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Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking

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Abstract A common approach used for introducing algebra to young adolescents is an exploration of visual growth patterns and expressing these patterns as functions and algebraic expressions. Past research has indicated that many adolescents experience difficulties with this approach. This paper explores teaching actions and thinking that begins to bridge many of these difficulties at an early age. A teaching experiment was conducted with two classes of students with an average age of eight years and six months. From the results it appears that young students are capable not only of thinking about the relationship between two data sets, but also of expressing this relationship in a very abstract form.

Keywords Algebraic thinking · Elementary students · Visual growth patterns · Semiotics · Student thinking · Teaching actions

1 Introduction

Mathematics activity is seen as the domain of reasoning about objects and their relations, and involves examining and investigating the truth of claims about those objects and relations (Carpenter et al. 2003). The power of mathematics lies in relations and transformations that give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning in the research literature (Johnassen et al. 1993; Sfard 1991). Thus the focus of mathematics teaching should be directed to fostering fundamental skills in generalising, and expressing and systematically justifying generalisations (Kaput and Blanton 2001). Traditionally, elementary schools place little emphasis on relations and transformations as objects of study. It appears that, as

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Malara and Navarra (2003) argued, classroom activities in the early years focus on mathematical products rather than on mathematical processes. One response to this is the introduction of algebraic thinking in the elementary classroom. Early algebra is not the same as algebra early. Algebraic reasoning in the early years includes a deep understanding of the mathematical structure of arithmetic expressed by language and gestures using concrete materials and representations. The language in particular appears to be an area in which many elementary students experience difficulties (e.g., Warren 2006).

A common activity that occurs in many early years' classrooms in the Australian context is the exploration of simple repeating patterns using shapes, colours, movement, feel and sound. Typically young students are asked to copy and continue these patterns, identify the repeating part, and find missing elements; a focus is on single variational thinking where the variation occurs within the pattern itself. Little activity occurs with visual growth patterns. Yet approaches for introducing algebra to young adolescents (12–13 years) build on early explorations of visual patterns, using these to generate algebraic expressions (Bennett 1988).

Patterns used in formal introductory algebra experiences are predominantly visual growth patterns. Students are asked to form the relationship between the patterns and their position, and use this generalisation to generate steps in the patterns for other positions, that is, they are asked to reconsider growing patterns as functions (i.e., as a relationship between the pattern and its position) rather than as a variation of one data set (i.e., as a relationship between successive terms within the pattern itself). Commonly this involves generating the visual representation, recording data in a table (the position and number of elements at that position), and from the table identifying the relationship between the two data sets. This is distinct from pattern recognition used in mathematical induction, a prominent proof technique in discrete mathematics (Harel 2001). The focus in the former is on ascertaining the functional relationship between data sets and exploring the concept of a variable.

Past research has indicated that many young adolescents experience difficulties with the transition to patterns as functions (Redden 1996; Stacey and MacGregor 1995; Warren 1996; Warren 2000). These difficulties include the lack of appropriate language needed to describe this relationship, the propensity to use an additive strategy for describing generalisations (i.e., a focus on a single data set), and an inability to visualise spatially or complete patterns (Warren 2000). Students also fail to link the position number to the pattern (MacGregor and Stacey 1996; Warren 1996), express the generalization in natural language (Redden 1996), and converting the pattern to a table of values and searching for relationships within the table results in increased cognitive load (Warren 1996). These difficulties persist in higher levels of mathematics with many older students focusing on result pattern generalization (regularity in the results) as opposed to process pattern generalization (regularity in the process) (Harel 2001). However, young students are believed to be capable of thinking functionally at an early age (Blanton and Kaput 2004). The conjecture on which this research is based is that the difficulties that occur with adolescent students stem from a lack of early experiences in the elementary school that support such approaches. In the Australian context not only have young adolescents had very little experience with visual growth patterns in the elementary school but also little experience with arithmetic being more than a process used for finding answers. This research investigates teacher actions that begin to assist young students to view and describe visual growth patterns in terms of their positional relationships. The specific aims of this research were to investigate instruction that begins to assist young students (1) to create unknown steps/positions in growing patterns, and (2) to articulate the generality of the visual growth pattern in terms of its position in the pattern.

