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Concept image revisited

Erhan Bingolbali · John Monaghan

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Abstract Concept image and concept definition is an important construct in mathematics education. Its use, however, has been limited to cognitive studies. This article revisits concept image in the context of research on undergraduate students' understanding of the derivative which regards the context of learning as paramount. The literature, mainly on concept image and concept definition, is considered before outlining the research study, the calculus courses and results which inform considerations of concept image. Section 6 addresses three themes: students' developing concept images of the derivative; the relationship between teaching and students' developing concept images; students' developing concept images and their departmental affiliation. The conclusion states that studies of undergraduates' concept images should not ignore their departmental affiliations.

Keywords Concept image · Concept definition · Derivative · Institution · Undergraduates

1 Introduction

Concept image and concept definition (hereafter referred to as 'CI & CD') is now an old construct in mathematics education but it has weathered the years well and continues to be cited in the literature (e.g. Przenioslo 2004; Giraldo 2006; Nardi 2006). It is, in our opinion, an important construct. It appears that when it was first introduced there was a widespread belief that if mathematics teachers/lecturers got their definitions right, then the concepts behind the definitions would, by careful tutor explanation and student diligence, become transparent to the student. If this interpretation is correct, then the authors of the construct contributed to our current understanding that while a tutor's definition of a concept may evoke correct associations for some students, many students will generate, amongst some intended associations, unintended *concept images*. Whatever the interpretation, students' concept images became an object of study.

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E. Bingolbali (✉)
Gaziantep Universitesi, Egitim Fakultesi, Ilkogretim Bolumu, Gaziantep, Turkey
e-mail: bingolbali@gantep.edu.tr

J. Monaghan
University of Leeds, Leeds, UK

The construct CI & CD was born in an era where individual and cognitive theories of learning dominated English language mathematics education publications and it was a *child of its time*: it focused on individual student mathematical constructions and the theories of learning alluded to/developed by early authors were all cognitive theories of learning. This is an observation, not a criticism. What is surprising is that as social theories of learning have gained ascendancy, CI & CD remains the preserve of cognitive theorists. This need not be the case and, as we shall argue in this article, the construct can be used in interpreting data in what may be called a *socio-cultural* study.

The story behind this article is that we embarked on a study of first year Mechanical Engineering and Mathematics undergraduate students' understanding of the derivative. The data from the end of semester 1 post-test was, frankly and in our opinion, amazing—Mechanical Engineering students were thinking in terms of *rate of change* whilst Mathematics students were thinking in terms of *tangents*. We explored this further, examining our lecture observation data and by constructing new questions for students to answer in semester 2 which allowed them to exhibit their preferences for forms of knowledge about the derivative. These data suggested reasons for the *cognitive shift* with regard to the derivative concept we noticed in the two groups of students: the teaching *privileged* specific forms of thinking about the derivative in each of the groups of students; many of the students *positioned* themselves to their perceptions of their departments' ways of thinking about the derivative.

The next section considers the literature; mainly the literature on CI & CD but also selected literature on social theories of learning, on teaching and on students' understanding of the derivative. We then briefly outline the study that informs this article, describe the two calculus courses and provide summary results from data collected. Section 6 has three themes: students' developing concept images of the derivative; the relationship between teaching and students' developing concept images; students' developing concept images and their departmental affiliation.

2 Literature review

We mainly focus on the CI & CD literature but also consider learning, particularly student learning of the derivative. These are widely researched areas and our review is, for the sake of brevity and focus, selective.

CI & CD emerged as a public construct in a PME article (Vinner and Herschowitz 1980) and in the next year the first journal article on the construct, Tall and Vinner (1981), appeared (in this journal). This article introduced all the current terminology around the construct used today and is by far the most referenced article on the construct. Concept definition is straightforward to describe, it is the form of words/symbols used by the tutor/course notes/textbook to define a mathematical concept. We are less interested in concept definition than we are in concept image in this article because we are not purely cognitivist theorists, i.e. from our standpoint factors such as students' departmental affiliation *enter into the mix* that results in students' concept images.

The notion of concept image is less straightforward to describe. Tall and Vinner (1981) describe it as the total cognitive structure associated with a concept in an individual's mind. It includes *mental pictures*, associated properties and processes as well as strings of words and symbols. It is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences natural numbers, might be "squaring makes the number bigger". It is unlikely that all knowledge that forms a concept image (mental pictures, associated properties and processes) will be simultaneously brought to bear by students in their actions in mathematical tasks. Tall and Vinner call the knowledge that is brought to bear by a particular student, at a particular time, and on a particular task, the *evoked concept image*. Of course, as for most important constructs in mathematics education, it is extremely difficult to gain insight into students' evoked concept images. All we can do is to find what we regard as valid ways to observe students and interpret the data.

