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The completeness property of the set of real numbers in the transition from calculus to analysis

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Abstract This paper focuses on teaching and learning the set of real numbers **R** and its completeness property at the university level. It studies, in particular, the opportunities for understanding this property that students are offered in four undergraduate correlative courses in Calculus and Analysis. The theoretical framework used in the study draws on concepts developed in the Anthropological Theory of Didactics, especially the notions of praxeology and mathematical organization. The paper shows different expectations concerning the same notion (**R** and its completeness) through different school levels, and intends to bring up some reflections about the transition from Calculus to Analysis.

Keywords Real numbers · Completeness · University level · Institutional rapport · Mathematical organization · Praxeology

1 Introduction

The research presented in this paper is focused on the undergraduate programs (“licenciatura”, in Spanish) in *Pure and Applied Mathematics* at the *School of Exact Sciences* of the University of Buenos Aires, Argentina. The University of Buenos Aires is a public university, where students pay no tuition fees. The licenciatura programs offered in the *School of Exact Sciences* aim at educating theoretical thinkers, and preparing them for research at the graduate level (Master, PhD). On average, students complete their licenciatura in mathematics in 6 1/2–7 years, which includes writing a thesis comparable with a Master of Science thesis in the North American system.

Some of the analyses included in this paper have also been presented in my article, ‘Análisis institucional a propósito de la noción de completitud del conjunto de los números reales’, *RELIME*, volume 9(1), 31–64. They are presented here with the permission of the publisher.

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The first year is a preparatory one and courses are common for all students planning to enter the *School of Exact Sciences*. It consists of six general courses, two of which are mathematics courses: *Calculus* and *Linear Algebra*. After completing these courses, students enter the *School of Exact Sciences* and those who want to specialize in *Pure and Applied Mathematics* start taking mathematics courses only. The failure rate is high in the first two years of the School of Exact Sciences, something that has been accepted as quite natural by the institution. Students who fail normally retake the courses until they succeed, but some end up changing the direction of their studies, or dropping out of the university.

In the study, I look at four correlative mathematics courses, in Calculus and in Mathematical Analysis. One of these courses (we call it *Course I*) belongs to the preparatory year of studies; the other three (*Courses II*, *III* and *IV*) belong to the first two years of the mathematics program at the *School of Exact Sciences* proper; they are taken during the second and third years of the licenciatura program. At the time of the study, the failure rate in *Courses II* and *III* was usually around 40%; in *Course IV*, the failure rate was about 65%. Precise data for *Course I* was not available, but, in my experience, students aiming at specializing in mathematics, rarely failed this course.

The high failure rate especially in courses where a conceptual transition from Calculus to Analysis is made, led me to study how this transition is taken into account in planning and organizing the mathematical practices through a longitudinal study concerning the four courses. To make the research manageable, I focused on a topic. I chose the property of completeness[1] of the set of real numbers. The set of real numbers (which I will denote here by **R**) is a common topic in all the four courses and, above all, it is a concept that is at the core of Analysis.

2 Aims of the research

The analysis of mathematical practices cannot be performed in isolation, and this is taken into account by the *Anthropologic Theory of Didactics* (or ATD, Chevallard 1997, 1998, 1999, 2002a, b; see also Sierpinska 2005; Laborde and Perrin-Glorian 2005; Barbé et al. 2005), the general framework of the present study, which proposes a structure for modeling institutional practices. In this theory the notion of institution is considered in a broad sense, for instance *Courses I* to *IV* can be considered as different institutions. Chevallard called "praxeology" a model of a practice, and suggested that a praxeology be composed of descriptions of four fundamental elements of each practice (the four "Ts"): the tasks the practice is set to accomplish; the techniques used to accomplish the tasks; the technologies used to justify the techniques, and the theories whose aim is to provide a broader theoretical foundation for the technologies (Chevallard 1998).

From the point of view of techniques, tasks can be grouped into types and kinds. A solution of a task can be generalized to a "technique" if the task is seen as just an instance of a whole class of tasks, so that the technique can be applied to solving other tasks of the same type. For example, a task such as, "prove that this (particular) sequence converges" can be seen as belonging to the type of tasks which ask the student to prove the

[1]The word "completeness" refers here to the property of **R** that can be stated as follows: every non-empty and upper-bounded set of real numbers has a least upper bound that belongs to **R**. There are other equivalent characterizations, which are listed in the Appendix. The word "continuity" refers to the analogous property of the straight line.

convergence of sequences; this type of tasks is, in turn, one of the broader kind of "prove that" tasks.

For Chevallard, mathematical tasks, types of tasks and kinds of tasks are not something "naturally given" by, for example, some mathematical theory to which they refer; rather they are institutional constructs and they depend on the institution in which they are used and practiced Their identification and description, relative to the institutional characteristics, is, indeed, an important object of didactics of mathematics as a domain of research (Chevallard 1998) .

Given a type of task, the actions to be performed for accomplishing it receive the name of techniques. Algorithms, for instance, are a particular kind of technique. In a given institution, relative to a type of task, there is a limited range of techniques that are considered acceptable. An institution may exclude certain techniques as belonging to other institutions. The discourses that describe, explain and use some kind of rationality to justify the accepted techniques constitute the technology. The discourses vary from institution to institution, and they also evolve within the same institution. The arguments that rationally justify the technology constitute the theory, which takes, with respect to technology, a similar role that technology takes regarding the techniques. A set of types of mathematical tasks, techniques, technologies and theories constitutes what has been called a *mathematical organization.*

In terms of these notions, the aim of this paper is to present and analyze those aspects of the mathematical organizations in *Courses I–IV*, which were closely related with the study of **R** and its completeness property, and to examine their differences and evolution through the courses. I assume that this analysis will allow me to grasp the different expectations concerning the same notion through different school levels and to understand the transition from Calculus to Analysis in the program I am studying.

3 Reference mathematical organizations for the concept of completeness of R

This section looks at some reference mathematical organizations that may have underpinned those of the particular educational institution I am studying here. In the first subsection, I look at how real numbers are usually conceptualized in school curricula. In the second, the historico-epistemological reference is briefly outlined.

