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AN EXPERIMENT IN TEACHING RATIO AND PROPORTION

ABSTRACT. This paper summarizes our analysis of the complexity of ratio problems at Grades 6 and 7, and reports a two-year experiment related to the teaching and learning of rational numbers and proportionality in these grades. Two classes were followed and observed. Part of the teaching material was common to both classes, mainly the objectives and the corpus of ratio problems in a physical context. But in one class, here called "Partial-experiment", the learning environment was exclusively a paper-pencil one and the teacher followed his usual method in designing and conducting teaching sequences. In the other class, here called "Full-experiment", the teaching was based on a framework, emerging from our analysis of complexity of ratio problems, involving precise guidelines and a specific computer environment. Using a pre-test and a post-test, we observed clear progress in both classes compared to a sample of "standard" pupils. Our comparative pupil-oriented study indicates more complete improvement in the "full-experiment" class, i.e., a better acquisition of fractions and their use for solving usual proportionality problems. The average pupil's progress is greater in the "full experiment", with the pupils who were initially high- or low-level attainers benefiting the most from the "full-experiment".

KEY WORDS: learning/teaching in secondary school, magnitude, quantity, ratio, proportionality, rational number, fraction, number line, double scale

1. ANTECEDENTS

It is well known that teaching and learning fractions, ratio, and proportionality in the middle grades are very complex processes. Given their importance in mathematics, it is essential to study these concepts as carefully as possible. There is an enormous amount of research and resources related to the domain. Mathematics curricula in many countries have taken account of the related research. The French curriculum for grades 6 and 7 includes all sorts of problems involving ratios. However, their objectives lead teachers to believe that proportionality should be acquired by the end of grade 7, activities for grades 8 and 9 being only applications and extensions, such as the linear function.

Full achievement in this field at the end of grade 7 seems ambitious: As early as 1976, the CSMS assessment of 15-year-old pupils showed less than 42% success rate in solving simple problems of enlargement. Hart (1984, p. 5) noted a percentage of additive errors between 25% and 40%, depending on the presented figures and the involved data. In 1981, Hart and the CSMS Mathematics Team, after the chapters of their testing and

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evaluation programme related to fractions, ratio, and proportion (pp. 66–81; pp. 88–101) concluded that: "*all children* [from 11–16] *make some progress but it is very slow"* (1981, p. 217). It is therefore not surprising that the French results at PISA 2003 show an important failure in the 15-year-old population in solving common proportionality problems.

With "modern mathematics", the question arose: Is it better to learn numbers and operations before applying them to magnitudes, or to study magnitudes first in order to construct numerical concepts? Nowadays mathematics education gives magnitudes a central place in learning, particularly in the domain of ratio. But during the 70's, magnitudes were dropped from many curricula, with regrettable consequences (Rouche, 1997, pp. 41–43; Brousseau, 1986, pp. 75–95).

The question remains how precisely to deal with magnitudes, and, beyond them, with the physical (in the sense of "stemming from the real world") aspects involved in ratio problems. Many classifications are based on Kieren's five sub-constructs (1980, pp. 134–136): Part-whole relation, Quotient, Measures, Ratio, and Operator. Therefore, physical as well as mathematical aspects are present in these classifications. See for instance the classification reported by Alatorre and Figueras (2005, p. 1): "*One of the issues is the context; a possible classification according to it is in rate (extensive quantities), part-part-whole, and geometrical problems; in turn, part-part-whole problems can be mixture [...] or probability problems*." Here "context" refers either to a physical sub-construct as "Part-whole", or to a physical experience as "Mixture", or to a mathematical domain as geometry. Thus, the notion of context includes, but does not differentiate explicitly physical and mathematical aspects.

Vergnaud (1983, pp. 128–140) carefully identified three different types of problems according to their mathematical structure: "*isomorphism of measures, product of measures, and multiple proportion*". For solving the former, he mainly distinguished two correct procedures, also from a mathematical standpoint: Scalar procedure, and Function procedure.[1] On the other hand, he emphasizes that: "*Most procedures used by students... had a physical meaning*" (1983, p. 142) or that "*these procedures cannot be explained by pure numerical properties. Numbers are magnitudes.*" (1983, p. 146).

All authors, particularly those quoted above, generally agree that the "build-up" strategy (e.g. 300g sugar for 4 people, 600g sugar for 8 people, another 2 people is half of 4 people, so that we need 750g sugar for 10 people), very often used by pupils, "*is a relatively weak indicator of proportional reasoning*" (Lesh et al., 1988, pp. 104–105). Piaget et al. (1968) referred to this stage as "pre-proportionality", and describe *"the true understanding of proportion as a formal operation task*" (Hart and CSM team, 1981, p. 90). So that "*students should be encouraged to move from building up, [..] to more sophisticated forms of proportional reasoning based on between- and within-relationships (Karplus et al., 1983) as they progress through middle school and prepare for algebra courses*" (Clark, 2005, p. 2). What could be the step from interwoven physical and mathematical considerations, present in the build-up strategy, to relevant formal thinking?

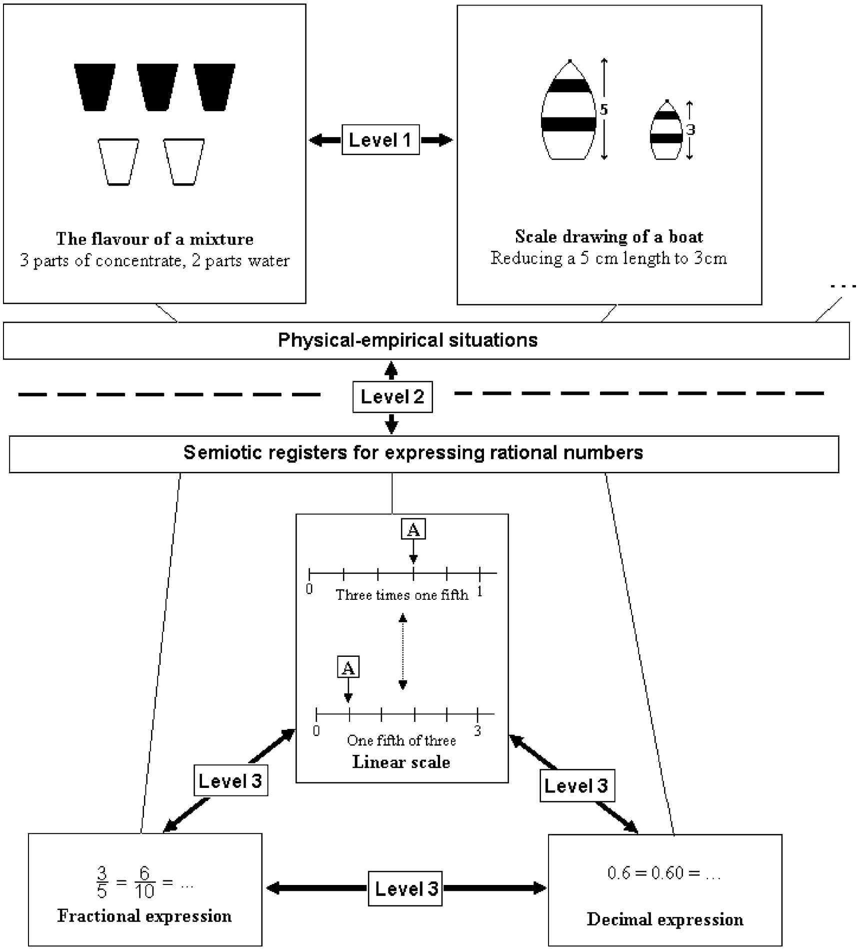
Along with the physical and mathematical, there is a third component. This would be the way of representing rational numbers. For instance, expressing and processing ratios may lead to using either fractions or decimal numbers. Lesh et al. (1988, pp. 95–96) specify seven types of proportion-related problems. The seventh type is presented as follows: "*A ratio [..] is given in one representation system, and the goal is to portray the same relationship using another representation system.*"

Authors like Duval go so far as to put representation at the core of conceptualisation (1995, p. 61). He describes the mathematical activity in terms of processing within and between semiotic systems or "settings" (Bloch, 2003), which he terms "semiotic registers". Duval (2000, pp. 61–63) writes: "*we do not have any perceptive or instrumental access to mathematical objects... as for any other object or phenomenon of the external world...*", "*...the only way of gaining access to them is using signs, words, or symbols...*". It does not mean that we can do without real-world experiments for teaching mathematics. It only means that, at a certain point of learning, pure mathematical objects must be considered and processed, if only because they can be dissociated from a particular context (e.g., sharing) and applied in many contexts (e.g., mixture, enlargement...).