The 1981 article focused on limits and continuity and the construct was taken up by mathematics educators, who met at PME conferences, who were interested in *advanced mathematical thinking* (AMT). This set the tone for CI & CD to be used by researchers interested in higher mathematics and/or older academic stream students. This is clearly a potentially fruitful area of application as definitions are arguably more widely used in high school and undergraduate mathematics than they are in the early years of schooling, but the construct has been applied to early learners too (Gray et al. 2000). It should be noted that it is unlikely that the application of this concept in AMT studies *just happened*; it is more likely that the construct grew from discussions within the informal PME AMT group.

CI & CD is a robust construct with regard to theories of learning in that it basically states that students bring all sorts of ideas to bear when they work on mathematical tasks, and no one disputes that. Prior to this year (2007) CI & CD had not, to our knowledge, been taken up by researchers who attend to culture, history, the institution or issues of identity, i.e. broadly socio-cultural researchers, but there appears to be no principled reason for this. Indeed, the construct can be viewed in terms of Vygotsky's (1934/1986) *complexes*, a "phase on the way to concept formation" (ibid., p.112) and the original view of concept images as developing differentially over students through a multitude of experiences is essentially a contextual viewpoint. The nearest we have found in the mathematics education literature to authors who critically attend to context referring to this construct is Yackel et al. (2000) article on sociomathematical norms in an undergraduate differential equations class (and they simply note Tall and Vinner (1981) in a list of authors for whom their work complements) and Pinto and Moreira (2007) who examine the topic tangent lines in a vocational school with regard to participation in communities of practice, but the concept image construct was merely the starting point for this research.

Nardi has a number of recent articles on CI & CD (these articles are listed in Nardi 2006). This arises from her work with university mathematics teachers whom, she states, 'catch on' to this construct because it is accessible, appears relevant and they can relate it to their teaching practice. She notes that CI & CD generally attends to cognitive issues but, like us, notes:

> I believe that subsequent phases of theory-building around this (and other) theoretical construct needs to focus on re-conceptualising it in ways that its initial inception could not possibly relate to back in the 1980s, when it made its first 'public appearance', and re-embed it in the remarkably richer contemporary theoretical landscapes of the field.

Returning to older CI & CD articles Vinner (1992) notes that the teacher plays an important role in the formation of students' concept images, though he does not explore this interaction/development empirically. We do not dispute this. Indeed, it appears as obvious to us that in all but a few instances where a student is learning completely alone (if this is possible) that the tutor will, intentionally or inadvertently, suggest associations which different students will explicitly or implicitly appropriate. We further believe that a teacher is not a *lone agent* but, knowingly or unknowingly, adapts her/his teaching to the class and/

or institution s/he is teaching in. We consider a teacher-calculus and an institution-calculus study shortly but first briefly attend to studies on students' understanding of the derivative.

In the history of mathematics education, studies on students' understanding of calculus ideas are comparatively recent but there was a *boom* of articles in the 1980s. The original CI & CD authors wrote many of these articles, e.g. Tall (1985), Vinner and Dreyfus (1989), and Vinner (1982). Articles dealt with limits (Davis and Vinner 1986), infinitesimals (Tall 1980) and graphic (especially computer graphic) approaches (Tall 1986) amongst other things. Differentiation was dealt with alone (Orton 1983a) or jointly with integration (Orton 1983b) and/or differential equations (Tall and West 1986). The upshot of cognitive studies in the 1980s and early 1990s was that calculus was a very difficult subject (Tall 1991), that there were many ways to view it (Schwarzenberger 1980), that students generated many unexpected interpretations (Tall 1991), and that computer graphics (and other computer software) could be used to transform learning and teaching from being procedure-dominated to being more conceptually-orientated (Leinbach et al. 1991; Steen 1987).

The 1990s saw the rise of mathematics education studies which adopted a variety of what may be called 'social theories of learning', studies where culture and/or artefacts and/or institutional aspects and/or identity were central features. We end this literature review by considering two studies which focus on social aspects of learning calculus: Kendal and Stacey (2001) and Maull and Berry (2000). These are interesting studies to focus on, for our reasons in this article, because they deal with students' understanding of calculus, pay essential regard to the conditions that learning takes place under and because the authors appear to have come to vaguely socio-cultural positions as a result of research rather than commencing their research with a particular social theory of learning in mind.

Kendal and Stacey (2001) use the term 'privileging' to describe the impact of teachers' emphases on students' learning differentiation. They borrow this term from Wertsch (1991) who described how different forms of mental functioning dominate in different contexts. They examine how two teachers (teacher A and teacher B) taught differentiation using a hand held computer algebra system, which made numerical, graphical and symbolic representations of the derivative readily available to their Year 11 classes. Although the teachers planned the lessons together they made differing pedagogical choices regarding aspects of differentiation to emphasise: Teacher A privileged rules and exhibited a strong preference for symbolic representation whilst teacher B privileged conceptual understanding and student construction of meaning; teacher A privileged graphical-numerical connections whilst teacher B privileged graphical-symbolic connections.

