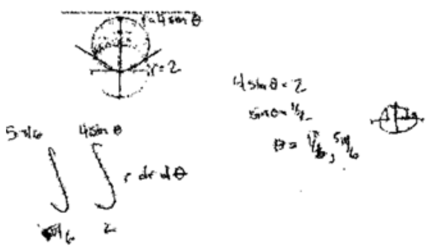
Answer from S1.



Answer from S2.



Answer from S3.



Answer from S5.

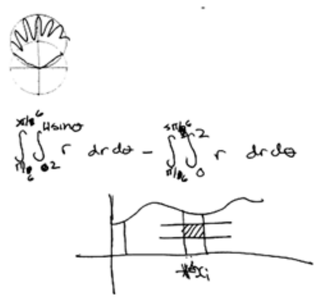


Fig. 5 Expected answers and actual student answers of question 2

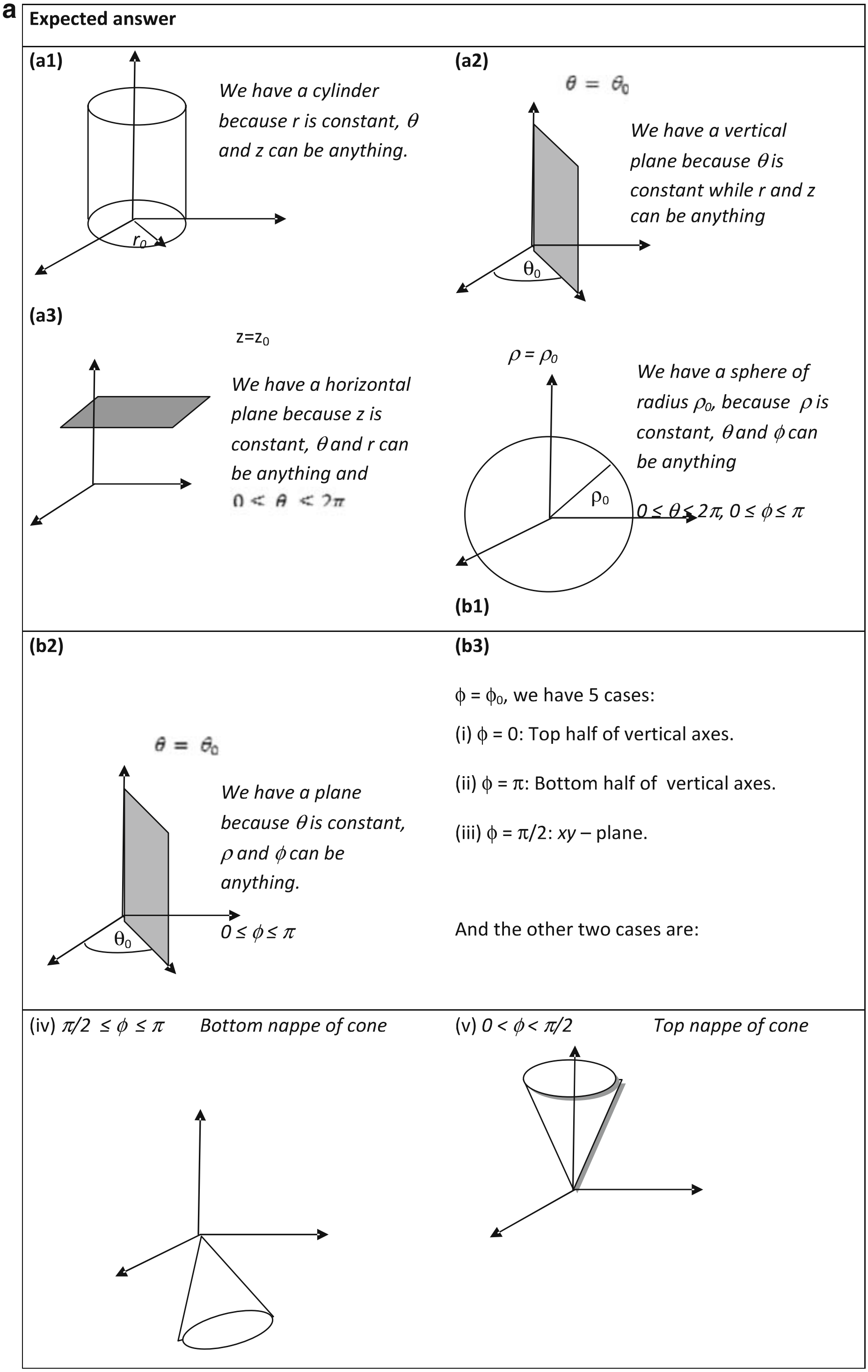


Fig. 6 **a** Expected answers of question 3. **b** Actual student answers of question 3