3.1 Two theoretical levels in the conceptualization of R

The set of real numbers is usually conceptualized on two levels. On one of these levels, the focus is on the necessity to introduce numbers other than ratios of integers (i.e. rational numbers). Non-periodic expansions are studied; as well as the representation on the line of some special irrational numbers like square roots of integers, π and e. At this level, the existence of real numbers is not regarded as a problem to be studied. The second level conceptualizes **R** as a set, endowed with arithmetic, order and completeness properties, seen as the natural domain of Analysis. These properties become essential when real functions and sequences are studied and not only general observations about them are made but their systematic theoretical justification is sought. The first level of conceptualization of real numbers is usually considered as sufficient in secondary school mathematics teaching and in some Calculus courses. The mathematical practices developed in the university program I am studying here are aimed at the second level.

Research in mathematics education has focused primarily on the first level. The second level attracted a lot less attention and I haven't found works related specifically to teaching and learning the concept of completeness of **R**. There exist, however, didactic studies related to the more general issue of transition from Calculus to Analysis. Authors have tried to identify the differences between the two domains that could explain students' difficulties in Analysis. For example, Maschietto (2002) points to the introduction of a local point of view as a distinctive feature of Analysis: What matters is what occurs in a neighborhood of a point, without much concern for what occurs at a global scale or at an isolated point. I think that completeness is a sort of local property: while it is not a property of *one* number but of *a set of numbers*; it is not an entirely global property. Artigue (1998) highlighted the "reconstructive" aspect of Analysis, or, indeed, its concern with theoretical justification of the techniques developed in Calculus. Teaching Calculus is often based on pre-constructed[2] notions which are given sense in the context of specific tasks and techniques for solving them. In Analysis, these notions acquire the status of objects by means of definitions.

Certainly, one can consider the notion of completeness as a product of theory-building processes in mathematics. I would say that the straight line and its continuity function as pre-constructed notions in Calculus courses; they are presented as self-evident and they support several practices. The naïve idea of real numbers as "all the numbers" functions in a similar way. A reconstruction of these notions is needed to start a further, deeper study in Analysis, in which an explicit formulation of completeness plays a central role in proving certain theorems.

3.2 Historico-epistemological reference

Different epistemological statuses of the property of completeness/continuity can be pointed out if we look at the history of mathematics (Bergé and Sessa 2003):

- An *implicit attribute*, as it was considered in Euclid's *Elements* in Proposition 1, Book I, where the point of intersection of two circles is used. Euclid proved each step in this proof using definitions, postulates, and common notions preceding this proposition, except for the existence of the common point of the two circles. This is more than an omission: without this assumption, the proposition cannot be deduced from that which precedes it. To justify the existence of the intersection point of two circles, a circle and a line, or two lines, a postulate that makes explicit the continuity of the line is needed. Another example of an implicit use of continuity/completeness can be found in the works of Girolamo Cardano (1501–1576) who deduced the existence of a solution for the cubic equation $x^3+q=px^2$ by finding values of x such that $x^3+q<px^2$ and other values such that $x^3+q>px^2$, and taking for granted the existence of an intermediate value of x for which the two expressions are equal. It is believed that Rafael Bombelli (1526–1572) went through these arguments and offered a justification using the continuity of the line in a drawing (Zariski 1926). Continuity/completeness was implicitly used in the development of Calculus in the 17th and 18th centuries: the kinds of problems that were studied during this period did not require making that property explicit. In solving the problems, mathematics was used as a tool for modeling; the aim was to compute a height or a distance whose existence was empirically guaranteed by the phenomena that were studied.
- An *intuitively accepted, explicit attribute*, in the works of Augustin-Louis Cauchy (1789–1857) and Bernard Bolzano (1781–1848). This was the beginning of the process

[2] By "pre-constructed notions" Chevallard (1997) meant those whose existence is taken for granted, with a representation that does not allow one to operate or make proofs.

of “arithmetization” of analysis, which found its most explicit expression in the works of Cantor and Dedekind later in the nineteenth century. Arguments based on graphical representations were put in question, and an attempt was made to reconstruct Analysis based only on arithmetical concepts. Particularly, results such as the Intermediate Value Theorem were justified using argumentation that explicitly used the property of completeness in any of its forms, namely the convergence of fundamental sequences, the existence of a unique intersection of nested closed intervals whose lengths tend to zero, the existence of the limit of bounded monotonic sequences or the existence of the greatest lower bound. However, depending on the situation, one of these statements was considered to be naturally true, and the others were deduced from it. That is, although completeness was explicitly expressed, it was eventually accepted as a natural property.

– *An explicit property that can be proved*, in the works of Richard Dedekind (1831–1916) and George Cantor (1845–1918) who recognized that, in previous expositions of the differential calculus, some properties that were attributed to the numerical system could not be justified based on arithmetic only. They formally constructed a numerical system based on rational numbers – a numerical domain accepted as arithmetically well-founded – in which these properties were provable. I recognize in this step a major change in the status of these notions. In his introduction to *Continuity and Irrational Numbers* (Dedekind 1963; first published in German in 1872 as *Stetigkeit und irrationale Zahlen*, Vieweg und Sohn: Braunschweig), Dedekind described his deep concern about finding a purely arithmetical foundation for Calculus. He explained how dissatisfied he was with the use of geometrical interpretations in proofs, for example, in proving that “every magnitude which increases constantly but not unboundedly, necessarily tends to a certain limit value”. He had nothing against using such arguments for didactic purposes. On the contrary, he was saying that such references are useful and even necessary in introductory teaching of the differential calculus, “if one doesn’t want to waste too much time”. But he refused to call such practices “scientific” and considered it necessary to work on the elaboration of a “purely arithmetic and fully rigorous foundation of the principles of infinitesimal analysis”. Dedekind took the geometric line as a pivot for the construction of a numerical system: he created – by means of what he named cuts – a way for accurately characterizing the continuity of the line, and he elaborated an analogous characterization for the numbers. This characterization offered a way for proving Analysis theorems that had never been justified beyond their geometrical evidence, or that had been accepted as natural. Indeed, this constituted the motivation for his work. In my view, this is a veritable reconstruction of pre-constructed notions: he made completeness operational and explicit, and he defined, for the first time, the continuity of the line. Cantor’s construction, on the other hand, consisted of the creation of one numerical system such that all fundamental (or Cauchy) sequences have a limit belonging to this system. Cantor needed this set to prove the uniqueness of the coefficients in developing functions in trigonometric series (Cantor 1871). That is, his goal was to obtain a well-defined set for proving other theorems of Analysis. From the point of view of this status, continuity and completeness are distinguished and separated as two different but closely related notions: continuity corresponds to the line, while completeness is formulated for the set of numbers. Continuity and completeness were conceptualized in axioms in the works of David Hilbert (1862–1899). Hilbert created two axiomatic systems: one for geometry that includes axioms of continuity (Hilbert 1971); and the other for defining **R**, that includes axioms for completeness (Hilbert 1899). The status of axiom comes from the necessity of defining **R** by means of a minimal set of statements. This is understandable if we keep in mind that this was happening within the axiomatization program of Hilbert at the beginning of the twentieth century.