Kendal and Stacey link these instructional differences to students' performance on differentiation items. Students of teacher A, who privileged routines, did better on items concerned with formulating the problem in terms of differentiation at a point. Students of teacher B, who privileged conceptual understanding, did better on items concerned with interpreting the derivative. The results show that students' performances were strongly influenced by the aspect of the derivative privileged by their teachers.

Maull and Berry (2000) examine first and final year (mechanical) engineering and mathematics undergraduates alongside postgraduate students and professional engineers through a questionnaire which sought to elicit student understanding of key mathematical concepts including the derivative. They found that mathematics students displayed greater preferences for both tangents and rate of changes aspects than engineering students at the entry. However, by the final year, the groups' reponses differed: engineering students showed greater preferences for rate of changes aspects of the derivative than mathematics students and both engineering and mathematics students displayed preferences for tangents aspects of the derivative almost at the same degree.

They also found that mathematics students displayed preferences for tangents aspects over rate of change aspects of the derivative. They do not go into reasons for the emergence of this difference and call for further research. They suggest exploring whether differences between mathematics and engineering students' developing understandings are related to the department to which they belong. Our work goes a little way towards addressing this issue.

3 The research

Our research investigated first year Mechanical Engineering (ME) and Mathematics (M) students' conceptual development of the derivative with particular reference to rate of change and tangent aspects, and included an examination of the influence students' departments may have on their knowledge development (Bingolbali 2005). Most, but not all, students had been introduced to the derivative in High School. The research was conducted in a large university in Turkey. Data were collected by a variety of means: quantitative (pre-, post- and delayed post-tests), qualitative (questionnaires and interviews) and ethnographic (lesson observations and 'coffee house' discussions). The study adopted a 'naturalistic' approach (Lincoln and Guba 1985)—situations were not manipulated nor were outcomes presumed. We report on a subset of this data in this article; the Appendix shows the questions, Q1–Q6, we report on.

We present a number of test results in this article. An important set of tests were pre-, post- and delayed post-tests. The tests items were developed through a detailed examination of the relevant mathematics education literature (e.g., Orton 1980, 1983a; Amit and Vinner 1990; Ubuz 1996) and various calculus textbooks (e.g., Hockey 1969; Bittinger 2000). Prior to administering the pre-test students sat a basic test on the derivative—as mentioned not all students studied the derivative in High School and these students were excluded from further testing. Some students missed one or other of the three tests. This left 50 ME and 32 M student who completed all three tests, in October, January and April/May. Test questions addressed 'rate of change' and 'tangent' aspects of the derivative in graphic, algebraic and application formats. The pre-test showed no significant difference between ME and M students' performance. In the post-test both groups improved their performance but ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions. In the period between these two tests, the calculus modules were observed and copies of students' notes were made to see how the topic 'the derivative' was taught in each department.

We found that 'overall, ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions' in data analysis in February. This suggests a fundamental shift in cognition over a one semester period of time. We regarded it as an important result and set about designing data collection tools to explore this matter further. We designed further interviews with lecturers and several 'student preference' questions. The focus of the article and space does not permit us to detail all of these tools in this article. We have selected one student preference question, Q6, which we believe sheds light on ME and M students' developing concept images of the derivative.

4 The calculus courses

In presentations of this work it emerged that brief expositions of the two calculus courses did not provide sufficient insight into the mathematics covered in the courses for the audience/readers to fully appreciate points which were later raised. For this reason we devote this section to descriptions of the courses.

Although they were both called ‘calculus’ the content covered only functions and differentiation; integration was covered in a later course. The ME calculus module, three 45 minute lessons per week, devoted 15 h (20 lessons) to the derivative. The M calculus module, six 40-min lessons per week, devoted 24 h (36 lessons) to the derivative. Both calculus courses were observed and compared with students’ notes to gain insights into which aspects of the derivative were ‘privileged’. University mathematics lessons vary in form: some are ‘lectures’ where the lecturer lectures and the students take notes; others are more akin to High School lessons where students sometimes interact with the lecturer and attempt examples as well as take notes. These Turkish courses were of the latter form.