b

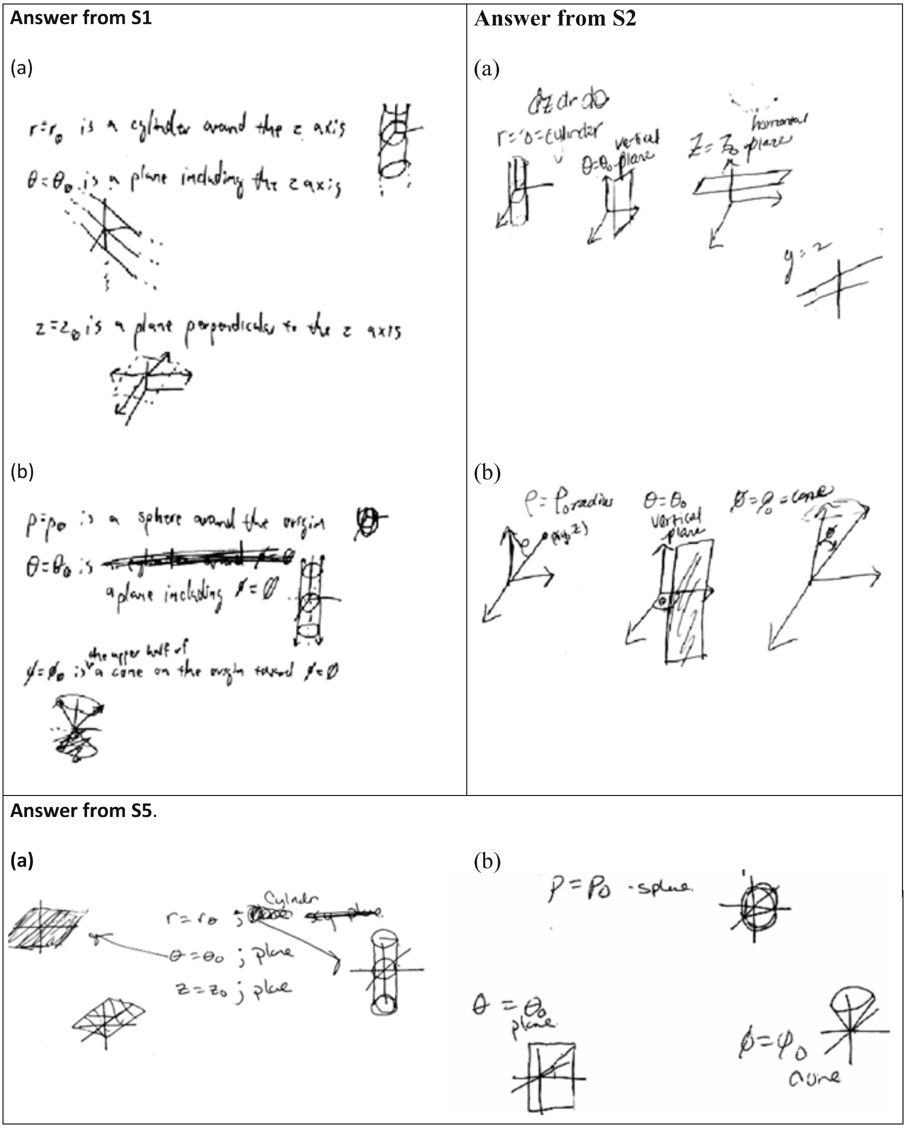


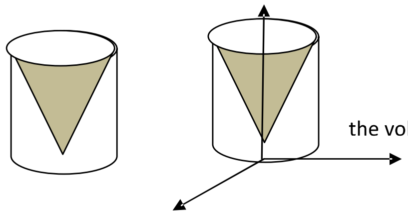
Fig. 6 (continued)

a

Expected answer.