The definition of **R** and its property of completeness allowed mathematicians to work in a numerical system having the same attributes as the points of the line (order, density and continuity) where well-known theorems of Calculus (like the convergence of monotonic and bounded sequences or the Intermediate Value Theorem) are provable and where every decimal expansion corresponds to a well-defined number. Analyzing the evolution of the notion of continuity in the history of mathematics, and how it is linked with the modern notion of completeness, one realizes that completeness/continuity appears in multiple forms, at various levels of explicitness, using different statements and it is given different statuses in the theory. Indeed, rigor and precision in the definitions, and logical certitude in proofs are at the heart of these changes of status.

4 The study: Sources of data and methodology of research

Four courses, *Análisis del Ciclo Básico Común*, *Análisis I*, *Complementos de Análisis II*, *Cálculo Avanzado*, were studied. In this paper, as already mentioned, I will refer to them as *Courses I*, *II*, *III* and *IV*, respectively. Each such course lasts one semester of 16 weeks, with 10 h/week of lectures and problem-solving sessions. Attendance is not mandatory for students. To pass the problem-solving part, students have to succeed at two partial examinations; once this is done, they take a global final examination. During the semesters the students' concern is to pass the problem-solving part of each course. They base their study on solving several sets of mathematical exercises and problems (in what follows we will call them sets of tasks) that are prepared by the instructors of the problem-solving sessions and the professors who give lectures. These sets of tasks, arranged in the format of a course-pack, are rather stable over the semesters. In a course, there are about 8 sets of around 18 tasks each, on average. Students are not given marks for accomplishing these tasks. In problem-solving sessions, instructors may solve some of these tasks on the blackboard or give some explanations they consider suitable for helping students to solve them. Solving the tasks and verifying the correctness of their solutions is the responsibility of the students. They can ask for help in the problem-solving sessions and they can consult several textbooks that are usually proposed as reference bibliography. Students know they will be required to solve tasks of a comparable or higher difficulty in the exams, therefore solving these sets of tasks gives them an idea of the minimal requirements for passing the course. Indeed these sets of tasks constitute an important reference not only for students, but also for instructors of the course and other courses. Instructors mainly prepare their problem-solving sessions taking these tasks as a guide. Some emblematic tasks are evoked sometimes by students, professors and instructors as examples of what was done in a course. Thus, the sets of tasks that correspond to each course play an essential role in the courses in the university program I am studying.

The following were used as sources of my data about the mathematical organizations of the courses:

- *Course I*: The syllabus, all sets of tasks, a recording of the two lectures that were concerned with real numbers, and a report with recommendations for instructors written by the professors in charge of this course.
- *Course II*: The syllabus, all sets of tasks, photocopies of students' notebooks with hand-written notes of the lectures and problem-solving sessions concerned with real numbers, interviews with two instructors and my own experience as an instructor in this course.

- *Courses III* and *IV*: The syllabus, all sets of tasks, photocopies of one student's notebook with hand-written notes of the lectures and problem-solving sessions and my own experience as an instructor in these courses.

For each course, considered as an institution in the sense of the ATD framework, the following procedures were used.

- I went through the sets of tasks, and selected, for analysis, those that, implicitly or explicitly, involved the property of completeness of **R** in any of its forms (for different forms of this property, see the Appendix), and that, in my opinion, belonged to the second level of conceptualization of **R** mentioned above. For instance, the task "Prove that $\sqrt{3}$ is not rational", has not been selected. Also tasks of the type, "Find the limit of..." were not selected (since the existence of those numbers was assumed here). But I did choose tasks which required analyzing the existence of a limit (since the existence of numbers under certain restrictions is, in my view, linked to completeness).
- When possible, I organized the selected tasks into types of tasks.
- For the chosen tasks or types of tasks, I analyzed possible techniques of solution and selected those that were the most likely to be treated in each course, based on the above mentioned available data.
- I considered the technologies as the whole set of explanations about the selected techniques: theorems, properties or some definitions that were used and accepted in each course, as well as the discourses justifying the techniques, based on the available data. I included the deduced technologies whenever their elements were made explicit, whether in the formulation of the tasks or in the discourse of the instructors or professors.
- Theories are supposed to explain, justify or give a frame of reference to technologies. In this study, whenever technologies are given by theorems, properties or definitions, theories are considered as a frame of reference, as if they were the titles of textbook chapters where the presented technologies were included, when they were made explicit. Even though they complete the description of the mathematical organizations, their role in this study is less relevant than the role of the other "Ts" of the framework.

Cases where justifications and explanations are not made explicit, may appear as having no technology or theory.

I present the tasks or types of tasks, techniques, technologies and theories for each course, preceded by a brief description of how **R** was introduced in the course.

After studying the four courses, I was able to deduce what aspects of **R** and completeness were taken into account in each course. I am not performing an analysis of the teaching approaches here; nevertheless, some teaching approaches are briefly mentioned for giving additional information about the courses.