1.1 Theoretical perspective

The theoretical framework that guided this research was the theory of semiotics. A semiotic perspective on mathematical activity provides an alternate lens through which to view the teaching and learning of mathematics. It is driven by a primary focus on signs and their construction (Ernest 2002). It is concerned with the production of signs and their use with a particular focus on the personal appropriation of the underlying meaning embodied in the relationship between signs. In contrast to cognitive and constructivist perspectives on the development of learner schemas or cognitive structures, a semiotic perspective offers a focus on socially displayed semiotic activity. In this perspective the reading of texts, making sense of tasks, computations, language, gestures, imitation and mental imagery all have semiotic functions. The underlying belief is that when people communicate through all sorts of signs (both idiosyncratic and conventional) knowledge emerges, with the individual continually interpreting and reinterpreting these signs (Peirce 1960).

The semiotic relation is inherently triadic, linking an object, a representation (called by Peirce the representamen) and the interpreter (referred to by Peirce as the intepretant) (Peirce 1960). Signs are concrete things, tokens or marks that by themselves have no meaning. It is in their interpretation that meaning exists. In this definition signs and representations are synonymous. Peirce (1960) distinguished three different types of signs, each dependent on and differing because of one's interpretation of their relationships with the object. These were icons, indices and symbols. *Icons* possess the closest physical resemblance to the object itself, a similarity to or an analogy of the object. *Indices* are associated with a particular object but do not necessarily have the characteristics of or provide an analogy of the object. Indices can exist without the object but it is their relationship to the object that gives them meaning. In their simplistic form they can act as a pointer to the object itself. *Symbols* are representations that fulfill their function regardless of any similarity or analogy with their object and equally regardless of any factual connection therewith, but solely and simply because they will be interpreted to represent the object. Something is an index or icon only if it functions as such, and this function relies on the interpretation of the interpreter. Both icons and indices are essential when introducing anything new within the mathematical discourse (Otte 2006).

Within this framework, language descriptions and symbolic notation systems both are designated as symbols. Vygotsky (1934/1986) regarded signs as tools that were capable of influencing one's inward behavior and the behavior of another. Semiosis involves the process of going beyond particular signs to more and more complex representations incorporating new signs and generalizations, an evolving process (Otte 2001). The central notion of knowing is about semiotic activity (relating signs, objects and interpretations) and critical awareness rather than mental representations (Otte 2006).

Knowledge emerges as a social product when students communicate about an object through all sorts of signs that are continually interpreted and re-interpreted by the individual. Saenz-Ludlow (2001, 2006) suggested that a semiotic view of psychological activity straddles the belief that cognitive activity is primary to the construction of knowledge (Piaget 1970) and the belief that social interaction mediated by symbolic tools plays a fundamental role in the psychological activity of the individual (Vygostky 1934/1986). Presmeg (1997) claimed that when one recognizes the structure of the system one engages in, explains this structure to others by such means as encoding it in a diagram or applying some overarching framework then mathematics exists. Thus signs play a dual role in cognition. On one hand they are a means of dealing with the object of knowledge while on the other they are social, where we find a niche for meaning. In semiotics neither the cognitive domain of the individual nor the social interaction is primary. Both coexist and support the evolving construction of meaning (Radford 2001).

From this perspective the focus shifts from specifying cognitive behaviors to characterizing mathematical experiences by taking into account the social and cultural aspects of this activity (Ernest 2002). Thus the essence of the methodology is to encourage student–teacher and student–student interactions to provide a range of opportunities for students to construct, symbolize and explain their understandings. From this perspective the teaching process has two related dimensions, namely, the instructional planning to be carried out by the teacher, including opportunities for participants to share meaning, and the social organization of the classroom setting so that peers contribute to each other's learning. Thus the teaching and learning process can be seen as a process of semiosis where the teacher and students become both contributors and interpreters.

In this research the functional relationships represented by the visual growth patterns were the objects of the signs. The representations (the signs) generated to assist in the interpretation of the objects (e.g., the external representations including diagrams, drawings and arrangements of concrete materials, and verbal argumentations about the representations and mathematical ideas) were the tools used to influence student's behavior and the interpreters were the students themselves and the researchers.