2. Our analysis of the complexity of ratio problems in the middle grades

For us, a wide variety of physical-empirical situations and representations of mathematical objects are needed to learn ratio and proportionality.[2] We think it is important to give a specific place to each, and thus to figure them separately. And then, the question of articulations arises. Consequently, we obtain the configuration illustrated by Schema 1.

The upper part of this schema deals with physical situations. Only two examples are given, but others are presented further. The lower part of Schema 1 deals, in the mathematical domain, with diverse representations of rational numbers and processing. As Duval (2000, pp. 58–59), we distinguish mathematical objects from their representations. For this reason, there are two different frames for numerical expressions: fractional writing and decimal writing. A fractional writing or a fraction $\frac{a}{b}$ (a and b being integers, $b \neq 0$), is one representation of a rational number. A decimal writing, finite or infinite and periodic, could be another representation of the same rational number.



Schema 1. Physical and semiotic diversities around a rational number.

We distinguish two kinds of articulation: articulation within a frame, and articulation between frames. Double bold arrows represent the latter.

Within the frames of physical-empirical situations, there is no room for showing an example of processing. A processing for the mixture situation could be: 3 parts concentrate and 2 parts water have the same flavour as

6 parts concentrate and 4 parts water. Level-1 articulation is related to work on physical quantities of different types. It mainly consists in the determination of common and distinctive characteristics.

Between the physical and the mathematical domains, Level-2 articulation can be connected with "*the type of multiplicative relations which exist between the problem quantities*" (Harel et al., 1991, p. 127). These relations can be expressed by using relevant representations of rational numbers.

Within the frames of the mathematical domain, an equality corresponds to a "treatment" in Duval's sense (2000, pp 63–64), e.g. $\frac{3}{5} = \frac{6}{10}$, or $0.6 = 0.60$. More generally, the transformation of an object within a given register is a *treatment*, as shown by the dotted arrow in the frame of the linear scale. The bold double arrows, in the lower part of Schema 1, are Level-3 articulations and represent "*conversions*" in Duval's sense (ibid). They transform the representation of a mathematical object into a representation of the same object in another register, e.g., $\frac{3}{5} = 0.6$.

This schema emerges from our analysis of complexity, reported in a previous paper (Adjiage, 2005), which we summarize in four points below.

1. We, as others, draw all possible conclusions from Vergnaud's work (1983–1988): "*Vergnaud coined the term "Multiplicative Conceptual Field" (MCF) to refer to a web of multiplicatively related concepts, such as multiplication, division, fractions, ratio and proportions, linearity, and multilinearity*" (Post et al., 1998). Thus, we consider rational numbers, ratios, proportionality to be strongly articulated notions to be integrated in a unique teaching schema.
2. But, in order to articulate them later, we must first separate the physical or empirical component from the pure mathematical component.
3. We use Duval's theory (1995, pp. 75–81; 2000, p. 61–64) to describe the mathematical domain. According to this theory (1995, pp. 75–81), the teaching would take into account different "semiotic registers", each of them revealing specific aspects of the same object. We have retained three rational registers: linear scale (a number line with resources such as subdividing, sliding along the line, zooming... – see e.g., 3.3.2.2), fractional writing, and decimal writing.
4. The articulations within and between the physical and the mathematical components are the privileged means for conceptualising the involved objects.

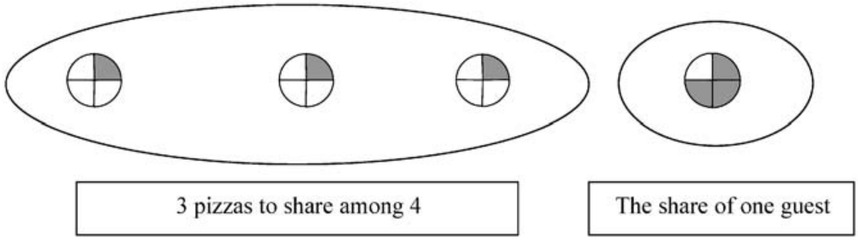
Having pointed out the physical component, it is necessary to describe it precisely. Three parameters: the number of magnitudes involved, the number of underlying objects, and the states in which the objects are examined (Adjiage, 2005, pp. 100–102), produce the following six physical-empirical ratio situations: "ratio of two heterogeneous quantities" (e.g., speed, flow, production...); "measurement"; "mixture"; "frequency"; "enlargement" (or "dilation"); "change of unit".[3] Hereafter, what we call "*context*" or "*variable of context*" will refer to these situations. In agreement with Tourniaire and Pulos (1985, p. 190), a survey conducted by Adjiage (2005) concluded the pertinence of this variable.

These situations could remind the reader of some of the five subconstructs of Kieren (1980, pp. 134–136). Actually, our categories: "ratio of two heterogeneous quantities", "measurement", and "enlargement" can refer to those of Kieren: "Ratio", "Measure", and "Operator", while others like "Mixture" or "Frequency" can refer to a "Part-whole relation" (3 baskets made for 5 attempts) or a "Ratio" (3 baskets made for 2 failed), or even an "Operator" (The success ratio of a basketball player is $\frac{3}{5}$. How many baskets will he probably make with 25 attempts?). The main difference between the two approaches is that we refer to six physical experiences whereas Kieren refers to "*five ideas of fractional numbers... as a basis for a rational number construct*" (1980, p. 134), disregarding whether those "ideas" are related to the mathematical or to the physical domain.

The linear scale register is central in the mathematical domain. We established (Adjiage and Pluvinage, 2000, pp. 52–61 and pp. 78–81) that it allows to:

- Consider a rational number, from the outset, as one number together with the first numbers encountered, i.e., whole numbers,[4]
- Highlight the link between the two needed integers that express a rational number in any system rather than these two integers separately;[4] this link is represented by the position, invariable in a scale change, of a point related to a coordinate system,
- Operate the three levels of separations/articulations presented in Schema 1.

Considering the mathematical domain separately from the physical, we would like to point out that we stand out from previous research. For instance Lesh et al. (1988, p. 112) state that: "*Fractions are special kinds of extensive quantities; they tell the size of a single object, for example, 3/4 pizza*". For us, $\frac{3}{4}$ is one of the thinkable expressions of a rational number, useful for interpreting a real-world problem, for instance sharing equally 3 pizzas among 4 guests. The pizza problem may be converted into a measure problem after a processing in the physical domain e.g.: cut each pizza into 4 pieces, put three of them together. Taking the whole pizza as a unit, one can easily recognise in $\frac{3}{4}$ a relevant numerical expression to give the final response: 3/4 pizza.



We think that pupils would be well-advised to identify the physical context, its objects, and the specific means of investigating it, separately from the underlying pure mathematical model. This separation might allow them to better articulate the physical characteristics of the problem, and the pertinent mathematical objects and procedures required to solve it. This leads us to the point 4.