5 The courses

5.1 Course I

Twelve sets of tasks were used in this course, that constituted the official set of problems and exercises students had to solve by the time this study was carried out. Their titles were: 1. *Preliminaries*, 2. *Real Functions*, 3. *Real Numbers*, 4. *Sequences*, 5. *Limits and Continuity*, 6. *Derivatives*, 7. *Mean Value Theorems*, 8. *Study of Functions*, 9. *Taylor theorem*, 10. *Integrals*, 11. *Applications of Integrals*, and 12. *Series*.

In lectures corresponding to the topic *Real Numbers*, the instructor first introduced the notion of interval, represented some sets on the number line and developed methods for solving inequalities with and without absolute value. She concluded her second lecture on this topic by introducing the definitions of lower and upper bounds as well as of the least upper bound (supremum) and greatest lower bound (infimum), as a means for better describing the solutions of inequalities, usually given by intervals or unions of intervals.

I selected several tasks from the sets of tasks 3. *Real Numbers*, 4. *Sequences* and 5. *Limits and Continuity*, which are described in Table 1.

Some remarks about Table 1 and Course I:

Representing numbers on the number line is a frequently used technique in this course. In lectures and problem-solving sessions, it was considered as something natural: whether or not each number has its representation on the line was not discussed, but assumed. In tasks of types 1 and 2 the technique was favored by the particular sets of numbers that were given: all the numbers the students had to find were integer numbers except for one case, where it was 1/2. The task presented in row 3 seems to be unique in its type, I cannot generalize it to a type of tasks.

In tasks of types 4 and 5 there was no reference to completeness; the stress was laid on the characteristics of the sequence and the function. Nevertheless the Intermediate Value Theorem would not be valid if the function, even if continuous, was not defined on a complete domain,

Table 1 Praxeologies related to completeness of **R** in course I

	Tasks or type of tasks	Techniques accepted by the institution	Associated technologies	Theories
1	To decide whether several subsets of **R** are or not upper or lower bounded.	To represent the set by extension, or in a drawing, and to find the answer by inspection.	The definition of upper or lower bounded subset of real numbers.	Real numbers.
2	To determine upper and l ower bounds, and supremum, infimum and maximum and minimum of some subsets of **R**.	To represent the set by extension, or in a drawing, or in a mental image, and to find supremum and infimum by observation. To see whether or not these numbers belong to the set.	The definition of supremum as least upper bound and infimum as greatest lower bound.	Real numbers.
3	To order the numbers sup a, sup B, inf A, inf B, where A and B are non-empty and bounded sets such that $A \subset B$.	To represent two sets on the number line (typically two intervals) one included in the other and to obtain the answer by inspection.	The definition of supremum as least upper bound and infimum as greatest lower bound.	Real numbers.
4	To find and justify the existence of the limit of a monotonic and bounded sequence or a squeezed sequence.	To check the hypothesis and to use the theorems that assures the existence of limit of such sequences.	The theorem that assures the existence of limit of a monotonic and bounded sequence, the Squeezing Theorem.	Sequences.
5	To prove that an equation has a solution in an interval.	To check the hypothesis and to use the Intermediate Value Theorem.	The Intermediate Value Theorem.	Limits and continuity.

and the existence of limits of monotonic and bounded sequences would not be assured in a non-complete domain. This condition usually remains implicit and consequently the meaning of these theorems (especially the Intermediate Value Theorem) is sometimes reduced.

The slots in the column of theories consist of Real numbers, Sequences and Limits and Continuity, as they were written in the titles of the sets of tasks. In this course completeness does not appear explicitly, it functions mostly as an implicit tool; its explicit mention is neither required nor used in accomplishing the tasks. Making completeness explicit was necessary for instructors (as stated in the written recommendations for instructors, for proving the Intermediate Value Theorem), but not for students. This is a first course of Calculus, which is intended, essentially, to impart technical knowledge on new objects more than to validate these properties and theorems. Accordingly, this course awards an important role to graphic representations, considering that they allow the lecturers to go quickly over several topics. In view of this, the inclusion of the concepts of supremum and infimum in this course is not quite justified: both notions are meaningful only in the context of proofs. Supremum and infimum are introduced in the set of problems for this course before studying continuous functions or limits; they appear as a goal in themselves. The instructor was trying to give these notions a meaning (even if it is an artificial one: to better describe the solutions to inequalities) for lack of situations where these notions would indeed be mathematically relevant.

5.2 Course II

Course II required Course I as a prerequisite. Seven sets of tasks were used in this course, that constituted the official set of problems and exercises students had to solve at the time this study was carried out. Their titles are: 1. *Sequences and Series*, 2. *Limits and Continuous Functions*, 3. *Differential Calculus*, 4. *Power Series*, 5. *Functions from* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}^m$, 6. *Differential Calculus in Several Variables*, 7. *Optimization of Functions from* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}$.

In the first lecture of Course II the axiom of supremum was presented, and it was used for proving that every non-decreasing and upper-bounded real sequence converges to the supremum. The Archimedean property was presented and obtained as a consequence of this axiom. The style of this lecture was mostly formal: definitions of upper bound, maximum, least upper bound; uniqueness of the least upper bound, axiom of existence of the least upper bound for an upper-bounded non-empty subset of **R**; definition of lower bound and greatest lower bound, and the first theorem: the existence of the greatest lower bound of a lower-bounded non-empty subset of **R**.

Several tasks of set 1 (*Sequences and Series*) involved completeness in one form or another. I present them in the first five types of tasks in Table 2. Set 2 (*Limits and Continuous Functions*) contained several other forms of completeness, they are presented in the last two types in Table 2.

Some remarks about Table 2 and Course II:

There is a similar task of type 1 in Course I, but, while in Course I students are asked "to decide", in Course II the instruction is to prove, that is, an explicit justification is required. In Course II, the task is: "Show that the set $A = \{n \in N : \exists m \in N, n = m^2\}$ is not bounded from above." In Course I, an answer based on the fact that the set of natural numbers is not bounded from above and taking for granted the inequality $n^2 \geq n$, would be considered acceptable. In Course II, students are expected to produce a more explicit proof, using the Archimedean property, as stated by the two instructors I interviewed. This constitutes a change in justification standards.