2 Method

The methodology adopted for the Teaching Experiments was the conjecture driven approach of Confrey and Lachance (2000). The conjecture consists of two dimensions, mathematical content and pedagogy linked to the content. The design aimed to produce both theoretical analyses and instructional innovations (Cobb 2000) though with one variation; that one of the researchers acted as teacher. In this type of research instructional design and research are interdependent (Cobb et al. 2003). It involves attempting to support the development of students' learning and investigating the processes and actions that assist this learning. Thus a hypothetical learning trajectory is postulated and conjectures are formulated about envisaged learning processes and specific means that might support these processes. The research occurred over a two lesson sequence. During and in between each lesson hypotheses were conceived 'on the fly' (Steffe and Thompson 2000) and were responsive to the teacher–researcher and the students. Instructional tasks were generated prior to the commencement of each lesson. During the lessons some tasks were modified according to the classroom discourse and interactions, with new representations being introduced in order to challenge students' thinking and encourage them to justify their understandings.

2.1 Sample

Two lessons were conducted in each of two classrooms from two middle socio-economic elementary schools from an inner city suburb of a major city. The sample, therefore, was comprised of 45 students (average age of 8 years and 6 months), a classroom teacher and 2 researchers. The lessons reported in this paper were those conducted by one of the researchers (teacher/researcher). The lessons were each of approximately one hour's duration. Figure 1 presents a sample of visual growth patterns explored in the 2 lessons.

The first lesson focused on copying and continuing simple visual growth patterns, describing the patterns in terms of positional language, and using this relationship to predict and create the pattern for other positions. In this instance the patterns chosen were those

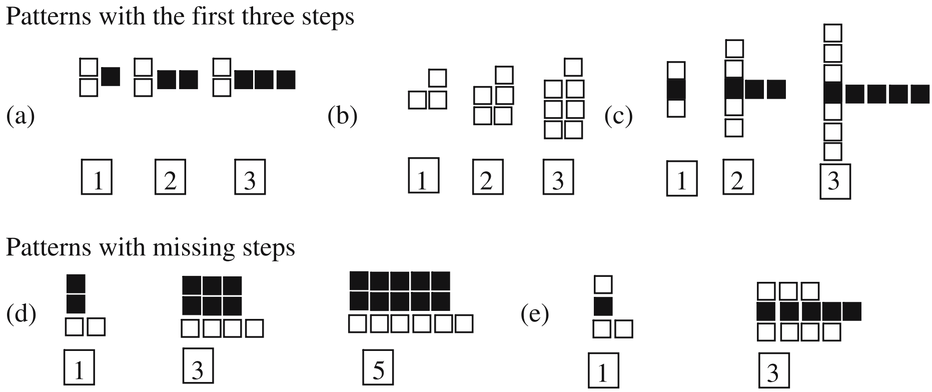

Fig. 1 Typical patterns presented in the two lessons

where the links between the pattern and its position were visually explicit. (e.g., a pattern where its width is its position and its height is always 2). The second lesson entailed re-examining some of these patterns, extending young students' language and thinking to describe and predict the patterns for any position, reversing the thinking (i.e., identifying the position when given the pattern). It was conjectured that these types of activity would assist students to focus in particular on the relationship between the position number and the pattern. Given the importance placed on describing the pattern in everyday language and its relationship to formal symbolic notation systems, the focus of these lessons was primarily on the development of this language. Hence it was decided not to record the data in a table of values but to ascertain if students could link the position with the construction of the pattern itself and describe this link in natural language. Warren (1996) found that converting a visual pattern to a table of values increased the processing load, making the task more difficult. In fact, recording data sequentially in a table appeared to encourage single variational thinking, finding relationships along the sequence of numbers instead of finding the relationship between the pairs, hence the omission of this step in this research.

2.2 Data gathering techniques and procedures

During the teaching phases, the other researcher and classroom teacher acted as participant observers. The lessons occurred sequentially. In each instance the other researcher and classroom teacher recorded field notes of significant events including student–teacher/researcher interactions. Both lessons were videotaped using two video cameras, one on the teacher and one focussing on the students who actively participated in the discussion. At the completion of the teaching phase, the researcher, teacher/researcher, and the classroom teacher reflected on the field notes, endeavouring to minimise the distortions inherent in this form of data collection, and come to some common perspective of the instruction that occurred and the thinking exhibited by the students participating in the classroom discussions. The video-tapes were transcribed and worksheets collected. The videos and participant observation scripts served to identify specific actions, specific use of representations and conversations that supported this learning. In order to ascertain how individual students were progressing along the learning trajectory pre and post-tests were administrated before the first lesson and 2 weeks after the completion of the second lesson. The delayed post test served to probe the robustness of the learning that occurred during the lesson sequence. Thus the data were two tiered, the first tier relating to the classroom