4.1 The M calculus course

After a review of functions as special sets of ordered pairs and an introduction to ideas of continuity, using ε−δ notation and ideas, the lecturer considered a secant, and then a family of secants, to an arbitrary function (presented graphically). The limit of this family, as end points converged, was presented as $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$. The derivative as a function (x in place of x_0) was then introduced and example derivatives were derived: $f(x) = x^2; g(t) = \frac{1}{t}; f(x) = \sqrt{x}, x \geq 0; f(x) = |x|$. The approach until this point has been informal, used a lot of graphs (with tangents shown) and has consistently used Δx notation, suggesting a ‘primitive infinitesimal’ approach to elementary calculus. The first theorem, that a function which is differentiable at a point is continuous at that point, then appears where, interestingly, Δx is replaced by h. The ‘physical meaning’ of the derivative is then very quickly considered via a recap of $v = \frac{s}{t}$ and the differentiation of an arbitrary function $s = f(t)$. Several theorems, $(f + g)'$, $(f - g)'$ and $(f \cdot g)'$ are proved and higher derivatives are introduced. ‘Linear differentiation’ is then considered; this is effectively informal non-standard analysis and the ‘increment theorem’ (Keisler 1976) $f(a + \Delta x) = f(a) + f'(a)\Delta x + \varepsilon(x)\Delta x$ is proved. The ‘chain rule’ is then stated and proved, implicit differentiation is informally covered and parametric differentiation $\left(x = f(t), y = g(t), \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}\right)$ is covered with examples. ‘Applications of the derivative’ (maxima and minima) are then considered at some length. The approach is quite graphical and left and right limits are considered: $\lim_{x \to x_1^-} \frac{f(x_1)-f(x)}{x_1 - x}, \lim_{x \to x_1^+} \frac{f(x_1)-f(x)}{x_1 - x}$. Rolle’s theorem and the mean value theorem are introduced, via graphs, and then proved. Here and elsewhere the conditions for the proof to apply are noted, e.g. a closed interval over which the function is differentiable for Rolle’s and the mean value theorems. Monotone functions are considered and a theorem on the sign of the derivative function is proved. Concavity is introduced and points of inflection are considered in some depth. L’Hospitâl’s rule is stated and informally proved. The course ends with a largely graphical consideration of rational and trigonometrical functions and asymptotes.

4.2 The ME calculus course

The course started with a review of functions and introduced continuity but without ε−δ notation. Like the M course it introduced the idea of the derivative at a point via a consideration of a family of secants. This exposition, however, was briefer than that in the M course and only one graph, of a non-specified function, compared to three in the M course, was presented. The derivative at a point a was defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Three theorems were then stated (concerning $(f \pm g)'$, $(f \cdot g)'$ and $\left(\frac{f}{g}\right)'$) and the theorems for $(f \pm g)'$ and $(f \cdot g)'$were non-rigorously proved. It was then proved that if a function is differentiable at a point, then it is continuous at that point. Example derivatives were then presented: $f(x) = x^2; f(x) = \frac{x^2+1}{3x^4-2x}$. The chain rule was then stated, proved and three examples presented. Implicit differentiation was presented as a method via examples. Without further ado the second derivative was introduced as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Two examples were given including $x = v \cdot t, \frac{dx}{dt} = v, \frac{dv}{dt} = a \Rightarrow \frac{d^2x}{dt^2} = a$. The derivatives of sin x, cos x, e^x and ln x were simply stated.

The above was covered fairly quickly. A similar period of time was then devoted to examples and problems: a mixture of rate of change problems, e.g. the rate of change of an area/volume, of a shape given by a formula, with respect to time (NB metric units were provided but no units were used in the M course) and various trigonometric derivatives derived (interestingly radians were used but why they are used was not addressed). Maxima and minima were introduced via definitions, e.g. $c \in [a, b]$ is a minimum point if $\forall x \in [a, b], f(x) \leq f(c)$, and a graph. A proof that if c is a maximum or minimum point in an interval, then $f'(c)=0$ is provided but this proof lacks rigour since no conditions on c and the interval are considered. Two examples are presented. Critical points are then considered and further examples are presented. Monotone functions are then defined and theorem-like statements, e.g. $\forall x \in]a, b[, f'(x) > 0 \Rightarrow f$ is increasing on $[a, b]$, are presented but not proved and examples are presented. Rolle's theorem and the Mean Value theorem are stated and proved, much as they were in the M course. L'Hospital's rule is stated but not proved for various forms (0 and ∞) and examples presented. Concavity and points of inflection are introduced graphically using sinusoidal and cubic functions. Three word problems are solved, the second concerning the rate of change of the volume of a prism. The course ends with a consideration of inverse functions, a proof that $(f^{-1}(y))' = \frac{1}{f'(x)}$ and inverse trigonometrical functions.

4.3 General comments on the two courses

With regard to rate of change and tangent and examples: the ME course lecturer specifically focused on rate of change issues for 133 min and provided nine rate of change examples but only focused on specific tangent issues for 10 min and provided no specific 'find the tangent line' examples; the M course lecturer specifically focused on tangent issues for 85 min and provided seven examples focused on tangent lines but only focused on specific rate of change issues for 11 min and provided no specific rate of change examples. Of course, and hopefully as the above course descriptions show, it is not possible to completely divorce tangent and rate of change approaches because differentiation concerns rate of change; but when we consider the privileging in the courses, then the overall comparison of the two courses in this paragraph show different foci.

With regard to theorems and proofs both lecturers presented 20 theorems: the ME course lecturer proved 10 of these whilst the M course lecturer proved 17. With regard to definitions the ME course lecturer introduced ten and the M course lecturer introduced 14 definitions.

