JULIAN WILLIAMS AND GEOFF WAKE

METAPHORS AND MODELS IN TRANSLATION BETWEEN COLLEGE AND WORKPLACE MATHEMATICS

ABSTRACT. We report a study of repairs in communication between workers and visiting outsiders (students, researchers or teachers). We show how cultural models such as metaphors and mathematical models facilitated explanations and repair work in inquiry and pedagogical dialogues. We extend previous theorisations of metaphor by Black; Lakoff and Johnson; Lakoff & Nunes; and Schon, to formulate a perspective on mathematical models and modelling and show how dialogues can manifest (i) application of 'dead' models to new contexts, and (ii) generative modelling. In particular, we draw in some depth on one case study of the use of a double number line model of the 'gas day' and its mediation of communication within two dialogues, characterised by inquiry and pedagogical discourse genres respectively. In addition to spatial and gestural affordances due to its blend of grounding metaphors, the model translates between workplace objects on the one side and spreadsheet-mathematical symbols on the other. The model is found to afford generative constructions that mediate the emergence of new understandings in the dialogues. Finally we discuss the significance of this metaphorical perspective on modelling for mathematics education.

KEY WORDS: case studies, discourse, mathematical genres, metaphors, models, modelling, translation

1. Introduction and background

Previous studies have shown how workplace systems structure practices in ways that can make them difficult for outsiders to access, thereby causing communication 'breakdowns' (Pozzi et al., 1998; Wake and Williams, 2001; Williams et al., 2001). In our previous paper (Williams and Wake, this issue) we showed how the mathematics of workplaces can become 'black-boxed' in instruments and artefacts and in a division of labour which is held in place by rules (in turn often regulated by aretefacts and instruments). Our original study[1] (Wake and Williams, 2001) involved a dozen workplace field studies in which we explored with workers, college students and their teachers aspects of a worker's daily activity that include elements of mathematics. In this paper we will show how communication repairs can sometimes be constructed so that outsiders can make sense of the 'black-box' and come to understand the mathematics inside. We highlight the role of particular cultural tools in these repairs, i.e. models and metaphors, and develop an argument for conceptualising mathematical models as metaphors

Educational Studies in Mathematics (2007) 64: 345–371
DOI: 10.1007/s10649-006-9040-6

– we argue that this perspective on modelling can clarify and broaden its significance for mathematics education. Thus we ask:

‘How can models and metaphors mediate communication between students, workers and teacher-researchers?’

Of course we are interested in consequences for pedagogy and curricula in mathematics education, especially for the motivation of learning and for the better teaching of uses of mathematics. Mathematics education has long had a significant interest in models and modelling. The whole corpus of work of Freudenthal and the Freudenthal Institute has shown how models and modelling problems drawn from the culture can be significant in the service of abstracting mathematics and building new mathematics, through ‘horizontal’ and ‘vertical’ mathematisation (Freudenthal, 1983; Gravemeijer et al., 1999; Streefland, 1991; Treffers, 1987). Well-known work by Cobb and colleagues has benefited from this approach, and the design of pedagogical models and problems that can serve as tools in classroom practice is now well documented (Cobb et al., 2000). However, in addition to the Freudenthal approach with a focus on the elementary and main school curriculum, there has been a persuasive development in mathematics education recognising modelling as a vital applied mathematical, problem-solving skill (Blum, 1991; Niss, 1996). This trend developed an approach to mathematics education, pedagogy and assessment, through the International Conference of Teachers of Mathematical Modelling and Applications (ICTMA), initially focused on Higher Mathematics and High School, but later extended though all the school phases. For example, see ICTMA proceedings over two decades: most recently the 11th Conference (Lamon et al., 2003). Indeed these two trends have had some cross-fertilising influences.