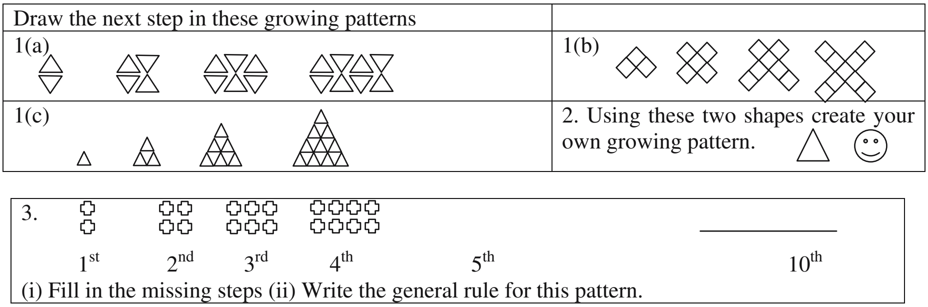

Fig. 2 Growing pattern questions on the pre and post test

learning and the second tier focussing on the individual students within this classroom. The pre and post tests were comprised of three questions as shown in Fig. 2.

Questions 1 and 2 were included to ascertain students' understanding of visual growth patterns while question 3 probed their ability to predict further positions in the pattern and describe, in general terms, the relationship between the pattern and its position. These tests mirrored the types of activities and discussions that occurred during the teaching phase. The delayed post test served to ensure that the responses reflected student's own thinking, rather than simply recalling the discussions that ensued during the teaching phase or recognising the items from the pre-test.

3 Results

The results of the pre and post-tests indicated that there was growth in students' understanding of growing patterns and in their ability to describe in general terms the relationship between the pattern and its position (Table 1).

As these results indicated, at the beginning of the teacher phase many students experienced difficulties in simply continuing and creating growing patterns. Even though after the two lessons, many more students were successful in these activities, there were still many who exhibited some difficulties with these tasks. Responses to question 3(i) indicate that by the completion of the two lessons significantly more students were able to correctly draw the pattern when given differing positions.

The responses to question 3(ii), the question relating to writing the general rule for a simple growing pattern, were categorised. The responses fell into 7 broad categories ranging from descriptions that gave no indication of the relationship between the pattern and its position to responses that specifically related the pattern to its position.

Table 2 illustrates the categories with some typical responses and summarises the frequency of responses for each category for the pre-test and the post-test.

Table 1 Frequency of response to growing pattern questions on the pre and post test

	Pre test					Post test				
	1a	1b	1c	2	3(i)	1a	1b	1c	2	3(i)
Incorrect	16	21	21	24	5	8	16	13	20	1
Correct	29	22	21	19	31	36	28	31	24	44
No answer		2	3	2	9	1	1	1	1	0

Table 2 Responses to question 4(ii): write the general rule for this pattern

	Category	Pre test	Post test
Responses that increased in frequency in the post test			
7	Relationship between position and pattern—visual description (The top and bottom row of the stars is the same number as the step)	1	12
6	Relationship between position and pattern—total number of tiles (Each step number $x2$=number of * or It's double the number of the step.)	2	7
4	Relationship within the pattern itself (Always the same as the tops as the bottom.)	0	3
2	No relationship (You do your original number.)	2	4
Responses that decreased in frequency in the post test			
3	It grows—Single variation (The patterns keep on growing and growing.)	6	3
5	It grows in twos—Quantifying single variation (Goes up by two. One more on each end.)	22	12
1	No response	12	4

The responses that refer to some interaction between the position and pattern (multiple aspect responses: category 7 and 6) increased from 3 to 19, and the responses that focused on single variation (single aspect responses: category 5 and 3) decreased from 28 to 15. The results of a chi-squared test ($\chi^2_1=15.45\ p<05$) indicated that this difference was statistically significant, that is, the differences of this magnitude are unlikely to occur by chance.

An examination of the lessons gives some insights into the teaching actions that assisted this growth in understanding. The next section describes some teaching actions that supported this growth and thinking that hindered the generalisation process.

3.1 Supporting processes

The use of concrete materials The use of concrete materials appeared to assist many students to ascertain the missing steps in the pattern. A number of students, when completing the accompanying pen and paper worksheet, recreated the pictorial pattern with the tiles and then used the tiles to create the 5th and 10th step. They then drew a picture of their solution on the worksheet.