Duval (1995, pp. 36–44; 2000, pp. 63–64) distinguishes "treatment" and "conversion" because the latter, unlike the former, entails a break in the means of representing and processing, and thus in thinking. Conversion between semiotic registers is a cognitive operation essential for accessing a unique mathematical object beyond its diverse expressions. While Duval only considers the mathematical component, we consider the physical one in terms of within- and between-processing.

3. General frame of the experiment and methodology

In order to test our framework, we designed a teaching experiment in which no change related to the content and objectives of the current French curriculum for grades 6 and 7 would be introduced, while the activities were specified by an explicit determination of goals, situations, and learning environment, either paper-pencil or computer sessions. We were interested in the learning progress, in both the results and the methods, we can expect from this teaching.

Two classes, often referenced hereafter as "*the followed classes*", were involved in this experiment. The two followed classes belong to a public secondary school located in a small town, 30 km to the north of Strasbourg. The results obtained by these pupils at national mathematics achievement tests allow us to consider this group, at the beginning of the observation period, as a standard sample of French 6th-graders.

The same experienced mathematics teacher[5] conducted the two classes during the two years of the experiment. He was an active participant in the experiment, particularly involved in the choice of the activities. For all other mathematical topics than ratios and rational numbers, he conducted his classes in the usual way.

The teaching relied on a common base of objectives and ratio situations, but it differed from one class to the other in environment and method. In one class, hereafter termed "*partial experiment*" (PEx), the experimentation was led by the teacher in his usual method and environment (blackboard and paper-pencil). In the second class, hereafter termed "*full experiment*" (FEx), a specific method accorded to Schema 1 was designed, and the environment included computer sessions.

Our main research questions are: Are the common elements introduced in both classes sufficient to generate clear progress in using rational numbers and proportionality? If a gap remains, is it the result of the dissimilarities in the two learning environments? Do the specific choices and implementation that we made in the full experiment have an observable qualitative impact?

Our main hypotheses are:

1. Varying the ratio situation contexts according to a systematic classification allows pupils to perform relevant proportionality procedures and thus obtain better results.
2. Working systematically within and between the physical and the mathematical domains involved in ratio problems, according to Schema 1, helps pupils to better process rational numbers, identify common underlying features in ratio problems, build up a model of proportionality, and perform advanced strategies.

3.1. *Common features to PEx and FEx*

Four general objectives, in accordance with the French curriculum, were established before the experiment started. At the end of the experiment, the pupil is expected to be able to:

1. Express a rational number using one of the three registers: fractional writing, decimal writing, and linear scale. Convert the expression of a rational number from one register to another.
2. Resort to appropriate registers to interpret and then solve a ratio problem, falling or not within proportionality, whatever the context and numerical data.
3. Compare rational numbers, insert a given rational number between two others, and locate it between two decimal numbers.
4. Decide if a rational number can be written as an integer, as a decimal number.

Both classes were supposed to investigate the mathematical and physical world of ratio through appropriate working sessions. They were submitted the same corpus of physical-empirical ratio problems, based on the six types of situations mentioned in 2. We also resorted to the variables which *"influence the type of proportional strategy likely to be used"* (Tourniaire and Pulos, 1985, p. 190): numerical complexity of the data (Tourniaire and Pulos, 1985, p. 188); relationships within the data – the involved ratios may be whole numbers, rational numbers, the latter being smaller than 1 or not (Noelting, 1980, pp. 227–228); the nature of the question – working out a fourth proportional, a scale factor, comparing ratios (Harel et al., 1991, pp. 125–126).

3.2. *Teaching in PEx*

In the partial-experiment class, the teacher proceeded in his usual way, mainly a constructivist approach. Mathematical concepts are built up in a "tool-object dialectic"[6] (Douady, 1984, pp. 14–17). This means that an already-studied mathematical being is regarded and used as a tool for solving a problem. During the solving session, it becomes apparent that this tool is insufficient for finishing the task. It must then be regarded as an object to be analysed and improved, and possibly be replaced. Now, this new object can be mobilised as a new tool, useful to solve a class of problems... until one reaches its limits and starts a new "tool-object" cycle. A moment of institutionalisation follows, and then, problems for training and applying are proposed to the pupils. Let us point out, first, when solving a ratio problem, these pupils worked in an exclusive paper-pencil environment, second, the physical and the mathematical domains were not systematically separated and articulated as in the full-experiment class, and third, as usual, fractional and decimal writing were introduced before the number line.

3.3. *Teaching in FEx*

In FEx, Schema 1 was implemented. The articulations between and within the mathematical and the physical domains were systematically investigated through specific sessions. The linear scale was the first register to be introduced for representing and processing rational numbers and it was considered as a privileged tool for interpreting and processing ratio problems. In order to lead the pupils to test the resources of such a register, it was necessary to provide them with the possibility of investigating it by trial and error. A computer environment was then desirable, so that the involved operations, e.g., subdividing or erasing would not be too taxing. Therefore, we designed and developed the software series ORATIO and NewOra (Adjiage and Heideier, 1998). Note that these series are not only devoted to the linear scale, but also to the fractional and decimal registers, so that FEx could investigate mainly the mathematical domain in this computer environment.

3.3.1. *The software series ORATIO and NewOra*
ORATIO is designed for introducing rational numbers. It is composed of twenty computer-programmes in two sets, and a database. The first set is made up of fourteen programmes, proposing "treatment" tasks. It allows pupils to investigate the three retained registers separately. When investigating the second set, made up of six "conversion" programmes, pupils are invited through specific tasks to connect the three registers with one another. Thus, they may acquire flexibility by choosing an expression that fits the interpretation of a ratio problem and possibly change it for greater convenience during the processing.

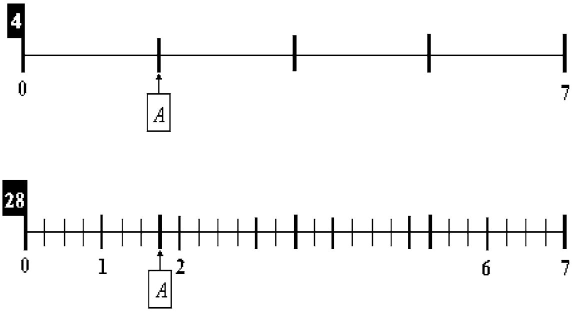
NewOra is about quotients and proportionality in the linear scale register. It is presented after ORATIO, when pupils have been trained to "treatment" and "conversion" tasks.

3.3.2. *The linear scale*
As this register is central in our framework, it seems useful to specify some of its features.

3.3.2.1. The linear scale in ORATIO. In this series, the number line is equipped with a single regular scale. The coordinate system may be the interval [0, 1] or any unit (or not) interval. The programmes submit two kinds of task: placing a rational number, and comparing rational numbers. The coordinate system is initially segmented into n sub-intervals. The number n is called the *fractioner*.[7] It appears above the lower bound, on a black background. In this register, a rational number is expressed by an arrow, pointing to a graduation of a coordinate system (see Examples 1, 2, and 3). The main resources are: re-subdividing, zooming, moving, if the case requires, beyond the initial screen limits. The only way of modifying the current subdivision is to change the *fractioner*, so that the user cannot content himself with observation: processing with whole numbers is the only path to success (see Examples 1 and 2).

3.3.2.2. The linear scale in NewOra. In this series, the number line is mostly equipped with a double regular scale, but under certain circumstances, it may be a single scale. In addition to those of ORATIO, the main resource is marking the graduations corresponding to integers.