More recently, the mathematics education community has begun to take the concept of *metaphor* seriously (English, 1997; Nunez, 2000; Pimm, 1987; Sfard, 1994). This follows an extensive development in cognitive linguistics, which situates cognition in language use in general and tropes in particular (Lakoff, 1987; Lakoff and Johnson, 1980, 1999). Taking their standpoint in embodied cognition and drawing on previous work in cognitive linguistics, Lakoff and Nunez (2000) have shown how one can build up mathematics from metaphorical extensions and conceptual blending of the four ‘grounding metaphors’ underpinning and embodying arithmetic. Their ‘building up’ of mathematics involves metaphorical constructions and representations at each stage. In the case of the number line model for instance, one observes how metaphorical talk about number is spatial, as in ‘between 24 and 25’, or ‘as near to zero as you like’.

Their approach has been criticised from mathematical and psychological points of view, and especially for suggesting that there is just the one, perhaps overly simplistic 'cognitive linguistic' and in particular 'conceptual-metaphoric' route to mathematical constructions (Goldin, 2001; Schiralli and Sinclair, 2003). We take Wartofsky's (1979) view that mathematics, models, analogies and metaphors all lie on a spectrum of representations: i.e. they all involve the presentation of one thing as if it were, or in terms of, something else. Linguistic representations provide an important route into mathematics, as do images and graphical-spatial representations: language and metaphor are perhaps particularly salient for declarative knowledge and concepts. We conclude that metaphors and models certainly have and always have had significance to mathematics teachers and teaching.

In this paper our analysis of naturalistic discourse between workers and outsiders is situated in part in this theoretical framework. By way of a concrete example, we will be considering the work of an engineer, Dan, who works in a large industrial chemical plant. One of his tasks involves the use of a spreadsheet formula which we present here in its entirety and with which we fully expect to mystify the reader as were we ourselves (at first sight anyway):

{{{{{"2nd INTEGRATING READING"-"0600 INTEGRATING READING"}
+{{{"2nd INTEGRATING READING"} – {"1st INTEGRATING
READING"}}/T_2}*TIME4}}/100000}/3.6*CALCV*1000000/29.3071}

Each day workers routinely collect meter readings at certain times; they fill in boxes on a recording sheet with these readings. Dan takes the values from the sheet and inserts them into his formula. Thus, for instance, "0600 INTEGRATING READING" is a value taken from the recording sheet, which the operatives originally read from a meter.

The result of this spreadsheet calculation is a value that is used to order gas from the supplier to keep the plant functioning over night. The reader might like to consider this practice, and try to make some meaning of it mathematically. How does this practice compare with the kind of mathematical practices we might expect to observe in a College classroom, e.g. the solution of a quadratic equation? To be slightly contentious for the moment, it might appear remarkably similar. As far as the engineer's operatives are concerned: they read and insert data into a proforma (c.f. College students identify coefficients of the quadratic and insert into a formula) – the mysterious formula (constructed elsewhere) being used to compute a result of interest only to someone else: the manager / boss (c.f. examiner/teacher). Both College student and operative are, in this account, essentially instrumental, alienated by being 'black-boxed' from the meaning of the mathematical activity they are involved with (Williams and Wake, this issue).

To understand what is at stake, what the *engineer* and his manager are doing, how the engineer came to construct this formula, and how it might need to be adapted in the future as circumstances change, is a problem and a challenge, however. This requires a certain amount of work process knowledge: knowledge about how this practice fits into the production process and its activity system. Before he goes off shift, Dan needs to estimate the amount of gas the plant will have consumed by the end of the 'gas day' (i.e. 0600 the next day). The gas consumed has to be accurately estimated, because the gas supplier expects to supply and get paid for that amount, and there will be a financial penalty if the real consumption is outside some tolerance of the estimate. As Dan says, millions of pounds are involved. This is a very hierarchical production system, with each worker assigned a role and rules of operation guiding the dance of the whole production community. The manager is the decision-maker and seeks to minimise costs incurred. His decisions and motives control the community accordingly. The engineer has the technical task of handling the data and reporting, and also instructing the workers under him as to what data to collect. Dan 'owns' the spreadsheet, but is also accountable to the manager. Dan tells us that his calculations have to be very accurate, and if he gets this wrong he may be dismissed. For Dan's operatives the spreadsheet is a black box, a mystery, and we find some evidence that Dan is happy to maintain his exclusive ownership of it, and hence his expert role.