Tasks of type 2 occurred also in courses I and III, and I will compare the forms in which they appear in the three courses later on, after having described Course III.

Table 2 Praxeologies related to completeness of **R** in course II

	Tasks or type of tasks	Techniques accepted by the institution	Associated technologies	Theories
1	To prove that certain subsets of **R** are unbounded.	To propose any general number *M*, and to show that there are elements of the sets greater than *M*.	Archimedean property.	Real numbers, axiom of supremum.
2	To find maxima, minima, supremum and infimum of several subsets of IR, justifying the answers (most of the subsets were given by the values taken by different sequences).	Most of the questions can be answered by analyzing the existence of limits and the existence of lower and upper bounds that may belong to the set.	The definition of supremum as the least upper bound and infimum as the greatest lower bound. Monotonic and bounded sequences converge to the supremum or the infimum. Maxima and minima are supremum and infimum that belong to the subset.	Real numbers, axiom of supremum, real sequences.
3	Prove that the set $A = \{a \in Q : a^2 < 2\}$ has no maximum nor supremum in **Q**.	There are several different techniques accepted by this course, they are described in the section on Course II.	The associate technologies are presented below together with the different techniques.	Real numbers, axiom of supremum.
4	To determine supremum, infimum, maximum and minimum of some subsets of **R** and to represent them on the real line (the sets are the solutions of inequalities and equations).	To solve the inequalities and equations, to represent the solutions and to conjecture the required values from the representation of the sets.	The definition of supremum as least upper bound and infimum as greatest lower bound. Maxima and minima are supremum and infimum that belong to the subset.	Real numbers.
5	To analyze the convergence of several sequences.	To check whether or not the sequences are monotonic and bounded, or to squeeze them between two convergent sequences.	Every monotonic and bounded sequence of real numbers has a limit in **R**, the Squeeze Theorem.	Real sequences.
6	To prove that some functions are surjective using the Intermediate Value Theorem.	To use the definition of limit and the Intermediate Value Theorem for claiming the existence of a preimage for a given value in the codomain.	Definition of limit, continuity, the Intermediate Value Theorem.	Continuous functions.
7	To prove that an equation (polynomial, trigonometric) has any solution in a given interval or in **R**.	To check the hypothesis of the Intermediate Value Theorem and to use it.	The Intermediate Value Theorem.	Continuous functions.

Some of the types of tasks in Table 2 ask for a justification – that will involve explicitly any aspect of completeness – and others can be answered by a direct application of a theorem or by observing graphic representations. That is, there are two different standards of validation mixed in the same list of tasks assigned to students. For instance, tasks of types 2 and 4, both ask the students to find the maximum, minimum, supremum and infimum of sets, the difference being that type 2 tasks require justification, and type 4 tasks do not. There is a risk, therefore, that students will have trouble understanding the mathematical culture of this course: Are proofs *always* necessary? Sometimes? When? What does this depend on? If this depends on whether or not the question explicitly asks for a proof or justification, proofs could be perceived as something irrelevant from the mathematical point of view; a result only of the teacher's whim[3].

The task in row 3, which I cannot generalize into a type of task, demands: "Prove that the set $A = \{a \in Q : a^2 < 2\}$ has no maximum nor supremum in **Q**." The students know that **R** contains other numbers than **Q**; now this problem is useful to distinguish **Q** from **R** from the point of view of their properties, and not only from the point of view that **R** contains also the irrationals. There are two main possibilities in proving what is asked. One is to consider only rational numbers. Then to prove A has no maximum, we take an element of A and prove there is a bigger one; to prove A has no supremum we take the set of positive rational numbers whose square is bigger than two and prove it has no minimum. The set A does not have supremum, even though it is non-empty and upper-bounded, that is, the axiom of supremum does not hold for **Q**, and consequently results depending on this axiom do not hold either. For instance, continuous functions taking values of different sign in a closed interval may have no zeroes. Students could perceive thus the differences between having and not having this axiom. Another possibility for proving what is asked is to consider A as a subset of **R**, to find the supremum ($\sqrt{2}$) and to prove that it does not belong to **Q**. Proving only with rational numbers could facilitate perceiving the relevance of the axiom, which is not the case for the second proof.

One of the interviewed instructors said about this problem: "I solved this exercise on the blackboard. In the next class many students came to me asking questions about the details of the proof. In doing it, I assumed that we knew almost nothing [...] I did it all (manually). I only required that it be possible to choose a positive rational number between zero and another rational number, for instance, by dividing the number by two. In the next class students were asking: 'could you please tell me what I am supposed to prove, and what not? In this exercise, what can we assume [as known]? That $\sqrt{2}$ is irrational? It seems that we cannot, but why? We have already learned that...' " (My translation from Spanish).

This instructor used only rational numbers in his proof, and this was confusing for the students who had already used irrational numbers in other exercises. Every proof assumes certain things as known, or already proved. But the choice of those things may appear quite arbitrary to the students – a part of the didactic contract with a particular teacher – and not depending on the mathematical organization alone.

In tasks of types 5, 6 and 7 the property of completeness passes mostly implicitly, hidden in the hypothesis of the Intermediate Value Theorem and the property of convergence of a monotonic and bounded sequence.

Finally, most of the tasks that ask for a justification have one common feature: the results that are to be justified are not "counter-intuitive"; they can all be "seen" as obvious from the

[3]For the expert or for an advanced student this would not pose a problem. In fact, there are situations in mathematics where the experts or advanced students just compute something or are convinced by a graph without worrying about proofs. When they have doubts about something or they have to communicate to others they stop and think about the assumptions and how well they are founded. Such situations happen naturally side by side and this does not shake mathematicians' understanding of the relevance of proofs in mathematics, because they have control over what they do. For a novice, there is the risk of perceiving validation as something external, that depends arbitrarily on the formulation of the tasks, instead of perceiving it as an internal need, a characteristic of the mathematical work at a certain level.

perspective of geometric representations. But their theoretical justification always requires completeness or some of the statements that can be deduced from it. The purpose of the justifications is not to convince, but rather to explain, to understand the mathematical reasons beyond the obvious, and to make explicit the coherence of a mathematical theory. As stated by the instructors, students often feel uneasy with these tasks; they not only have to learn how to make proofs; they also have to accept that writing a proof is a legitimate task they can be assigned in the course.