An analysis of the ME and M departments' mid-term and final calculus course examinations indicates that lecturers reinforced the different emphases in their teaching in examinations questions: ME examinations included rate of change but not tangent questions whilst the opposite was the case for M examination; M examinations included proofs but ME examinations did not include proofs.

5 Results

The results are presented in three sections. First, results related to rate of change and tangent items from pre-, post- and delayed post-tests are presented. Second, results particularly concerned with concept images are provided. Finally, the results related to student preference question (Q6) are presented.

5.1 Questions 1 to 4

We report on two tangent (Q1 and Q2) and two rate of change (Q3 and Q4) questions. We first present a brief descriptive analysis of the questions based on correct/incorrect responses and then provide the results of statistical analysis conducted on these questions. Table 1 shows the pre-, post- and delayed post-test results for Q1–Q4.

Before commenting on these results we wish to point out that the situation was not quite as 'bad' as these figures suggest. The table only shows correct responses. Responses that were not correct include partially correct responses as well as incorrect responses and non-responses. In Q1-a, for instance, 34% of ME and 25% of M provided partially correct answers in the pre-test and 20% of ME and 19% of M did so in the post-test.

The results show no noticeable difference in the performance of the ME and M students in the pre-test in any of the questions. The results also show, with the single exception of ME students for Q1-b, that both groups of students displayed improved performance in the post-test. The delayed post-test performance for both groups of students is broadly consistent with their post-test performance.

Without exception, M students' performance in the post-test and the delayed post-test was better than ME students' performance on tangent questions and ME students' performance in the post-test and the delayed post-test was better than M students' performance on rate of change questions. These results are not 'soft measures', they report on what students got right and wrong. They do not provide 'fine detail' on students' concept images but they do suggest a fundamental difference in students' understanding of the derivative. We refer to this analysis of these results as the 'emergent trend': M students consistently performing better on tangent items and ME consistently performing better on rate of change items in the post- and delayed post-tests.

We subjected these results to an appropriate statistical test of significance, the Mann–Whitney *U* test. The test was applied to the sum of students' improvement score on the rate

Table 1 Correct responses (percentages) to questions 1–4

Items	Pre-test		Post-test		Delayed post-test	
	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)
Q1-a	22	22	32	42	42	56
Q1-b	10	18	4	34	10	34
Q2-a	28	22	50	63	50	69
Q2-b	28	13	34	56	38	56
Q3-a	24	22	84	53	74	53
Q3-b	18	22	64	53	76	50
Q4-a	56	41	70	44	78	44
Q4-b	6	6	22	13	36	22
Q4-c	8	6	30	22	60	19
Q4-d	8	3	28	9	52	13

of change and tangent items from: pre-test to post-test; from pre-test to delayed-post test. This test produced a significant difference between ME and M students on: tangent questions from pre-test to post-test (p=0.003) and from pre-test to delayed post-test (p= 0.002); rate of change items from pre-test to post-test (p=0.027) and from pre-test to delayed post-test (p<0.001). Strong supporting evidence that the 'emergent trend' reflects a socio-cognitive phenomena.

5.2 Question 5

Q5 simply asked students 'What is the meaning of a derivative? Define or explain as you wish'. It was designed to access students' concept image of the derivative over time. Repeated rereading of students' responses generated five response categories: rate of change (RC); tangent (T); rate of change and tangent (RC&T); other explanations (OE); and not attempted (NA). Before presenting the results (Table 2) we explain how we allocated students' responses to these categories and give examples of students' response for each category.

> RC was allocated to a student's response when 'rate of change' or 'rate of increase' was cited, e.g. "The derivative tells us at what rate something is increasing or changing."
>
> T was allocated to a student's response when 'tangent' or 'slope of the tangent' was cited, e.g. "The derivative is the slope of tangent line at a point on a curve."
>
> RC&T was allocated to a student's response when both RC and T criteria were met, e.g. "Derivative can be defined as the slope of the tangent line and rate of change."
>
> OE was allocated to a student's response when neither the RC nor the T criteria were met, e.g. "if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$."

The pre-test shows a roughly similar pattern of response between the two groups of students, though, interestingly, more M than ME students gave rate of change and more ME than M students gave tangent reasons. In both the post- and delayed post-test, however, students' responses were consistent with the 'emergent trend', M students citing tangent and ME citing rate of change.

5.3 Question 6

This question was, as mentioned, only administered once, at the time of the delayed post-test. The question presented the core ideas of both rate of change and tangent approaches to the derivative in the words of two imaginary students, Ali and Banu. Students had to

Table 2 Students' responses (percentages) to question 5

Response	Pre-test		Post-test		Delayed post-test	
	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)
RC	6	13	30	16	36	25
T	28	16	26	41	26	41
RC&T	4	0	16	16	20	6
OE	30	31	18	22	10	16
NA	32	41	10	6	8	13

choose an interpretation and give reasons for their choice. We present the results in two parts: their choices (Table 3) and reasons for their choices (Table 4).