Now the reader understands the purpose of the activity, and to some extent its division of labour and rules, let us return to the formula again and see if it can be understood. The readings substituted into the formula are gas meter readings at the beginning of the gas day, 0600, and later at two times quite close together (actually a time $t =$ T2 apart). These two 'integrating readings' are used to calculate the *rate* of gas consumption, and hence predict the total gas day's consumption at this same rate over the time remaining until 0600 next morning, "TIME4". We translate the formula into academic 'College mathematics' below.

So far we have seen how different workplace mathematics appears from a College mathematics point of view. Above we performed a 'translation' as it were between different languages. But these are not differences a linguist would recognise as such, they are more like different Bakhtinian 'genres' and 'social languages' *within* a national language, i.e. patterns of communication and styles of language used for particular purposes, within particular types of setting and communities.

For Bakhtin et al. (1986) 'genre' is the 'horizon of expectations brought to bear on a certain class of text-types' (p. 428), and includes

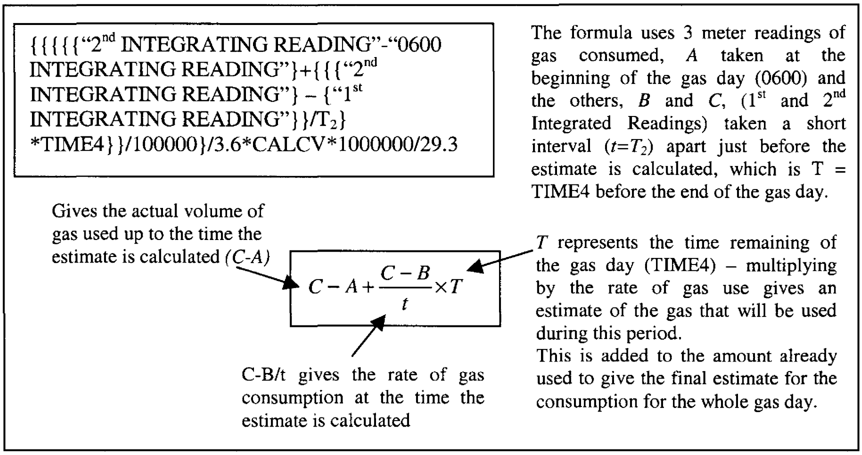


Figure 1. Translating the workplace spreadsheet formula into College algebra.

everyday genres such as 'the personal letter' whose conventions are almost universally shared within a culture (Gee, 1996; Halliday, 1978; Halliday and Hasan, 1985; Wells, 1999; Wertsch, 1991). We talk of workplace *genres* of mathematics because each technical workplace may develop its own conventions and terms, though it may also draw on elements of more widely recognised social, cultural forms of mathematical language, 'engineering mathematics', 'spreadsheet mathematics' and so on.

This particular workplace mathematics genre has been shaped by many local features:

- the aim of the activity is efficient production, and the need to minimise gas costs due to mistaken estimations of gas needs during the 'gas day' accounting period;
- the local instruments which measure gas consumed at various times, in various units, and the recording sheet the operatives use to record relevant values;
- the particular jargon of the workplace: the 'gas day', the naming of the variables as 'integrating readings' etc;
- the technology of the spreadsheet and language used to address it (only partly local);
- the division of labour that allows (maybe insists) that Dan has the technical expertise and job of 'calculation', his 'lads' are operating in the dark, the manager interprets and reports, and holds them all to account.

In Williams and Wake (this issue) we suggested we can view College mathematics as 'another genre' of mathematics, shaped by an international mathematics community, a national curriculum and assessment culture and also by more local systems of the professions, pedagogy and schooling. Thus the formula we created above to clarify meaning and structure of the workplace spreadsheet formula for you, the reader, is structured by an academic genre of mathematics which has emerged historically within academic practices of scholastic mathematics: it values elegance, formality and certain conventions which afford the perception of mathematical structure (we notice at once its linearity in T for instance).