The specific concrete materials utilized were tiles as iconic signs for the visual growth patterns and small cards with position numbers as indexical signs for where each pattern was situated within the pattern sequence (see Fig. 1, patterns a–c). Teaching entailed enacting the relationship between these signs. Students were encouraged to create the pattern with their tiles and to identify, then name the steps in the growing pattern, placing under each the appropriate position card. This cycle continued until the students had placed the cards marked with 1st, 2nd and 3rd beneath the first three steps.

- T/R: What do you think this might be? (Pointing to the first group of tiles)
- Class: 1st
- T/R: What do you think this might be? (Pointing to the second group of tiles)
- Class: 2nd

It was conjectured that this sequence assisted students to focus on the two different aspects of the pattern (the steps of the pattern and their position in the pattern) and to begin physically to link the two signs used to represent these components. The actions explicitly focused on (a) interpreting a tile as an iconic sign for the pattern, and (b) encouraging students to view the sign for the position as an indexical sign, that is, a sign pointing to the position each component of the visual growth pattern holds within the overall pattern.

Patterns where the relationship between the pattern and position were explicit These types of patterns appeared to assist students to describe verbally the relationship between the pattern and the position, for example, it is twice the step number, it is the same as the step number, it is one more than the step number.

These discussions consisted of four distinct stages, each assisting the children to refine their description of the relationship between the two sign systems (the tiling pattern and the positional cards) to ensure that the interpretation of their natural language description (the symbolic sign) represented the object (the functional relationship represented by the visual growth pattern). The first encouraged students to proffer explicit descriptions of the pattern itself. The following excerpt exemplifies the types of discussion that occurred in the classroom context that supported this thinking for the visual growth pattern exhibited in Fig. 1, pattern b:

- Kyla: Just made it taller each time.
- T/R: You made it taller each time. And how much taller?
- Kyla: 1 row
- T/R: By 1 row. What else do you know about the one row?
- Kyla: It gets bigger.
- T/R: It gets bigger by how many?
- Kyla: 2.
- T/R: By two. Excellent. It gets taller each time by one row and it gets bigger each time by two. Did everyone have that sort of pattern?

Initially students were examining each successive step in the pattern to describe how the pattern was changing. They were only considering the iconic sign for the pattern and ignoring the indexical sign indicating the position of the component within the pattern itself. In this instance their language description for the pattern considered only the variation that existed in the pattern itself. The next section exemplifies the second phase in the discussion, beginning to relate the two signs, the iconic and indexical signs.

Explicit questioning to link the position to the pattern For the pattern presented in Fig. 1, pattern b when asked to describe the 4th step, one student responded that is was 9 tiles. Explicit questions needed to be asked to ensure that the students connected the pattern's shape to its position. These questions were of the form—What does the pattern look like? How many columns? How many in each column? For the 3rd step, how many on the left, how many on the right? The questions explicitly related the position to the pattern's visual components.

- T/R.: What about the 4th one?
- John: It has 9.
- T/R: So can you tell me what the 9 tiles look like?
- C: Um.
- T/R: Can you describe what the 9 tiles look like. It's what?
- John: 5 on one side and 4 on the other.

- T/R: How is this linked to the position number?
- John: This one is the same 4th (gesturing to the left), and this one is one more (gesturing to the right).

The above excerpt exhibits how the use of questioning and justification assisted John to begin to refine his verbal description of the pattern so that it started to bear a closer relationship to the object. From this excerpt John initially identified the iconic sign for the pattern in its holistic form. The teacher's questions began to assist him to deconstruct the sign into its parts and then relate the parts to the indexical sign pointing to its position in the pattern. Finally, John exhibited the beginning stages of semiotic activity by expressing his interpretation of the relationship between the iconic sign and indexical sign in everyday language, the beginning of the emergence of a sign as a symbol. The third phase required students to extend their thinking from small position numbers to large position numbers.

Generalising from the pattern in small position numbers, to large position numbers It was found that to articulate the relationship between the position number and the visual pattern in general terms, students needed to discuss the relationship for increasingly larger positions, for example, describe the 10th step and the 20th step. Classroom observations indicated that many students needed physically to place the signs for these positions on the table and physically construct the patterns before they were able to contribute to the discussion. Most students successfully completed this task. To ensure that they were linking the pattern's position to the pattern itself, several more discussions ensued: each time the step number increased, for example, what would the 100th step look like? 1,000th step, 3,000th step?