- *In the case of single scale*, the user is invited to work out the fractional expression of a rational number A. The *fractioner*, here 4, indicates the initial subdivision of the coordinate system [0, 7].
- *In the case of double scale*, the user is invited to work out images and pre-images, the general form (not given to the pupils) being a linear map $y = ax$, (x, y, a being rational). In Example 2, values of x are represented



Above, a question: 7÷4

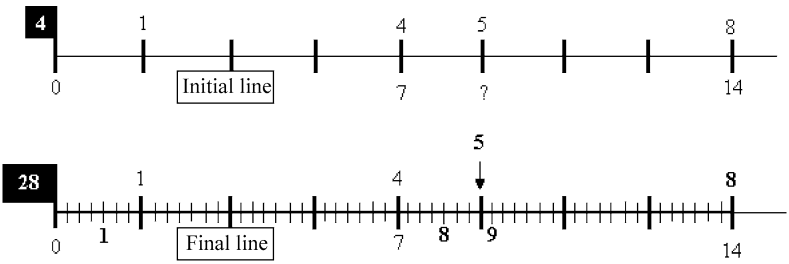
Or using NewOra resources:

"Work out a fractional expression of *A*"

Below, an answer: Seven fourths or $\frac{7}{4}$

The user has chosen to divide each sub-interval of [0,7] into 7, in order to locate the unity on the scale, and thus any integer, so that he can easily locate *A*, first between 1 and 2, and then give a fractional expression of A: $1+\frac{3}{4}$ or $\frac{7}{4}$

Example 1. Using the single scale in NewOra.



Example 2. Using the double scale in NewOra.

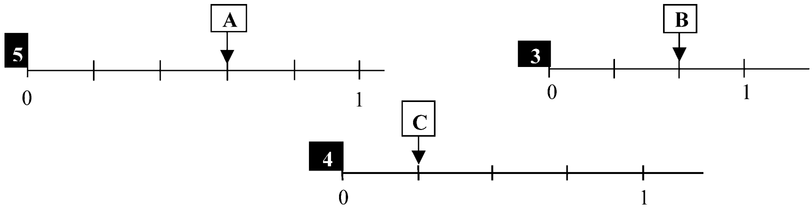
on the upper scale, values of y on the lower. Given 4 and its image 7, and thus the images of 8, 12 and so on (initial line), the user is invited to work out the image of 5 (and, possibly, e.g. the reverse image of e.g. 8).

In our example, the user has re-subdivided each sub-interval of the initial line into 7 in order to locate first the unity, and then any integer, on the lower scale. The *fractioner* has turned to 28 on the final line. Now, it is easier to locate the image of 5, first between 8 and 9, and then give the following fractional expressions: $8 + \frac{3}{4} = \frac{35}{4}$

3.3.3. *Specific sessions in the full experiment*

We have now presented the specific environment that makes it possible to implement our teaching framework in FEx. We can therefore describe five moments, staggered over a few days, of an instructional sequence in FEx.

Moment 1. Solving a pure mathematical problem by using ORATIO software. In Example 3, pupils are invited to experience the pure numerical potential of the linear scale register apart from the fractional and decimal registers. Note that, once mastered, the latter provide an interesting, easier, alternative to the linear scale.



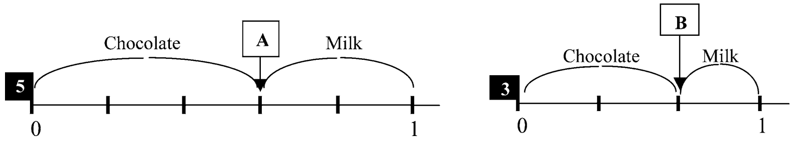
Example 3. Comparing A, B, C, according to *Gradu4* in ORATIO. The three-segmented lines are not presented on the same scale. Pupils cannot base their reasoning on measurements, but on *relative thought* and calculation.

Moment 2. Solving a pure mathematical problem, similar to the previous ORATIO-NewOra-problem, in a paper-pencil environment.

Moment 3. Solving a physical ratio problem (Example 4), structurally and numerically similar to that of Moment 2. The means of solving are the pupil's choice. Some of them spontaneously recognise, in this new statement, a different expression of the Moment 2-problem. Others do not.

Example 4. Solving a mixture problem: We are preparing a chocolate drink, a mixture of chocolate[8] and milk. Recipe A uses 3 parts chocolate for 2 parts milk. Recipe B uses 2 parts chocolate for 1 part milk. Which mixture will be more chocolaty?

Moment 4. *The impulse to convergence*: The teacher reiterates, at the beginning of the lesson, the terms of the scientific debate (Legrand, 2000): "How are Moment 2-and-Moment 3-problems different, how are they similar"? Thus the pupils are asked to make explicit the connections between given mathematical representations on one hand, and given quantities and their relationship on the other hand. Resources of *Gradu4* can be helpful to solve the mixture problem, as shown in Example 5.



Example 5. One interpretation of the mixture-problem, using the linear scale register.

Moment 5. *Institutionalisation in the sense of Douady* (1984, p.16). For instance results such as: "7 divided by 4 is equal to seven fourths ($7 \div 4 = \frac{7}{4}$)"; "Given an enlargement in which a 4cm length becomes a 7cm length, then any length to be enlarged has to be multiplied by $\frac{7}{4}$." Note: this enlargement situation was actually submitted to FEx in another Moment 3-session. The double scale problem shown in Example 2 was the mathematical problem of the related Moment 1-session.

3.4. *Pre-test, post-test and control test*

In order to assess our experiment, we elaborated three different tests: a pre- and a post-test, each one composed of 35 items, given to both FEx and PEx, as well as a shorter control test of 6 items given to 121 representative 7th-graders (not including the two followed classes) and 110 prospective school-teachers. The latter are not specialists in mathematics, but, as they are recruited after the first university diploma (three-year study), they can be regarded as a sample of educated adults and their performance can be considered as reference results. The control test required the solving of six items, referring to the six types of proportionality problems that we have distinguished, and sharing the following common characteristics:

- in each item the data are whole numbers
- processing them requires proportionality

The control test was submitted to the 7th-graders and the prospective teachers during the school year. To both followed classes, we gave the pre-test in September 2001 at the beginning of the experiment, and the post-test in April 2003 about one month after the end of the experiment. A condensed form of these tests is presented in the Appendix.

4. General results

4.1. *The control test*

We present here only a brief survey of this study, which is related in Adjiage (2005). The average observed results were:

- Teacher-trainees, success rate: more than 5 out of 6 (success rate: 86%)
- 7th-graders: less than 1 success out of 6 (success rate: 15%).

Concerning the 7th-graders, the precise average success rate was 0.90 out of 6, and the standard deviation was: 1.31. We will use these values in the comparison of the general results (see Table I).

Furthermore, the findings show an important variation between the procedures used by the two tested populations. In the pupil population, changes are frequent from one item to the other, and they are very rare in the teacher-trainee group.

4.2. *The general results of the followed classes taken as a whole*

The pre-test and the post-test, given to the followed classes, contain a subset of six items (framed labels in Appendix) having the same characteristics as the six items of the control test. Thus, comparisons between the three tests are possible. Those items are in Pre-test (grade 6): PW21, PW22, DIL3, HET1, MIX1, FREQ1, and in Post-test (grade 7): CV09, PW02, PW04, DIL2, FREQ1, HET1. All items could not be exactly the same in the pre-test and in the post-test, and consequently not exactly the same in the control test, so, we will only consider the results as rough indicators of the attained level.

In this section, we take account of the results of FEx and PEx together. There were pupil arrivals and departures within the two school years, but the total remained 47. We have verified that the results of the pupils present throughout the two years do not differ significantly from those reported here. These results are in Table I together with those of the control test given to 7th-graders. For each pupil, the score is the number of items successfully completed among the six considered.