In contrast, the workplace genre has certain qualities of local situativity: these prove very helpful to Dan in supporting his re-construction of the formula for our benefit as we shall see. The terminology used for variables for instance are redolent of the sources of the data, and absent from our translated version in the academic genre: these variables therefore have to be defined by us in order for the formula to recover its meaning as a workplace practice, and so to re-contextualise it (Van Oers, 1998).

We have now explained this formula for the reader, but how did we come to make sense of it at the time? We now examine the communication breakdown and its repair in some of our data, and we find ourselves drawn to investigate cultural models and metaphors in our quest to understand these.

2. How workers (and we) use metaphors and models in helping make sense of mathematics in practice: the machine tool programmer

A fine example of the power of metaphor in translation arose in Steve's metalwork shop where a machine tool is programmed to punch holes in, and 'nibble' metal plates (for more see Williams and Wake, this issue).

One line of his program reads as follows:

```
X 25.   Y 172.5   T12   G90
```

This line commands the drill to punch a hole in metal plate: to 'move' through certain distances across (X) and down (Y), to select an appropriate tool (T) with which to cut a hole. Presumably the original author of the programming language in use here was a software designer – we could say another kind of mathematician - making use of notions of Cartesian axes for vectors but adopting conventions particular to this genre of mathematics, such as the use of a decimal point after the whole number 25, and the 'simultaneous' execution of all the commands appearing on one line of program (curiously, in this programming language, the command G90 is taken to be executed before X and Y commands are executed, even though it appears later in the instruction).

In explaining the last command, G90, which was baffling to the researcher, Steve said:

Steve: G90 switches it back to thinking from nought-nought

Researcher: Yes

Steve: So from here it's thinking: 'Oh, ... I'm going from nought-nought, I'm going to go one hundred and seventy two and a half down, twenty five in, ...'

Steve's use of a metaphorical device (– it's thinking, 'Oh, I'm going...' –) here is of interest. Speaking formally he might have said something like "The subroutine G90, according to the manual, resets the machine's co-ordinate frame from relative to absolute references for the commands X and Y."

Instead, quite typically of our case studies, he adopted an anthropomorphic machine-as-person, in fact servant-being-instructed metaphor. Thus, Steve's program is formally, perhaps, a text of characters which will become binary coded and stored as strings of bits, which later become electrical signals to the machine that order mechanical actions. But the program is read or understood as 'a series of orders to a servant/robot who understands them and acts accordingly'. This particular type of servant can read or interpret coordinates in relative or absolute modes, and G90 tells the servant to switch the mode.

In the above case, Steve reads "G90" as an 'instruction' to the machine thenceforward to 'interpret' movement 'commands' differently, i.e. to 'think from nought-nought'. Thus he sees it as changing the machine's way of 'thinking'. He may be aware that G90 is a subroutine, and even how this routine works, but if so he gives no indication of this: he speaks of it as any other instruction, its meaning is in its function, expressed effectively through the metaphors employed. It is uncertain whether or how his understanding of the code reflects that of the programmer who originally invested mathematics in it: his discourse suggests that it may be quite different. Once the programming language has its technical attributes codified in a manual, it becomes a tool in a different activity, such as Steve's: the object of activity is new, and thus the language acquires new meanings. For Steve this meaning is principally expressed metaphorically.

Being told to 'G90', our servant metaphorically responds conversationally:

"Oh ..."

as though momentarily surprised by this command. Why surprise? Because the command G90 forces our servant to re-interpret what had previously been (mis-) understood – reading the line of the programme from left to right- by the commands X and Y. And so now the servant thinks:

"... I'm going from nought-nought".

Thus a conversational device, "Oh", here helps one to understand how the command is to be understood as a change of interpretation of the vector commands. It seems that Steve has situated his discourse metaphorically in a world where the computer has become a person in a dialogue, and this allows him quite effortlessly to make use of everyday dialogical strategies and devices drawn from this world. This metaphorical world of machine-as-servant therefore proves extensible to include implicit entailments that were not apparent at first sight, or even made explicit: the metaphor offers a somewhat open-ended set of resources to be drawn on as and when necessary.