- Brian: Um, we first did the—cause to work it out, cause the first one had 2 on the one side and then 3 on the other.
- T/R: On this one 2 and 3 (pointing to second step)?
- Brian: Yeah and so we thought the 20th one would have 20 on one side and 21 on the other.
- T/R: 20 on side and 21 on the other. Who saw that pattern? Ok so what does the 10th one look like?
- Evan: It's got 10 on one side and 11 on the other.
- T/R: 10 on one side and 11 on the other. Everyone's got that? What about now I am going to ask you a really tricky question, what about the 50th — what would it look like?
- Helen: 50 on one side and 51 on the other.
- T/R: very good, what about the 100th?
- Elise: 100 on one side and 101 on the other.
- T/R: Very good! What about the 1,000th?
- Adam: 1,000 on one side and 1,001 on the other.

While most students in the classroom context appeared successful in completing this task; on the post-test over half the sample reverted to a single variation description of the pattern, indicating just how prevalent this thinking is.

The fourth phase involved extending this thinking to the language of generality with the Teacher Research presenting a card with "nth" on it and asking:

- T/R: What about the nth one?
- Ben: 1 nth on one side and nth on the other.

- Ben: No nth and one! They both have nth but this one has one more.
- T/R: How many would there be all together in the 100th step though?
- Karen: 201.
- T/R: How do you know?
- Karen: Easy, cause there is 100 and 100 on each side is 200 plus 1 is 201.
- T/R: Very good. Yes that's another way of thinking about it. Anyone else think about it like that?
- T/R: Another way.
- John: Each step on each side plus one more on the outside.

Many students found it difficult to distinguish between ordinal language and cardinal language when describing the pattern for the nth position. Continual questions and discourse focusing on differing interpretations assisted them to refine their verbal descriptions, continually making them more in tune with the object they were endeavouring to describe.

In fact, all students when proffering their generalizations for this pattern on the worksheet avoided the use of *n* altogether. Some typical responses for this pattern are presented in Fig. 3.

Response (c) includes the student's written description together with an accompanying iconic sign to represent the generalization.

Using colour to represent different growing components of a pattern The use of two different coloured tiles assisted students to consider two growing components of the pattern and relate each to the position number (see Fig. 1, patterns c–e). In a discussion about 1(c) Alex commented;

- Alex: Well both sides (pointing to each of the white tiles that formed the two arms of the t shape), if you add both sides together and take off one that is the amount in the middle (pointing to the black tiles that formed the stem of the t shape).
- Alex: Each side is equal to where it is.
- T/R: the 40th step.
- Alex: Oh 1 side would be 40 and the other side would be 40 so you add it together (gesturing to the two arms of the t made from the white tiles) and take one off which would be 79 that's the middle (gesturing to the stem of the t made from the black tiles).

The colour assisted Alex to identify the two components of the iconic sign that represented the pattern. This process was seen as crucial in assisting him to generate a verbal description (a symbolic sign) that represented the object. Children who did not make this distinction experienced difficulty in reaching a description. Alex also physically separated the tiling pattern into its two iconic components as he proffered his descriptions. The gesturing was seen as crucial to assisting others to understand his thinking. It is conjectured that his

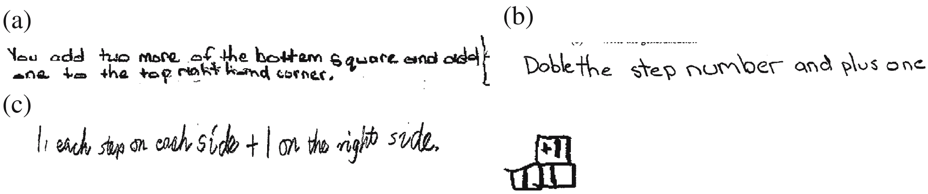

Fig. 3 Reponses to write my rule for pattern c

description which included the gesturing incorporated an iconic, an indexical and a symbolic dimension. It was in this interplay among these dimensions that the generalization not only began to become an object in its own right, but also allowed other students to gain insight into how he objectified the pattern. It could be conjectured that the use of colour began to assist Alex reach a process pattern generalization (Harel 2001).

Some other examples where colour appeared to assist students to identify the generalization are presented in Fig. 4.