These results are quite impressive, and the z-test we used for comparing the results of the control test and the post-test is obviously significant. This argues in favour of Hypothesis 1 (see Section 3). Nevertheless, we could not consider that we had completed the study at this point. Although the followed pupils obtain a better score than the standard pupils, they remain far from reaching the level of "educated" adults, whose mean is slightly

TABLE I
Comparing general results, followed classes vs. control group

Date	Scores out of 6 items	Mean	Range	Standard deviation
2001	Pre-test (47 pupils at grade 6)	0.77	[0, 2]	0.53
2003	Control group (121 pupils at grade 7)	0.90	[0, 6]	1.31
2003	Post-test (47 pupils at grade 7)	2.36	[0, 6]	1.72

more than 5 out of 6. Even if we use a less severe criterion, similar for instance to the 75% success rate used in the national French 6th grade-evaluation (Dupé et al., 1998, p. 6: *Méthodologie*), we cannot be satisfied: only 13 pupils out of 47 reach a score superior or equal to 4/6. Let us examine now the specific outcomes of the full experiment.

5. Comparative study of the results, FEx vs. PEx

In this section, we consider a study of individual results limited to the pupils who took both pre-test and post-test, i.e. 20 pupils in the full experiment class and 15 in the partial experiment. In the analysis, a first step consists in considering the success scores. In a second step, we consider the procedures used by the pupils.

5.1. *Analysis of success and failure*

All items have the same (unit) weight: each success is labelled with the value 1, and failure with 0. We present here two analyses:

- The statistical comparison of *means of the difference in scores* obtained by each pupil in the pre-test and the post-test (FEx: mean 10.85, variance 47.8; PEx: mean 6.6, variance 48.5)

Fisher's test of equality of variances is not significant; therefore the comparison of the observed means in FEx and PEx by Student's law is relevant. And the value of the observed centred reduced deviation is 1.794, which has a level of significance $\alpha(\text{s}) \approx 0.04 < 5\%$. So that we are allowed to conclude that FEx-pupils reached significantly better results than PEx-pupils.

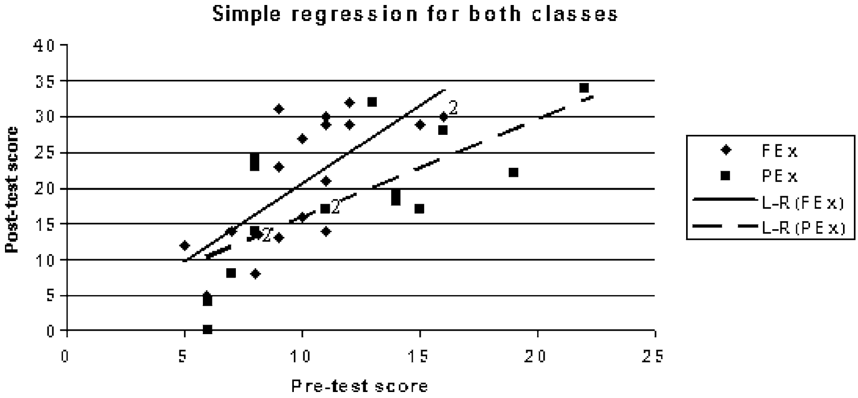
- The simple regression of Post-test score (y) on Pre-test score (x), separately in PEx and FEx. Although there were 35 items in Pre- and Post-tests, the highest scores reached by the pupils were 22 in Pre-test, and 34 in Post-test (Graph a).

In Graph a, when two individuals have the same coordinates, a number 2 lies near the corresponding point: two pupils in FEx obtained (8, 14), two pupils in PEx obtained (11, 17), two pupils in FEx obtained (16, 30).

In PEx, the regression equation is: $y = 1.389x + 1.986$

In FEx it is: $y = 2.170x - 1.081$

The simple regression of y on x leads in each case to a significant linear relationship between the two scores. Residuals are independent for each class, and only one pupil (pre-test score: 9, post-test score: 31) produces a lack of normality in FEx.



Graph a. Pre-test and post-test scores.

The gap between the coefficients of x in the two equations indicates more efficient learning in FEx. The FEx points generally lie above the PEx points. More precisely, if we put aside 4 low points of FEx: (8, 8), (9, 14), (10, 16), and (11, 14), and 3 high points of PEx: (8, 23), (8, 24), and (13, 32), the remaining points satisfy the following property: Each post-test score in FEx is higher than all post-test scores obtained by PEx pupils whose pre-test score was lower or equal. We suggest the following interpretations:

- A strong majority of FEx pupils made clear progress, but is it possible that the experimental environment, particularly its computer component, did not suit a few FEx pupils?
- In a class with a more standard educational environment such as PEx, the majority achieves slow, regular progress (Hart, 1981, p. 217), and a few pupils reach a higher level.

Our experiment did not include separate interviews of the pupils, so that we cannot state more precisely the preceding interpretations. That would be of interest in a later experiment.

5.2. *Analysis of procedures at Pre-test and Post-test, FEx vs. PEx*

First of all, we needed to select a sample of items from each test (Pre- and Post-). Each item of one sample may be matched to one item of the other sample. All items require explicit processing and explanation, cover different values of the diverse variables considered, may be compared to those of the control test, and open the field of possible procedures. Taking into account these criteria led us to retain five items in pre-test: DIL3, MIX1, FREQ1, DIL4, HET3, and five items in post-test: DIL2, MIX1,

TABLE II
Code for observed procedures

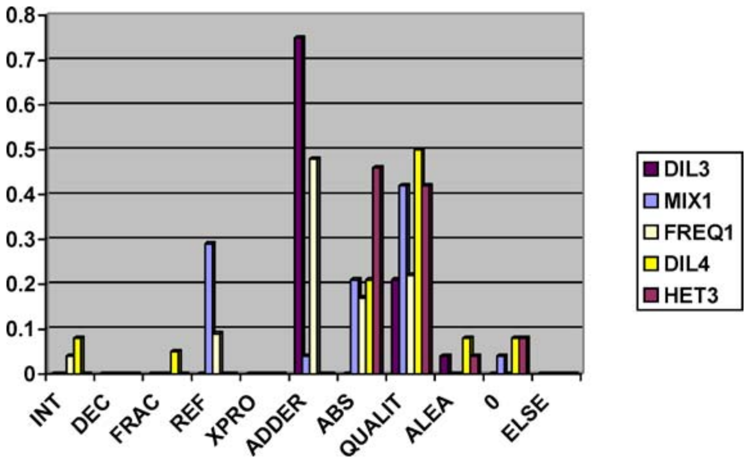
Code	Description	Examples out of the post-test items
INT	Using whole numbers to express a ratio	MIX1: $18 = \mathbf{3} \times 6$; $20 = \mathbf{2} \times 10$, then the proportion of girls is higher in class A. DIL2: enlarging the hat, $21 \rightarrow 30$. Now, $21 \div \mathbf{7} = 3$ and $3 \times \mathbf{10} = 30$. Then the base of the enlarged hat measures in cm: $14 \div \mathbf{7} \times \mathbf{10} = 20$.
DEC	Using a decimal to express a ratio	HET1: $3 \div 5 = 0.6$; $4 \div 7 = 0.571\ldots$ Then, liquid A is heavier than liquid B.
FRAC	Using a fraction to express a ratio	FREQ1: $\frac{2}{6} \times \frac{10}{10} = \frac{20}{60}$, $\frac{3}{10} \times \frac{6}{6} = \frac{18}{60}$, then player A is better than player B.
REF	Finding a common reference to compare	FREQ2: player B wins 18 matches out of **60**; player C wins 24 matches out of **60**. So, player C is the best.
XPRO	"Cross-multiplying"	DIL2: $21 \times ? = 14 \times 30$, therefore the base of the enlarged hat measures in mm: $\frac{14 \times 30}{21} = 20$.
ADDER	Adding the difference	DIL2: the height of the hat is enlarged by **9** mm (30–21), so the enlarged base measures in mm: $14 + \mathbf{9} = 23$.
ABS	Comparing only the first terms of ratios	MIX1: $20 > 18$, so, in classroom B, there are more girls than in A.
QUALIT	Qualitative reasoning	HET1: There are more litres of liquid B than liquid A, but also more kg. So, the two liquids weigh the same.
ALEA	Combination of data without apparent logic	
0	No response	
ELSE	Other wrong responses	

FREQ1, FREQ2, HET1 (Appendix). Table II shows a classification of the observed procedures taken from the test papers.