This students' uneasiness may be reinforced by the formal style of presentation in the lectures. In the course we observed, the axiomatic presentation of **R** did not give reasons why the axiom of completeness was included. These reasons are, indeed, of the same order as those that may lead someone to accept making proofs of intuitively obvious theorems.

5.3 Course III

Course II was a prerequisite for Course III, which included topics such as *Real Numbers*, *Norms in* $\boldsymbol{R}^n$, *Series*, *Topology of* $\boldsymbol{R}^n$, *Limits of functions*, *Riemann–Stieltjes Integral*. There were several tasks involving completeness, described in Table 3. In the lectures of Course III, **R** was officially defined as a set of rational cuts, and it was shown that every non-empty upper-bounded subset has a supremum; that every nested sequence of closed intervals with lengths tending to zero has an element in its intersection; and that every Cauchy sequence is convergent.

There was a distance between the contents of the lectures and the tasks for students. In the exercises for the students there was no mention of cuts or Cauchy sequences. These theoretical aspects were manipulated only by the instructor in Course III. It is only in Course IV that they would become also the students' responsibility.

Some remarks about Table 3 and Course III:

Let's consider the task in row 1:

Let $A \subset \mathbf{R}$ be a non-empty and upper-bounded set.

Prove that the following statements are equivalent :

1. s satisfies the conditions:
   (a) $\forall a \in A: \ s \geq a$
   (b) if $t \geq a$ for all $a \in A$, then $t \geq s$
2. s satisfies the conditions:
   (a) $\forall a \in A: \ s \geq a$
   (b) $\forall \varepsilon > 0 \ \exists a_\varepsilon \in A : s - \varepsilon < a_\varepsilon$
3. s satisfies the conditions:
   (a) $\forall a \in A: \ s \geq a$
   (b) there exists a sequence $(a_n)_{n \in N}$ in A such that $\lim_{n \to \infty} a_n = s$

This task can be considered as belonging to the type "to prove the equivalence of two or more statements". This is a new type of task relative to the previous two courses; now the goal is to prove two (or more) conditional statements. It is not a statement 1 or a statement 2 that have to be proved, but the conditional statement that 1 implies 2 and vice versa. The instructor solved on the blackboard 1 implies 2 by a direct proof and 2 implies 1 by contradiction and he left the equivalence with 3 as an exercise for students, suggesting them to prove 3 equivalent to 2 rather than 3 equivalent to 1. One technique for obtaining 3 from 2 accepted by this course is to construct a sequence, considering the hypothesis 2 and

Table 3 Praxeologies related to completeness of **R** in course III

	Tasks and/or type of tasks	Techniques accepted by the institution	Associated technologies	Theories
1	To prove the equivalence between two or more statements in the section of course II.	The associated techniques are described below.		**R** is a complete ordered field.
2	For sets $A \subseteq B$, to find and to prove the relations between sup (A), sup (B), inf (A), inf (B).	To conjecture the inequalities by means of a representation and to prove them using the definitions of supremum, infimum and inclusion of sets.		**R** is a complete ordered field.
3	To find and prove the relations between sup ($c \cdot A$), $c \cdot$sup (A), inf ($c \cdot A$) and inf ($c \cdot A$). To find and prove the relations between sup ($A \cdot B$) and sup ($A+B$) with sup (A) and sup (A).	To examine particular cases and to make a conjecture. To use the definitions of supremum and infimum and the definition of the sets $c \cdot A$, $A+B$, $A \cdot B$.		**R** is a complete ordered field.
4	To determine supremum, infimum and maxima and minima of some subsets of IR and justify.	The techniques are different regarding the sets that are presented.	The definition of supremum and infimum in any of their equivalent forms.	**R** is a complete ordered field.
5	To prove that the distance between $x \in \mathrm{IR}^n$ and a non empty set $A \subseteq \mathbf{R}^n$ is well defined by $d(x,A) = inf\{d(x,y), y \in A\}$	To show that the set $B = \{d(x,y), y \in A\}$ is not empty and bounded below, therefore inf B exists.	The definition of infimum is useful to prove that a distance is well-defined.	Topology in $\mathbf{R}^n$.

taking successively decreasing ε, for instance $\varepsilon = 1/n$. For proving 3 implies 2 it is necessary to use the definition of limit of a sequence. The explanations instructors usually give concerning this type of task tend to justify the task by the advantage of having equivalent characterizations, rather than justifying the used techniques, that is why the technology slot for this task in Table 3 is empty.

Solving the tasks in rows 2 and 3 (which I cannot generalize into types of tasks) certainly helps to develop some abilities in proving and manipulating definitions, and this is may be the reason why they are there, and not because they help in understanding **R** and completeness. Discourses and explanations of instructors are mostly based on the argument that proofs are important, and as in tasks of type 1, they tend more to justify the tasks themselves rather than the techniques.

The task in row 2 was also in Course I, but in Course III a proof is required. Also tasks of type 4 were in courses I and II, an example in the next section will highlight the difference.