We will seek to show how this can also be the case with mathematical models. That is, a mathematical model can also generate a world of 'metaphorical entailments', or mathematical implications not explicit at first sight, (see next section) and that this can be a key generative feature in communication and problem solving.

Lakoff and others have studied metaphors as mappings from a source to a target domain. Thus, after Lakoff and Johnson (1980) we have the following metaphorical structure:

source domain = Target domain
servant = computer-machine
master = programmer
series of orders = program

The utility of a source domain in metaphoric use is to provide a more concrete domain to map onto the relatively formal and abstract target: the *source* under-pins or 'under-stands' the *target*. The complex of implications of the metaphor in the source domain is now transferred creatively by Steve to explain effects in the target domain, thus:

Source domain	=	Target domain
'it' thinks	=	machine interprets
'oh'- surprise-	=	machine recognises (unusual) signal to re-interpret...
'I'm going from nought-nought'	=	via change of mode of interpretation of X and Y

Related metaphors include the conduit metaphor for communication (Reddy, 1993), and the brain-computer analogy. These metaphors and analogies involve many ‘entailments’ such as: the machine has a ‘memory’, programs have to be ‘interpreted’, and so on. Indeed, it would be all but impossible to discuss programming without deploying terms such as ‘language’, ‘interpreter’, and ‘memory’. Of course, metaphorical underpinning can also introduce false entailments into the Target domain, by over-extending an analogy (Spiro et al., 1989). For instance we may behave as if we believe that the computer will display the ‘common sense’ of a human servant. Perhaps the retention of the pronoun ‘it’ for the servant helps inoculate Steve against such a danger. The erroneous interpretation of the line of the program as if ordered from left to right – which was instrumental in causing surprise – perhaps arises from a false ‘left-to-right’ entailment of the English language, i.e. from the metaphor ‘program as written English communication’.

The traditional mapping or comparison approach to metaphor has a weakness that is evident in this example. The asymmetric mapping schema ‘from’ Source ‘to’ Target domain suggests that the computer is the more abstract frame which is to be understood by the more concrete master-servant domain, or compared one with the other via a common ‘ground’. But actually metaphor brings about an interaction (Black, 1962, 1993). As we know the concept of a person’s ‘memory’ is these days indissolubly linked to that of computer memory, as well as vice versa: thus we have the ‘brain is a computer’ metaphor, in which memory can be ‘working memory’, ‘short term memory’ or ‘long term memory’. Now the computer is used to understand the servant! Indeed the mind-is-computer metaphor is also referenced in talk of brain functioning colloquially, as in ‘I’d better re-compute’. This metaphor is essential to our understanding of human and computer ‘memory’ whose modern meaning has been constituted by a *recursive interaction* of meanings in the two domains. Thus, we see that Steve could have a conversation of some quality with the machine because the machine shares with Steve many common faculties: they both have a memory, they speak a common language, they interpret instructions, and they can even both be surprised. We will later argue that this quality of recursion is crucial also with mathematical models and representations. That is, the meaning of a model is generated recursively through its interaction with other domains and representations by other models and metaphors. Furthermore, Steve’s metaphorical way of speaking seemed to the researcher perfectly clear and unproblematic: perhaps the response “Yes” is the most remarkable part of this short discourse. The researcher got the point at once!

We therefore suggest that this metaphor is a ‘cultural model’, i.e. it is a widely distributed, inter-subjectively shared way of speaking about programming, machines and so on within a particular culture (Gee and Green, 1998; Holland and Quinn, 1987; Hutchins, 1995). Such models are effective mediators of communication as well as a means of thought. We argue that mathematical models can come to have a similar status in the culture; indeed we will argue that they must do so if they are to be of communicative value. This provides a motivation for placing 'models' in a common culture and so at the heart of mathematics education.