The results for the M students are consistent with the ‘emergent trend’, M students choosing Banu’s tangent description. The results for the ME students are evenly divided for Q6-b though some support for the ‘emergent trend’, ME students choosing Ali’s rate of change description, can be seen in Q6-a. We put forward a possible reason for this ‘even split’ with regard to Q6-b in Section 6.

We now attend to students’ reasons for their choices. Repeated reading of students’ responses produced five categories: real life & application; mathematical & scientific; department; instruction; and not categorised. Before presenting the results we explain how we allocated students’ responses to these categories and give examples of students’ response for each category.

‘Real life & application’ was allocated to a student’s response when their response made reference to real life or application, e.g.

ME 1: I would support Ali. I am thinking with an engineer mentality. This makes me tend to be closer to the practicality and the concreteness.

‘Scientific & mathematical’ was allocated to a student’s response when their response referred to mathematical or scientific ways of thinking, e.g.

M 1: Banu’s understanding is closer to mine because she explains it in a mathematical way.

‘Department’ was allocated to a student’s response when their response referred to a department or being an engineer or a mathematician, e.g.

M 2: Banu interprets the derivative from a mathematician’s perspective, and Ali interprets it from a physicist standpoint. At the end of the day, since I too am from mathematics department, I find Banu’s explanation closer to myself.

ME 2: Calculating rates of change seems to me more real. On the other hand what Banu says is not far away….But since I am going to be an engineer, Ali’s idea would be just different. Because I would be the one who makes mathematics concrete.

‘Instruction’ was allocated to a student’s response when their response referred to their calculus courses, e.g.

ME 3: We are using it in that way and learning it that way.

‘Not categorised’ was allocated to a student’s response when their response did not provide ‘appropriate reasoning’ to allocate it to another category, e.g.

M 3: I would support Banu because Banu’s is closer to my understanding.

Table 3 Students’ responses (percentages) to Q6

	ME		M	
	Q6-a	Q6-b	Q6-a	Q6-b
Ali (A)	51	49	19	13
Banu (B)	27	49	63	78
Both (A &B)	22	2	16	3
Not attempted (NA)	0	0	3	6

Table 4 Students' written responses (percentages) to Q6

Response	ME				M			
	Q6-a		Q6-b		Q6-a		Q6-b	
	A(51)	B(27)	A(49)	B (49)	A(19)	B(63)	A(13)	B(78)
Real life & application	29	0	29	0	9	0	6	0
Scientific/mathematical	0	13	0	29	0	25	0	28
Department	11	0	11	0	0	19	0	25
Instruction	9	4	7	0	0	19	0	9
Not categorised	9	9	9	20	9	6	6	22

NB Some responses fall under more than one category.

With regard to item 6a (respectively 6b), 29% (respectively 29%) of ME but only 9% (respectively 6%) of M students cited 'real life and application' in explaining their preferences for Ali's rate of change orientated interpretation. These results are consistent with what we have called *the emergent trend*, i.e. engineers are concerned with the real world. This is, however, not so evident with regard to being scientific/mathematical as 29% of ME students supported Banu for this reason (otherwise *the trend* is supported). Although the percentages in the "department" category are not large the pattern of responses is consistent—students' reasons are consistent with departmental practices, i.e. no ME (respectively M) student who chose Banu's (respectively Ali's) interpretation cited department in explaining their choice. In addition, some ME and M students mentioned classroom calculus instruction in explaining their preferences. We return to all these responses in Section 6.

6 Discussion

The literature on CI & CD and results from our study raise many issues about students' developing concept images of the derivative. We focus our discussion of these issues around the following three themes:

- Students' developing concept images of the derivative;
- The relationship between teaching and students' developing concept images;
- Students' developing concept images and their departmental affiliation.

We pay particular regard to evidence for claims we make because we feel that the interpretative nature of these claims leaves us open to the danger of making claims about students' concept images which are not supported by data or clear reasoning.

6.1 Students' developing concept images

We speak of students in general terms in this article, noting trends in patterns of responses over time but are aware that some individuals do not fit with this trend and that the developmental paths of individuals providing similar responses will not be identical. Research designs and tools, like all tools, have constraints and affordances. We can say nothing on microgenetic development but we can provide broad and detailed descriptions of groups of students over time. Data from all six questions supports the thesis that there was a development in students' concept images of the derivative over the first year of their studies. Responses to Q1-4, the emergent trend, provide evidence that ME and M students developed different understandings of the derivative that affected their performance on different kinds of questions. Q5 says nothing about performance but gives us an insight into concept images of the derivative at three temporal points in students' studies. Q6 provides evidence about student preferences towards two representations of the derivative.

Evidence from Q1-4 is the emergent trend, that overall, ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions. This was consistent in both post- and delayed post-tests and the base-line pre-test showed no significant difference between ME and M students' performance. It is, as mentioned, rather striking evidence but it provides little insight into students' concept images.