5.4 A common type of tasks in Courses I, II and III

Tasks of type 2 in the courses I and II, and tasks of type 4 in Course III are essentially the same, except for the fact that in courses II and III students are asked to justify their answers. It maybe useful to illustrate this analysis by choosing one task and comparing the techniques that each course would consider acceptable for accomplishing it. Let us consider, for example, the set $A = \left\{\frac{1}{n} : n \in N\right\}$. Representing this set on a number line and inferring, "by inspection", that inf A=0 and sup A=1, is a technique that would be accepted in Course I, but not in II and III, where the student is expected to reason more explicitly and formally. For instance: A is a bounded set; its elements satisfy $0 < \frac{1}{n}$, because $\frac{1}{n}$ is a ratio of two positive numbers; $\frac{1}{n} \leq 1$ as every natural number satisfies $n \geq 1$. Thus the set A is shown to be lower-bounded by 0 and upper-bounded by 1. Since $1 \in A$, the number 1 is the maximum. For proving that 0 is the infimum, the following arguments can be used:

1. showing that no positive real number can be a lower bound: given $\varepsilon > 0$, it is enough to take $n > \frac{1}{\varepsilon}$ for $\frac{1}{n}$ to be less than ε. In this case, the Archimedean Property is used, even if it remains implicit. This technique would be acceptable in Course II. In Course III, an explicit reference to the Archimedean Property would be required.
2. As the elements of the set form a decreasing and lower bounded sequence, then it converges to the infimum: therefore the infimum is 0. This argument would be accepted in both, Course II and Course III.
3. 0 is a lower bound, and there exists a sequence, included in the set, that converges to 0, therefore 0 is the infimum. This argument would be acceptable in Course III where this characterization of supremum was introduced.

Thus, the courses differ in the techniques and technologies that are considered acceptable for solving the same type of task; the differences reflect increasing justification standards.

5.5 Course IV

Course III is a prerequisite for Course IV. Topics included in this course are *Real Numbers*, *Infinite Sets*, *Metric Spaces*, *Rudiments of the Theory of Banach Spaces*, *Sequences and Series in the Complex Field*, *Differentiation in Euclidean Spaces* and *Concept of Differential Equation*. Students in this course will study more general spaces, and will know other complete spaces than **R**. In the lectures, the axioms of a complete ordered field were presented and it was shown that the set of rational cuts satisfies these axioms. That is, Dedekind cuts do not play the role of a construction, but that of a model in showing the consistency of a theory.

There are several problems on completeness in the official sets of tasks students have to accomplish. Kinds of tasks are mostly to prove, to conjecture and prove, to analyze, to give examples, to prove the equivalence of two or more statements. Problems regarding real numbers are in the first set of tasks, from where I selected one which would correspond to the type *to prove the equivalence of two or more statements*:

Show that in an ordered Archimedean field the following statements are equivalent[4]:

1. Every bounded sequence has a convergent subsequence.
2. Every Cauchy sequence is convergent.
3. If $(I_n)_{n=1}$ is a nested sequence of closed intervals whose lengths tend to zero, then there exists a unique element $x \in \cap_{n=1}^{\infty} I_n$.
4. Every non-empty and upper-bounded subset has a least upper bound.
5. Every monotonic and bounded sequence is convergent.

This task shows different aspects of completeness of **R** that, at first sight, do not seem to be linked. Starting from each of the statements, it is necessary to build an object that satisfies certain conditions. The proof of equivalence of several statements is a type of task that has already been introduced in Course III. Students must generate conditions in order to use the hypotheses. For instance, starting from 1 leads to building suitable sequences and subsequences from a nested sequence of closed intervals, or from a bounded set, or leads to wondering whether a Cauchy sequence is or is not bounded in order to be able to use the hypothesis. As well, starting from 2 leads to building a Cauchy sequence or subsequence from a bounded one, or from a nested sequence of closed intervals, or from a bounded set, or from a bounded and monotonic sequence again in order to use the hypothesis, and so on[5].

The instructor proved two of the conditional statements and left the others to students, suggesting an order to make the proofs. Potentially this problem facilitates the study of completeness, and the interpretation of each statement as an expression of this property. Reflections on these aspects won't come automatically from making the proofs, and opening the door to discuss them would help students to evolve in their conceptualizations. The study of complete metric spaces in this course may also facilitate seeing completeness of **R** from a more general perspective, observing that some of the expressions of completeness can be generalized while some others require specific properties of **R**, namely, its order properties.

6 Synthesis and discussion of the results

An analysis of practices was performed. What is its scope? What can be said about the mathematical organization in the different courses? And what does this analysis allow us to see regarding the transition from Calculus to Analysis? I address these questions in this final section of the paper.

[4]Remark: an ordered field K that satisfies one of the statements 1, 4 or 5 is necessarily Archimedean.

[5]Proofs available in Bergé (2004) (go to Anexos al capitulo 3.doc).

In the introduction, I assumed that analyzing the mathematical content of the courses in terms of "praxeologies", i.e. by identifying the types of mathematical tasks studied in the courses, and the accepted techniques, technologies and theories for solving them, will provide elements for modeling the mathematical organizations relative to completeness of **R**, specific for these courses. In short, I may say that, at the most elementary stage, justification based on observation or geometric intuition is considered sufficient; later on, students are required to prove things that may be obvious for them; finally, students are asked to prove relations that may be quite unobvious. I elaborate on this point:

Graphic representations and what can be inferred from them occupy an important place in Course I. This allows students and instructors to make quick progress in solving certain tasks in Calculus. The related techniques consist of representing sets on the number line, and producing answers to questions about their boundedness based on such representations and, occasionally, a direct application of some unquestioned theorems. The existence of numbers under certain conditions is not an object of study in this course; consequently, completeness is used only implicitly here. It remains hidden behind the obvious "continuity" of the graphic representation of the number line, and theorems such as the Intermediate Value Theorem – that assure the existence of some numbers. Course II raises the standards of validation, taking into account the fact that using the information obtained from a graphic representation can sometimes be insufficient. These standards of validation – sometimes regarding the same type of task that students had already solved by observation in Course I – seem to be raised mostly as a matter of changes in the didactic contract rather than by showing that geometric intuition and graphical representations can be misleading.

Tasks in Course III are less varied: they stabilize around the "Prove that" type. Moreover, the concepts of supremum and infimum change their status. In the previous courses they were numbers to be computed. In Course III, they are objects that can be manipulated, theoretically compared, added or multiplied, and are useful conceptual tools in defining the notion of distance or later on, the Riemann–Stieltjes integral. However, I surmise that the arithmetic of supremum presented in this type of tasks (see for instance rows 2 and 3 of Course III) does not make completeness any more meaningful or significant for the students, even though it does contribute to develop some new techniques. It does not make the notion of completeness any clearer, and provides little additional clues as to where this property can be used or is useful. In Course IV supremum and infimum are one more way of expressing completeness and their utilization in proofs is more conceptual than in Course III.