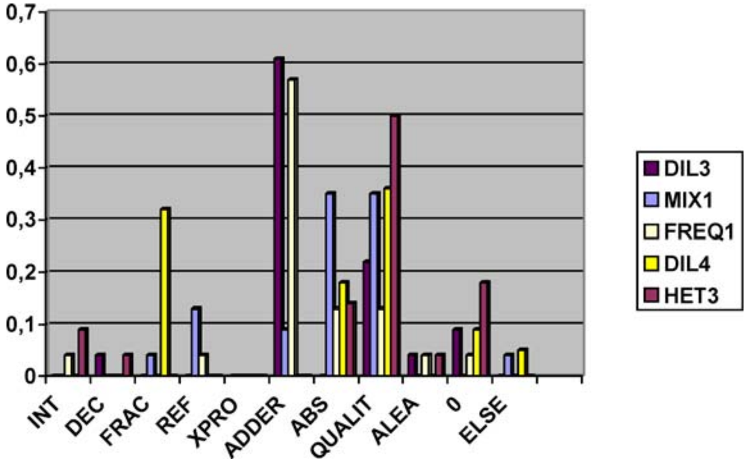
This classification takes up results from the previous research. First of all in distinguishing, among the correct procedures, those that explicitly mobilise ratios (INT, DEC, FRAC) from those that only implicitly refer to ratios (REF, XPRO). Secondly, in considering two usual wrong strategies: ADDER (Adding the difference) and ABS (Comparing only the first terms of ratios). This classification differs from the previous research in declining the "explicit ratio" strategy into three modalities, which are different ways of expressing and processing ratios: INT[9] (use of whole numbers), DEC[10] (use of decimal numbers), and FRAC (use of fractional expressions). For us, these modalities are not equivalent as indicators of "*sophisticated forms of proportional reasoning*" (Karplus et al., 1983). For instance, when comparing the success rates of players A and B in FREQ1 (post-test), a pupil writes: $\frac{2}{6} = \frac{20}{60}, \frac{3}{10} = \frac{18}{60}$, so $\frac{2}{6} > \frac{3}{10}$, thus Player A is better than player B, he indicates that he manages to go beyond the given numbers 2, 6, 3, 10, accessing the implicit links (between 2 and 6, 3 and 10) and their invariability in a scale change, and then comparing these links. If another pupil uses $2 \div 6 = 0.333\ldots$ and $3 \div 10 = 0.3$ and comes to the same conclusion, he only indicates that he takes into account the given numbers, uses a relevant operation (division), performs it properly, and knows how to compare two decimal numbers. As for INT, we consider it as an advanced form of a build-up strategy. We do not claim that using fractions is better than using whole or decimal numbers to solve a ratio problem. We state that the former appears as a better indicator of an advanced form of proportional reasoning.

The analysis of procedures at Post-test reveals important differences between FEx and PEx. These differences were not present at Pre-test. The Pre-test graphs (Graphs 1 and 2) are very similar. Particularly, both classes largely use wrong procedures (from ADDER to ELSE). A rapid survey of Post-test graphs (Graphs 3 and 4) shows two major phenomena. First, a positive global evolution of both classes toward the use of correct procedures (according to Hypothesis 1), second a strong dissimilarity. This dissimilarity is that the pupils in FEx are grouped around procedures mobilising fractions (acquisition of a powerful and privileged tool), whereas the pupils in PEx are dispersed over all the procedures Graphs 1, 2, 3, and 4 (no privileged tool performed).

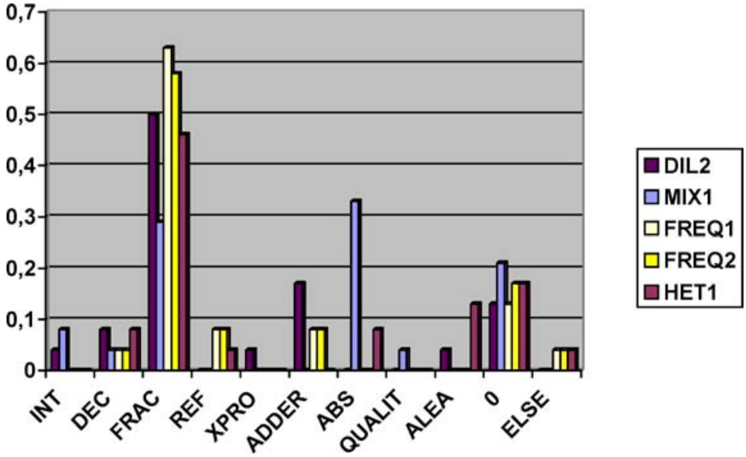
The study based on the control test (4.1) included an analysis of the procedures used, which shows the same type of opposition: regrouping of



Graph 1. (Pre-test, FEx) frequency (%) of use of the observed procedures, item by item.



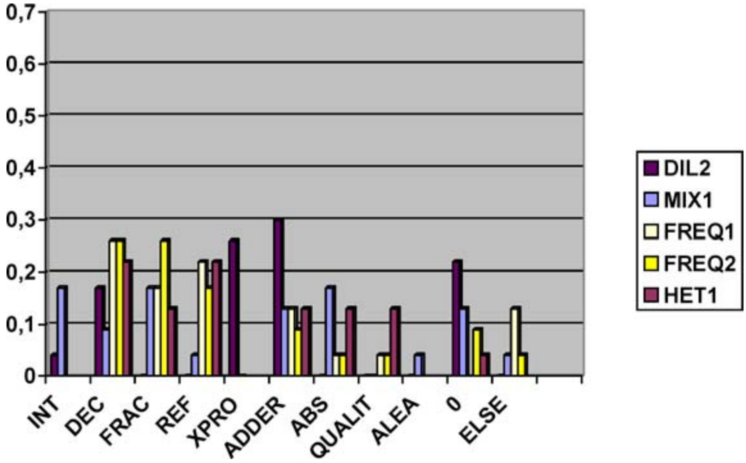
Graph 2. (Pre-test, PEx) frequency (%) of use of the observed procedures, item by item.



Graph 3. (Post-test, FEx) frequency (%) of use of the observed procedures, item by item.

the teacher-trainees, dispersion of the 7th-graders. But a clear difference exists between the "educated population" and FEx: For the former, the regrouping is around XPRO while it is around FRAC for the latter.

So, on one hand, FEx tends to approach an "educated population", whereas PEx remains nearer the general 7th-grader population. Pupils in FEx tend to use a unique procedure for solving different ratio and proportionality problems. This leads us to state that they are able to recognise a unique underlying mathematical structure, effacing the diversity of those problems, particularly the diversity related to the variable of context. On the other hand, knowledge of algebra can explain the difference between the "educated population" and FEx. Teacher-trainees master elements of



Graph 4. (Post-test, PEx) frequency (%) of use of the observed procedures, item by item.

algebra that allow them to use XPRO, but not as a blind procedure. Most of them do it in a clear-sighted way, as the precise examination of their papers showed: They first interpret the problem in terms of ratios and proportions.

The dispersion of PEx pupils over all the procedures does not allow us to state that they have recognised an underlying common mathematical model. This dispersion may be explained by more sensitivity to the command variables, particularly the variable of context. In this sense, these pupils seem to behave in accordance with what Vergnaud (1983, p. 142, 146, 149) reported about standard 14-year-old pupils: "*Children do not think of b, c, and a as pure numbers. They see them as magnitudes...*", so that their reasoning remains close to the physical-empirical context of the problem they are processing.