Using visual patterns that were not in sequence In the second lesson students were encouraged to explore visual growth patterns that were not in sequence (see Fig. 1, patterns d and e). It was conjectured that these patterns focused specific attention on the interpretation of and relationship between the two signs for the components of the pattern (the tiles as an iconic sign and the position number as an indexical sign) as students were initially asked to construct the missing steps and then encouraged to engage in conversations about how the position number was related to the pattern.

3.2 Hindering processes

Language used to describe the generalization Most students experienced difficulty in precisely describing a visual pattern. For example, when they created a 20 by 3 array, most described this as 60. With probing, some indicated it was 3 across, 3 rows of 20 and eventually 3 columns of 20.

Writing the generalisation as compared with saying it orally The classroom discussions indicated that these students found it much easier to verbalise the generalsation than to provide a formal written response. When asked to share their written responses for the pattern delineated in question 1(c), one child shared 'it increases by 2 every time' another 'always the same number on each side with one more' and two more said 'Each step has step number on each side with one more' and 'Just put them in groups of 2 one on each other and add one on the top'. The range of responses indicated that even though many could articulate the generalisation, when it came to writing many experienced difficulties, with the tendency to give responses that focused on the single variation of the pattern. Arzarello (1998) claimed that while some students use the same words as their teachers and peers they may give them different meanings for representing the situation, indicating that

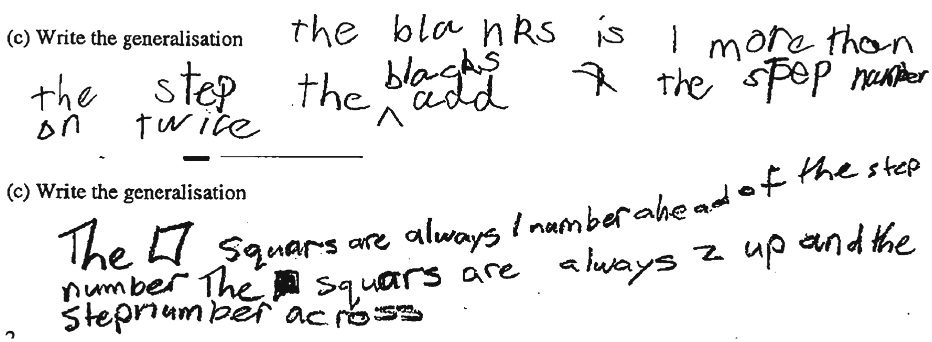

Fig. 4 Responses illustrating the use of colour to write generalizations for pattern d

for them the relationship between the signs and their mathematical meaning may be confused. This confusion is exemplified in their written responses (see response a in Fig. 3).

Completing patterns—single variation Their propensity to think of growing patterns as adding on the growing part to the preceding step impacted on their ability to create missing steps within the pattern and to create the step number when given the total number of tiles. For example, we presented visual patterns giving only 1st, 2nd and 5th steps. The students were asked to complete the missing steps. The most common strategy was to compare the 1st and 2nd step and continue adding on tiles to reach the 5th step. This single variation thinking (additive strategy) was best exhibited in the following example where they were given the 1st and 3rd step ( and ). Nearly all of the students gave  as the 2nd step. When challenged they simply recreated the 3rd step to fit their pattern. They not only ignored the signs for the position number but also continued the pattern by focusing on just the one component of the sign for the pattern itself, the 1st step. This suggests that perhaps there is a hierarchy in sign recognition, with the iconic sign being the easiest with which to engage and to interpret hence the propensity to return to describing visual growth patterns in terms of how the tiles change from position to position.

Reversing the thinking We also presented the total number of tiles and asked which step this represented. Most students found this very difficult. This appeared to occur for two reasons. First, it relied on linking the position to the step number, with which many struggled, and second in some instances it required a sound understanding of number patterns.

Expressing the generalisation in language Many students could not express the pattern in general language, and when using the language confusion occurred between the ordinal language and number of tiles.

- T/R: What if I had the nth position? What would the pattern look like?
- Jill: nth on the top and nth on the bottom.
- T/R Describe the pattern in terms of the number of tiles.
- Scott: n tiles on the top and n tiles on the bottom.

This again illustrates the many difficulties students experienced in distinguishing the ordinal aspect of the pattern and the cardinal aspect of the pattern. On the more positive note, there were at least five students in each class that could not only describe the generalities in correct mathematical language but also write these generalities using abstract notation systems (e.g., for the nth step there are n blue tiles and $n+1$ yellow tiles).