Q5, an open question which asked students to define or explain what the derivative means, provides insight into students' concept images. It is quite consistent with the emergent trend. Looking at the RC and T responses we actually see that more M than ME students in the pre-test presented rate of change responses whilst more ME than M students presented tangent responses. In the post- and delayed post-tests, however, this response pattern was reversed and reflected the performance indicated in the emergent trend.

Q6 was only administered at the time that delayed-post test was administered but the response pattern in Table 3 is consistent with the general trend, though less marked than in Q5 in the ME students. (We note in passing that something interesting is happening within the ME students, more than might be expected chose Banu's explanation, but we cannot say what it is. We believe it may concern viewing pure mathematics as 'proper mathematics' whilst retaining rate of change conceptions.) With regard to Q6-a half of the ME students found Ali's rate of change-oriented interpretation closer to their own understanding but 27% found Banu's tangent-oriented interpretation closer to their own view (see Table 4). The opposite, only stronger, was the case with M students. In Q6-b the preferences of the M students remain similar but the ME students are equally divided. Overall, however, the results show that ME students showed a stronger preference towards rate of change orientations and M students showed a stronger preference towards the tangent orientations.

The results to all six questions show a clear trend, that students' concept images of the derivative changed as they progressed from entry to the end of the first year: ME students' concept images of the derivative developed in the direction of rate of change orientations and M students' concept images developed in the direction of tangent orientations. The question of interest then, is 'what brought about these changes?' Of many possible things that may have brought about these changes we address two, which are grounded in our data and appear important, in the following subsections: teaching and departmental affiliation. We report on these separately but these two aspects are, almost certainly, intertwined.

6.2 The relationship between teaching and students' developing concept images

Vinner (1992, p.200) anticipated our work:

> The concept image... is shaped by common experience, typical examples, class prototypes, etc. with a given textbook and a given teaching, one can predict the outcoming concept images and can predict also the results of cognitive tasks posed to the students.

We closely monitored the classroom calculus practices that each group of students was exposed to; it was here where the students encountered 'typical examples' and experienced 'given teaching'. We focus on the analysis of ME and M calculus courses and discuss their potential contribution to the emergence of different concept images between ME and M students.

As shown in Section 4, the ME calculus course emphasised rate of change aspects of the derivative (including examples) and the M calculus course emphasised tangent aspects. In the words of Kendal and Stacey (2001), the ME calculus lecturer 'privileged' rate of change aspects whilst the M calculus lecturer 'privileged' tangent aspects of the derivative. Such teaching, as Vinner notes, 'shaped' their developing concept images of the derivative. Although we emphasise this rate of change/tangent difference we wish to note that this is not the only difference. Side by side with the tangent orientation of the M calculus course was a greater emphasis on definition and proof: 17/20 theorems proved in the M course compared to 10/20 in the ME course and 14 definitions in the M course compared to ten in the ME course. Our interpretation of the import of this is that the tangent (rate of change) emphasis of the M (ME) course was an aspect of 'different mathematics' presented to the two groups of students: theoretical mathematics to the M students and practical mathematics to the ME students.

So are we saying 'you get what you teach'? Well, yes, to a certain extent we are but we think the lived-in-experience of being an undergraduate mathematician or engineer is more complex than conditioned response to teaching stimuli. Vinner (1992) speaks of the 'common experience' of students, of which, the calculus classroom is only a part. ME and M students will share a number of common experiences as undergraduates but the experience that is unique to each group is their experience of their departments, to which we now turn.

6.3 Students' developing concept images and their departmental affiliation

Data which we consider provides us with insight on both concept images of the derivative and departmental affiliation are students' open responses to Q6. Note that this question does not state which department the students are from and this, to us, is a positive feature of the question as it require greater 'sense making' from the students than a similar question with "Ali is an engineer…" in it.

Table 4 summarises our coding of students' explanations for their choices. Some categories represent about a quarter of each student group and some categories have voids but there is no majority category and no category with all voids. The most representative category for the ME students who chose Ali's interpretation (respectively, M students who chose Banu's interpretation) was 'real life and application' (respectively, 'scientific and mathematical').

Although we established an 'objective' method for assigning responses to categories it was, at times, difficult 'at a gut level' to judge the category of the response of some ME students between 'real life and application'/'department' and of some M students between 'scientific and mathematical'/'department'. For instance, ME 1 was assigned to both 'real life and application' and 'department' categories as he mentioned both "real life" and "an engineer mentality" and ME 2 was assigned to both 'department' and 'real life and application' categories as he mentioned "an engineer", and "real" and "concrete". These two co-joined categories account for the majority of ME students choosing Ali and the majority of M students choosing Banu and are, we believe, intertwined in students' thoughts and experiences. 'Instruction', by comparison, accounts for much fewer responses. We do not claim that teaching is less important than departmental affiliation in the emergence of students' developing concept images but simply that students' perceptions of their department impact upon how they position themselves and their developing preferences and concept images.