5.2.1. *Mastering ratios and proportions?*

Up to this point, we have not taken into account the possible errors of the FEx pupils in the use of FRAC. We have only pointed out that they massively resorted to this procedure whether right or wrong. In Table III, S and F are respectively the success and failure rates among the FEx pupils who used FRAC.

It is obvious that the use of FRAC, which we consider, let us recall, as an advanced procedure, is strongly linked to success, except in HET1. How can we interpret this lone shortfall? In examining the papers of the 8 pupils who failed in HET1 using FRAC, we observe that they all made the same mistake: they compared $\frac{5}{3}$ and $\frac{7}{4}$ instead of $\frac{3}{5}$ and $\frac{4}{7}$, following the order of appearance of the involved numbers (5, 3, 7, 4) in the problem statement. Note that all of them succeeded in this non-pertinent comparison. They know

TABLE III
Success and failure among the FEx pupils who used FRAC

	DIL2		MIX1		FREQ1		FREQ2		HET1	
	S	F	S	F	S	F	S	F	S	F
	10	**2**	**7**	**0**	**15**	**0**	**13**	**1**	**3**	**8**
Together	12		7		15		14		11	

that ratios expressed by fractions are pertinent to interpret this problem, they know how to compare fractions, and these are important steps on the path to success. We must however admit that we have only partly succeeded in separating and articulating the physical and the mathematical domains, while our goals for processing rational numbers (Hypothesis 1) seem to be better reached. The bad result in HET1 proves that FEx pupils still lacked elements of modelling. Learning algebra the following year should help them to better apprehend the links between magnitudes and computation, and thus, lead them to correct this kind of misunderstanding. But, using for instance XPRO as a blind algebraic procedure, without solid references to ratios and fractions, could be an illusion as Comin (2000, p. 180) reported. Up to this point, our hypothesis concerning the FEx pupils is: learning algebra should complete their mastery of FRAC, in its calculating and modelling facets, and they should quickly attain the level of an educated population.

6. Conclusion

Let us point out what our work may bring to our subject. First of all, it provides a “compact” analysis of the complexity of ratio problems, relying on others’ research (considering: both rational numbers and proportionality as a whole, variables involved in proportional reasoning, analyses of pupils’ works), and differing from it in a few points (separate and articulate the mathematical and the physical domains, between and within them, consider the specificity of diverse expressions of rational numbers, and the difficulty of moving from one expression to the other). Second, it furnishes teachers at grades 6 and 7 with: a corpus of six types of ratio situations, necessary and sufficient at this scholastic level; a corpus of three registers for expressing and processing rational numbers; a tool for working within the articulations between and within the mathematical and the physical-empirical environment, i.e. the linear (single or a double) scale; a software series for facilitating the systematic study of the three registers, particularly the linear scale. Third, it furnishes an experimental test of our framework, which we summarize below.

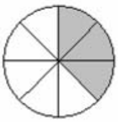
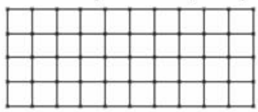
The two classes involved in the reported experiment make progress and tend to perform relevant procedures (see 5.2). Are the common elements introduced in both classes sufficient to explain this clear progress, as asked in our first question research and stated in Hypothesis 1? The comparison with a representative sample of 7th-graders shows results strongly in favour of the followed classes (see Table I). This leads us to admit that these elements had a positive impact on the results, even if other elements (as particular attention paid to the learners during any experiment) may be of some influence as well.

According to the results of Section 5.1, the learning in FEx appears to be more efficient than in PEx. The comparative analysis of the procedures used furnishes some indicators of a better achievement in FEx than in PEx. FEx-pupils give most obvious signs of recognising proportionality as a mathematical structure underlying the different ratio problems in context. They perform advanced scaling strategies around the FRAC procedure. These two facts argue in favour of Hypothesis 2: systematically working the separations and the articulations between and within the physical and the mathematical domains involved in ratios helps pupils to discern invariants and access the proportionality model. The systematic work within the different rational registers helps them to secure their fractional processing and thus leads them to resort to it more willingly. In return, processing fractions is well accorded to the valid treatments in proportionality, so that pupils that master fractions should better master proportionality. Briefly said:

- FEx approaches an "educated" adult population, both concerning the success rates and the procedures, more than PEx does. To use an image, we will say that during the observed period the average pupil in both classes learned how to swim in the "ocean of ratios", but the swimming styles are more efficient and similar in the full experimental class.
- FEx pupils, in spite of substantial acquisitions, have not yet finished learning. Another step, which could be learning algebra, seems necessary to reach the level of a reference "educated" adult population.

Some questions are still open after this experiment. Have we reached an optimum? Some pupils in the full experiment remained at a low level. But they departed from a point located rather far from what is expected at the beginning of secondary school (see: Pluvinage and Mallier, 1998, the proportion of pupils in this age group, whose literacy remains uncertain). Is it only a matter of time or will they, on the contrary, not get beyond "*numeracy*" and reach a situation which we could call *"ratio illiteracy"*? The next important step in the curriculum is *algebra*. It is known that learning algebra increases the ability to use fractions. But that is under the constraint of a sufficient initial level of acquisition. Will the knowledge of the linear scale help the pupils in this new learning process? We believe it to be the case, but our observations here do not yet allow us to affirm it.

APPENDIX: CONDENSED PRESENTATION OF PRE-TEST AND POST-TEST

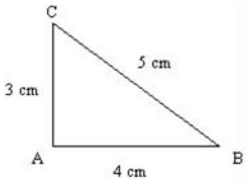
Initial questionnaire (grade 6, 2001)	Final questionnaire (grade 7, 2003)
1st sequence	**1st sequence**
Q1-2 True or false? $\frac{3}{5} < 1$ $\frac{3}{5} > 1$ $\frac{3}{5} < \frac{1}{2}$	**Q1-2** True or false? $\frac{4}{9} < 1$ $\frac{4}{9} > 1$ $\frac{4}{9} < \frac{1}{2}$
Q03 Find a suitable whole number: $\frac{8}{3} <\ldots< \frac{13}{4}$	**Q03** Find a suitable whole number: $\frac{21}{4} <\ldots< \frac{19}{3}$
Q04 Find a suitable whole number: $\frac{28}{5} <\ldots< \frac{32}{5}$	**Q04** Find two <u>successive</u> whole numbers: $\ldots < \frac{28}{5} < \ldots$ **Q05** and **Q06** Calculate the sums of fractions $4 + \frac{5}{6} = \frac{\ldots}{\ldots}$ $\frac{1}{2} + \frac{1}{3} = \frac{\ldots}{\ldots}$
CV01 to **CV06** Locate each fraction on the graduated number line (*a representation of [0, 2] with decimal graduation was given*) $\frac{7}{10}; \frac{1}{2}; \frac{4}{5}; \frac{8}{5}; \frac{16}{10}; \frac{1}{3}$ **CV07** to **CV11** Items about pie charts (four from fractions to pie parts, and the converse in CV11, *see figure below*)  Which fraction does the grey pie part represent?	**CV01** to **CV07** Locate each fraction on the graduated number line (*a representation of [0, 2] with decimal graduation was given*) $\frac{7}{10}; \frac{1}{5}; \frac{4}{5}; \frac{7}{5}; \frac{3}{2}; \frac{3}{4}; \frac{17}{10}$ **CV08** Which fractions are represented? (*Two representations were given: in the first each unit segment of [0, 3] was cut in six and an arrow was pointing at 5/6, in the second [0, 1] was cut in 18 and an arrow was pointing at 5/18*) $\boxed{\text{CV09}}$ What is the volume of 5 kg of honey? *A double graduated line number was given (upper: honey mass, under: honey volume)* 0 6 12 18 *Kg of honey* 0 4 8 12 *L of honey*
2nd sequence	**2nd sequence**
PW11 to **PW13** We recall that there are 60 minutes in 1 hour. Find the number of minutes corresponding to: $\frac{1}{10}$ h; $\frac{1}{5}$ h; $\frac{4}{3}$ h. $\boxed{\text{PW21}}$ and $\boxed{\text{PW22}}$ What are the fractions of 1 h represented by: 20 minutes? 50 minutes? **PW31** and **PW32** I want to divide 3 pizzas between 4 people. Represent each one's part and colour it in red. What fraction will each one receive? **PW4** Explain how you can cut a piece of 50g in a 250g-packaged butter. (*A perspective representation of a pack was given*) **PW5** How many roses are there in 8 bunches of 5 roses each? **PW6** Each of 50 children should eat a piece of cheese. How many 6-piece boxes will I buy?	**PW01** A can filled to the brim contains 27 litres. How many litres are there in $\frac{5}{9}$ of the total volume of this can? $\boxed{\text{PW02}}$ What fraction of the volume of the preceding can are taken up 18 litres? **PW03 and** $\boxed{\text{PW04}}$ The mass of a chocolate bar (*figure given: a 4×10-rectangle*) is 200g. Colour a piece that weights 80g, and indicate the fraction of the bar represented by this piece. 