(a) $z=\sqrt{x^2+y^2}$ *and* $z^2=x^2+y^2$ *is the equation of a cone (Quadric Surface). In this case: top nappe*

Two ways to set up: As the formula for volume of a cone is $\frac{1}{3}\pi r^2 h$, it is $\frac{1}{3}$ of a cylinder's volume.



the volume is:

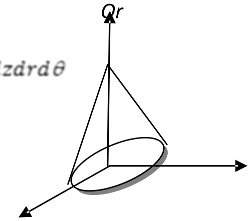
Then

$\frac{1}{2}\int_0^{2\pi}\int_0^{2}\int_0^{r} rdzdrd\theta$

"Upside Down"

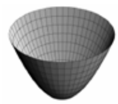
Or

$\int_0^{2\pi}\int_0^{2}\int_0^{2-r} rdzdrd\theta$

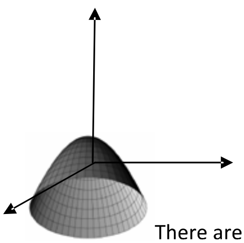


Either way we get $\frac{8}{3}\pi$

(b) $z=r^2=x^2+y^2$ *Paraboloid of Revolution.*



"Upside Down"



There are also two ways: (i) $\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} rdzdrd\theta$

Or (ii) Take the cylinder and subtract volume of "outside" of paraboloid.

$\pi(2)^2 4-\int_0^{2\pi}\int_0^{2}\int_0^{r^2} rdzdrd\theta$ *Either way: The volume is* 8π

Fig. 7 **a** Expected answers of question 4. **b** Actual student answers of question 4

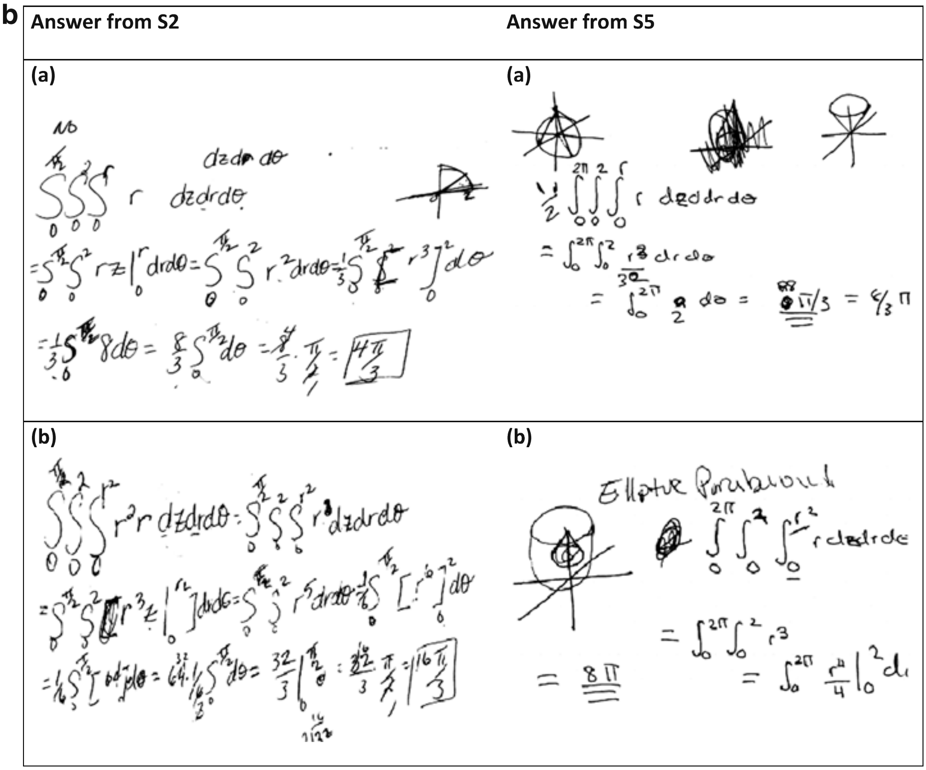


Fig. 7 (continued)

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