A global look at the Tasks or Types of Tasks columns in the tables, suggests that tasks evolve from computational and other technical exercises ("find", "decide", "determine") in Course I, to tasks requiring more and more sophisticated forms of justification: "justify" in Course II; "prove", "give a formal proof" in Courses III and IV. This forces the evolution the techniques in the same direction, from applications to more creative proofs.

Finding types of tasks was not always possible for me, as it was pointed out in several tasks through the study of the courses. The higher the theoretical level of the mathematics courses increased, the less I could generalize the tasks into types of tasks without falling into broad categories of tasks such as "to prove".

Technologies in courses I and II are described generally by properties, theorems and definitions that support computational and technical exercises; meanwhile in courses III and IV justifications and discourses of instructors tend to justify the tasks rather than the techniques, and no technology or theory is described in the strict sense considered by ADT.

I end this discussion with a brief parenthesis. The approach to completeness in courses II and III is done by means of the axiom of supremum. The supremum has the advantage that, once it is known, its use in proofs is versatile, taking into account the different ways of characterizing it. The disadvantage is that it must be defined as a new object of study for that purpose, something that is hardly understood at the beginning. It could be interesting to study the effects of introducing completeness by the statement that every monotonic and bounded sequence of elements of **R** is convergent in **R**: I hypothesize that since students have already work with sequences and limits, the emphasis would be located in the new subject, namely, the axiom guarantees the existence of a limit in the set.

7 Conclusions

My goal in this research was to find some clues for understanding the increase of failure rate as students went on from studying Calculus to studying Analysis. I now present some reflections that would not have arisen without the analysis of praxeologies of the courses.

In the historical development of mathematics as a research domain, completeness came into being as a tool for the construction of proofs in a context where the existence of numbers is no longer taken for granted. This genesis distinguishes completeness from other concepts in analysis. From this points of view, there is no reason to include completeness as a topic of study in an introductory Calculus course such as Course I, which awards little attention to theoretical justification. This concept acquires meaning and significance only when the validation of the mathematical work is at the centre of attention. Asking students to prove results that are obvious for them, as in Course II, may trigger an activity of validation in the classroom, but in this situation it may not be very easy to make students engage with the activity as mathematicians rather than just students accomplishing an arbitrary and meaningless school task. Still, looking at the practice from the institutional point of view, a positive aspect of the process can be acknowledged: In accomplishing the task of proving the obvious, one kind of unknown is eliminated: students, now conscious of the work to be performed, already know the result they are to prove, and can concentrate on constructing the proof. This is why I think that Course II has a privileged position in that it is preceded by Course I: the content of the statements to prove is not new for the students. There is the advantage of now being able to reflect and discuss, among others, such things as what to prove and what not to prove; what is assumed in proving, and what is not; of having an axiomatic system as a departure point for proving; and of taking a distance from intuition, to mention but a few. In the course of such discussions, there is a possibility of making a more conscious reference to questions such as what the axioms are, and why it is necessary to include the axiom of completeness. If the axioms are shown as the necessary departure point in order to prove some theorems, the students can understand more – not only about completeness, but also about how an axiomatic theory functions in mathematics, in general.

In Course III accomplishing the selected tasks requires training in making proofs in a methodic formal style, mostly regarding supremum. Formal style continues in Course IV, where different aspects of completeness previously studied are generalized and put in relation with some new others. In this course the notion of completeness can be identified as what has been called by Robert and Robinet (1996) a Formalizing, Unifying and Generalizing concept. The use of a meta level in teaching and learning such concepts has been discussed in several papers (Robert and Robinet (1996), see also Dorier 1995).

The notion of completeness of **R** inhabits both, Calculus and Analysis courses, but it takes more or less explicit forms with regard to theoretical justification. Understanding the changes of status that this property undergoes in passing from Calculus to Analysis requires a perspective that learners do not spontaneously take in accomplishing the tasks. Students are not naturally inclined to take this perspective, and this fact is not sufficiently seriously taken into account in the mathematical organization of the courses. Maybe the increase of the failure rate in the transition from Calculus to Analysis could be explained by the lack of this perspective in both students and course designers and teachers. Yet, I surmise, my analysis gives enough evidence that the sequencing of the courses and their content offer an advantageous institutional basis which could privilege a better organization of the transition.

Appendix

Different ways of defining completeness

There are several ways of thinking about and defining the property of completeness of **R**. We present a – non-exhaustive – list of different equivalent ways of characterizing it[5]:

1. Every bounded sequence of elements of **R** has a subsequence convergent in **R**.
2. Every Cauchy sequence is convergent in **R**.
3. If $(I_n)_{n=1}$ is a nested sequence of closed intervals of **R** whose lengths tend to zero, then there exists an unique element x in **R**, $x \in \bigcap^{\infty}_{n=1} I_n$.
4. Every bounded infinite subset of **R** has an accumulation point in **R**.
5. Every non-empty and upper-bounded subset of **R** has a least upper bound that belongs to **R**.
6. Every monotonic and bounded sequence of elements of **R** is convergent in **R**.
7. **R** is connected.
8. Every continuous function defined in **R** that takes values of different sign in a closed interval takes the zero value at an element of this interval.
9. Every cut of elements of **R** has a unique element of separation that belongs to **R**.
10. Every covering by open sets of a closed and bounded subset of **R** has a finite sub-covering.
11. Every decimal expansion is a number that belongs to **R**.

Each of these statements shows different aspects of the completeness of **R**. If they are thought of as hypotheses for proving theorems, each one involves different images, tools and ways of operating, that make us think of them as different conceptions (Artigue 1990) of completeness. Beyond the study of **R**, these properties reflect more general principles that go through all of mathematics: the construction of objects by approximation of others of a particular type; the completion of metric spaces, the individualization of an element by means of a nested sequence of closed sets, the attainment of extremes, etc.

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