4 Discussion and conclusion

As indicated by the results of the pre-test nearly half of the students could not complete the next step in simple growing patterns nor create their own growing pattern. This could be for two reasons. First, they had had limited experiences with visual growth patterns in the early years, or second, visual growth patterns are not as easy as they first appear. An examination of curriculum documents and commonly used classroom texts would suggest that the predominant focus in the early years is on repeating patterns. These students certainly did not experience the same difficulties with repeating patterns (due to space restrictions these data cannot be reported in this paper). By the completion of the teaching phase there had been some improvement in their ability to complete and create growing patterns, indicating that perhaps the difficulties did indeed stem from a lack of experience in this area. As indicated by past research, many young adolescents experience difficulties with the transition to patterns as functions. The inability to visualise spatially or complete patterns (Warren 2000) is a key impediment to this aspect. The impact that earlier classroom experiences have on this thinking requires further investigation. This research suggests that such experiences purposely built into elementary classroom experiences may indeed commence to address this impediment.

The results confirm the conjecture of Blanton and Kaput (2004) that young students are capable of thinking functionally. The results also suggest that there are a variety of teaching actions that support this thinking, namely, using concrete materials to create patterns, specific questioning to make explicit the relationship between the pattern and its position, and specific questioning that assists students to reach generalization in relation to unknown positions. Young students are not only capable of thinking about the relationship between two data sets but also of expressing this relationship in a very abstract form. Classroom discourse that explicitly focused on the functions of each sign system also appeared to assist, especially a focus on the position cards, as indexical signs pointing to the pattern's position. The results also indicate the power of semiotic activity in raising not only critical awareness of the interplay between icons and indices but also in directing the interpretations of the signs to assist in formulating explanatory hypotheses about the object itself. As suggested by Otte (2006) not only is the function of signs dependent on the interpretation by the interpreter but specific questions and actions can assist young students to reinterpret them in ways that lead to 'knowing'. Viewing the tiles as icons and position signs as indices was essential in introducing the notion of co-variational thinking into the mathematical discourse. The confusion between ordinal and cardinal language when endeavouring verbally to use n in their descriptions about the pattern could be due to interpreting the position sign in its iconic form rather than indexical.

While young students are capable of thinking functionally, it appears from this research, single variational thinking is perhaps cognitively easier or so entrenched in early experiences that a propensity to revert to this thinking is understandable. It was conjectured that not recording the data in a table would reduce the probability of this occurring. However, instead of looking for patterns in sequences of numbers, they appeared to look for patterns in the sequence of tiles, that is, instead of saying we keep adding on 2 for the sequence of numbers in the table, they said "we add on two tiles as we proceed along the steps". This thinking was so entrenched that some students were even willing to change the examples given to make them fit their sequential thinking pattern.

The interaction between the oral description of patterns and putting this description in written form also requires further investigation. While both are viewed as symbols from a semiotic perspective, it appears that some symbol systems are easier for young students to engage with than others. Many students exhibited an ability to express the generalisations orally, but many of these oral descriptions lacked precision. So while their oral responses appeared 'correct', one wonders how much 'filling in' the listener applies when hearing the responses to questions asked. A review of the video-tapes indicated that this was indeed the case, suggesting that the precision needed for correct written responses can be missing from classroom conversations. In this instance, gestures and manipulation of materials add to the conversations, elements missing from written responses. These students also appeared to lack some of the mathematical vocabulary needed to provide precise responses, words such as "row" and "column" and describing an array as 2 rows by 4 columns. Thus on many occasions they could model the functional relationship with concrete materials and could attempt to describe this relationship using imprecise language embellished with gestures. However, they often reverted to 'lower level' responses when asked to write their generalization in a written form (e.g., "add on 2" instead of "the number of tiles is double the step number"). This result could begin to explain the large variations in responses on question 3(ii) on the post test, a problem that nearly all could complete and describe orally within the context of the classroom discourse. The role of mathematical language and mathematical understanding in the elementary classroom needs further investigation.

This research commences not only to identify teacher actions that support examining growing patterns as functional relationships between the pattern and its position, but also to delineate thinking that impacts on this process. Many of the difficulties these students experienced mirror the difficulties found in past research with young adolescents. This result suggests that perhaps these difficulties are not so much developmental as experiential, as these early classroom experiences began to bridge many of these gaps.

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