3rd sequence

DIL1 From the money of school cooperative, $\frac{3}{5}$ are each year devoted to a travel. Complete a table of money spent for 1999- to 2001-travel. (*Cooperative amount is 3500F in 1999, 4000F in 2000, and 3400F in 2001*)

DIL2 The value of 1 Euro is about $\frac{20}{3}$ of 1F. Using this rule, find the value in F of 90 Euro; of 42 Euro.

$\boxed{\textbf{DIL3}}$ We want to draw the same triangle as ABC but enlarged. After enlargement, the side AB will have a length of 6cm. What will be the lengths of the sides AC and BC?



Initial triangle

DIL4 Store 1 announces a 20% discount sale, and store 2 reduces prices by $\frac{1}{4}$. Which store makes the best proposal?

4th sequence

The mass of 2 litres of honey is 3 kg.

$\boxed{\textbf{HET1}}$ Three pots are represented, the volume of each one is 1 L. Choose one of them and colour it up to the level attained by 1 kg honey.

HET2 Above a graduated scale (*[0, 9] with graduation in whole numbers of pounds, or halves kg*) draw an arrow pointing at the place corresponding to 5 L of honey.

HET3 Comparison of 500 g and 80 g of dry food respectively given to a 35 kg heavy dog and a 5 kg heavy cat.

$\boxed{\textbf{MIX1}}$ Compare the taste of chocolate in the two mixtures: 2 glasses of milk with 3 of chocolate, and 1 glass of milk with 2 glasses of chocolate.

$\boxed{\textbf{FREQ1}}$ We want to compare the results of an assessment in two classes. For the same problem, 3 pupils out of 5 succeeded in class A, and 2 pupils out of 3 in class B. Which class was more successful in solving this problem?

3rd sequence

COE1 to COE4

Items of the type $\mathbf{100} \xrightarrow{\div 2} 50 \xrightarrow{\times 3} \mathbf{150}$

(*Bold numbers given, and normal numbers are to be found*)

Data: 6 and 10; 2400 and 1800; 5 and 7; 35 and 7.

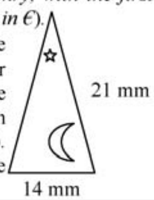
COE5 to **COE8** Complete the fractions:

$$6 \times \frac{\dots}{\dots} = 10;\ 2400 \times \frac{\dots}{\dots} = 1800;$$

$$5 \times \frac{\dots}{\dots} = 7;\ 35 \times \frac{\dots}{\dots} = 7.$$

DIL1A, DIL1B, DIL1C Among a 4000 € total amount for house repair, electricity repair represents $\frac{1}{4}$, painting $\frac{3}{8}$, and the complement is for masonry. Fill out the table of expenses (*for electricity, painting, and masonry, with the first row in fraction and the second in €*).

$\boxed{\textbf{DIL2}}$ We want to enlarge the illustrated clown hat. After enlargement the 21mm side will measure 30mm (the given figure does not respect sizes). What will be the length of the base?



4th sequence

MIX1 In class A, there are 18 female students and 6 male, and in class B, 20 female students and 10 male. In which class is the proportion of female students higher?

$\boxed{\textbf{FREQ1}}$ Tennis player A won 2 out of 6 matches against Serena Williams, and tennis player B won 3 out of 10. Which score is better?

FREQ2 A third tennis player C won $\frac{2}{5}$ out of her matches against Serena Williams. Who has the best score? A, B, or C?

$\boxed{\textbf{HET1}}$ The mass of 5 L of liquid A is 3 kg. The mass of 7 L of liquid B is 4 kg. We fill up two containers of the same size one with liquid A, the other with liquid B. Which is heavier?

FREQ3 There are 2700 houses on a Greek island: $\frac{4}{9}$ have heating, and among them, $\frac{3}{4}$ have electric heating. How many houses have electric heating?

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previous version to Kathleen Hart and to benefit from her observations and suggestions.

NOTES

1 If (Fi, Gi) is a sequence of ordered pairs of two proportional variable quantities Fi and Gi, so that $Gi = f(Fi)$, where f is a linear function, a procedure using a ratio as $\frac{F_i}{F_j}$ or $\frac{G_i}{G_j}$, without unity of measure, is termed "scalar procedure" or "between strategy" (Noelting, 1980, p. 234). A procedure using the constant ratio $\frac{G_i}{F_i}$ or $\frac{F_i}{G_i}$ is termed "function procedure" or "within strategy" (ibid).

2 We term ratio a multiplicative relationship between two physical quantities (e.g. 3 parts of concentrate, 2 parts of water), or two mathematical quantities possibly being numbers. Thus, a ratio can be associated with a number, possibly rational (e.g. $\frac{3}{2}$or $\frac{3}{5}$ in the case of the mixture above). We term proportionality a linear relationship between two variable quantities.

3 All these situations have been taken into account in other studies, for instance Harel et al. (1991, pp. 125–126) or Alatorre and Figueras (2005, p. 1), but we have never found the six of them in the same paper.

4 A classical introduction to rational numbers by fractions, often reinforced by a teaching that gives greater place to "pie parts", leads pupils to consider that a fraction does not represent one but two numbers (Hart and Sinkinson, 1989; Streefland, 1993, p. 114; Adjiage, 1999, p. 204).

5 Michel Barthelet, who participated for many years in IREM (Institut de Recherche sur l'Enseignement des Mathématiques) activities, besides his teaching charge.

6 In French: "La dialectique outil/objet".

7 In French: "Le fractionneur".

8 Kathleen Hart, who helped us to revise our paper (thanks a lot, Kath!), criticized this choice of situation, because there is something implicit in the question: All glasses of chocolate should have the same taste. Similar doubt does not arise for instance in the case of mixtures of fruit syrup and pure water. Actually none of the involved pupils asked for some explanation in the presented case, nevertheless this fact underlines how important the precise descriptions of the physical environment are.

9 Pupils who used a whole number to express a ratio were "marked" INT (e.g. 3 in MIX1 for 18/6). But pupils who used two whole numbers as consecutive operators, instead of one fractional operator (e.g. $(\div 7) \circ (\times 10)$ instead of $\frac{10}{7}$ in DIL2, to move from 21 to 30), were also "marked" INT.

10 Pupils of both classes, who used a decimal number in any item of the final questionnaire, also used in the same item the sign $\div$, clearly distinguished, in the official French curriculum for grade 6 (6^{e}), from the fraction bar.

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