The above analysis of students' responses to Q6 falls far short of direct evidence that students' developing concept images of the derivative are related to their departmental affiliations but it does, to us, suggest a 'departmental influence'. Such an influence is likely to be felt by lecturers as well as students and, in this case, departmental considerations are worthy of study in both teaching and in learning.

7 Conclusions

Mechanical engineering students' concept images developed in the direction of rate of change aspects of the derivative whilst mathematics students' concept images developed in the direction of tangent aspects. Further to this, it appears that students' developing concept images and the way they build relationships with its particular forms are closely related to teaching practices and departmental perspectives.

Considering studies with regard to individuals' concept images in the literature it appears that CI/CD studies of students' understanding of the derivative have focused on cognitive aspects and the individual mind. We do not dismiss cognitive studies, nor do we ignore the individual, but we feel that they must be seen in *context*, individual cognitive functioning is influenced by others, by the setting and by the way individuals position themselves in settings. From this perspective differing concept images of the derivative are not really surprising, they are simply interesting phenomenon to explore. We are aware that our investigation leaves much unexplained (how student 'positioning' develops as well as accounting for students who do not appropriate departmental stances, e.g., ME students who do not appropriate rate of change interpretations of the derivative). Future studies of undergraduate students' concept images, of the derivative and of other concepts, should not ignore students' departmental affiliations.

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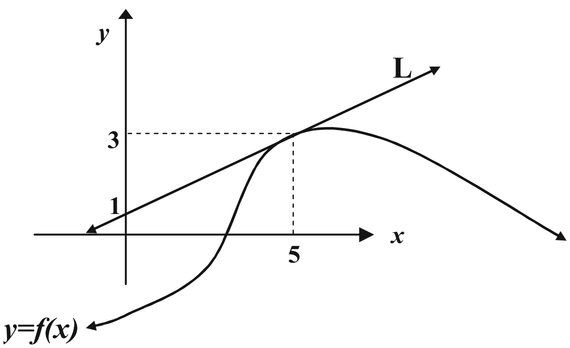
Appendix

Question 1 (Q1)

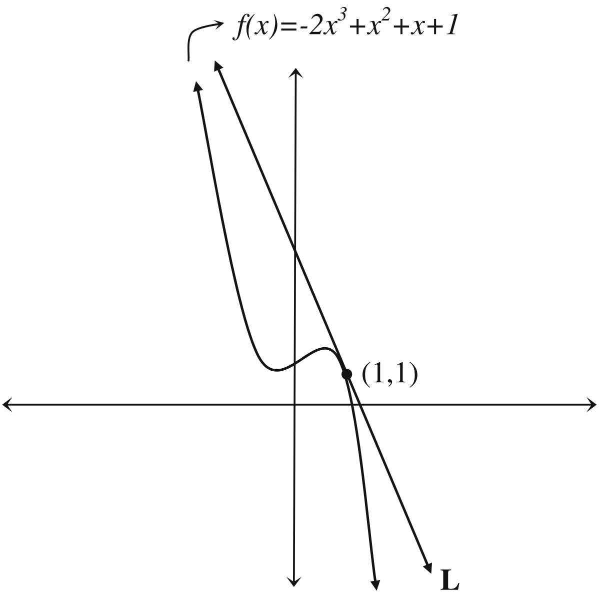
Line L is a tangent to the graph of $y=f(x)$ at point (5,3) as depicted in the graph below.



a) $f'(5)=?$
b) What is the value of the function $f(x)$ at $x=5.08$? (be as accurate as possible)

Question 2 (Q2)

a) Find the equation of the tangent to the curve $y = x^3 - x^2 + 1$ at (1,1)
b) Find the equation of L by making use of the graphs given below.



Question 3 (Q3)

Find the rate of change with respect to the given variable of the following functions at the values indicated.

a) $f(x) = x^2 - 7x$, when $x=3$
b) $g(x) = (x^2 - 1).(x + 1)$, when $x=1/3$

Question 4 (Q4)

For a certain period the population, Y, of a town after x years is given by the formula $Y = 1000\left(50 + 2x - x^2/6\right)$. Find:

a) The initial population,
b) Its initial rate of increase,
c) The time at which the rate of increase is 1,000 people per year,
d) The time at which the population stops growing and its value at this time.

Question 5 (Q5)

What is the meaning of a derivative? Define or explain as you wish.

Question 6 (Q6)

Two university students from different departments are discussing the meaning of the derivative. They are trying to make sense of the concept in accordance with their departmental studies.

Ali says: The derivative tells us how quickly and at what rate something is changing since it is related to a moving object. For example, it can be drawn on to explain the relationship between the acceleration and velocity of a moving object.

Banu, however, says: I think the derivative is a mathematical concept and it can be described as the slope of the tangent line of a graph of y against x.

a) Which one is closer to the way of your own derivative definition? Please explain!
b) If you had to support just one student, which one would you support and why?

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