In 'The model muddle' Wartofsky (1979) clarifies the meaning of 'models': for him the essence is in re-presentation. He suggests that a model begins with our first mimetic acts and first use of language, and 'we continue modelling by way of ... analogies, models, metaphors, hypotheses and theories' (p. 10). Mathematical models and even isomorphisms are representations at one end of a spectrum from abstract to concrete, but every act of drawing attention to similarity or comparison belongs on the spectrum at some point. Lakoff and Johnson (1980) expose the embodied nature of cognition in the body-metaphors employed in our language: 'we travel the path of life', 'love is warmth', 'relationships provide nutrition', 'time is a resource' etc. These metaphors re-present the abstract, formal, social (life, love, time, etc.) in more embodied concrete, perceptual and imagistic terms (journeying, warmth, resources etc.). Thus the concrete and familiar in language is ultimately, for them, experienced in our body in space and time, via perception and image.

In many instances these metaphors have become so deeply embedded in our culture that we have no other way of thinking about their targets: thus for many psychologists the brain IS a computer with a short-term memory and so on, while for computer scientists BASIC is *really* a language, to be interpreted and so forth. Once such metaphors have embedded themselves in the culture to the point they are 'dead' (in Black's term: see Black, 1993) we may become culturally trapped in their metaphorical entailments: acts of metaphorical deconstruction become necessary. The point about the anthropological (Holland and Quinn, 1987), or cognitive linguistic (Lakoffian) de-construction of cultural models and metaphors for us is that it diagnoses a historical reification process: the model or metaphor that once came creatively into being has been crystallised and may even be forgotten and dead. This reification and forgetting is double-edged, since meaning becomes implicit and unconscious. In our example, we can say that Steve explains to the outsider how the command G90 functions, and rather than recreate the knowledge of G90 as a subroutine, he mediated his explanation metaphorically, calling on a widely shared cultural model, i.e. our way of thinking and talking about programming as instructing.

To take another case by way of contrast, we note in the above example the use of Cartesian X and Y co-ordinates in the movement commands, which is a widely culturally shared mathematical model, shared by Steve with us by virtue of having been taught in school. Thus the referencing of

X and Y in the code serves as an effective model for the language-author and for its user, Steve. Clearly mathematics can provide important ‘cultural models’ in affording communication too.

Black (1993) contrasts dead metaphors such as those analysed by Lakoff with metaphorical work that creates something new in the moment of the speech that invents it: metaphor in poetry provides good examples of this. A living metaphor, one that is still doing new work, may still be in the act of establishing itself as a cultural model, with its ramifications being worked through in practice. A powerful technological example of this is the ‘generative’ metaphor of Schon (1995). He describes how a group of technologists (trying to develop a man-made-fibre paint brush that doesn’t ‘clog up’) come to see the paint-brush as a kind of pump. The brush pumps the paint onto the surface it paints. As this insight develops over the course of days and weeks, its entailments are worked out: e.g. the ‘pipe’ the paint flows through is formed by the brush fibres, with the hand-handle working as the active pumping mechanism. This ‘pipe’ needs to deliver a smooth flow of paint from the top to the bottom of the brush-fibres so as to prevent the paint becoming ‘gloppy’ and clogging, so a smooth gradient to the ‘pipe’ has to be engineered. This analogy proves vital in engineering the right shape and arrangement of the fibres. In this paper we will show how a mathematical model can provide entailments that are generative in a similar way, providing for insights and communication repairs in a powerful way.

What begins as a metaphorical insight, works out as a fully organised analogy, whence we imagine it might become common-place in the engineering community: brushes behave like so and so because ‘they are really just pumps’. Even though apparently ‘dead’, these analogies might still, through new interactions of the target domain with the source domain, give new life: ‘hey, why don’t we make a ‘real’ water pump out of fibres’. One might imagine a time coming when technologists in the business talk of brushes that pump faster and smoother than others, in ways that leave the metaphor tacit or finally dead. Subsequently, in deconstructing this cliche, a critic might reveal inappropriate entailments that provoke fresh insights into our assumptions about brushes. This is the life-span of metaphor, from creative, to living, to dead, to resuscitation and hence perhaps new creativity, via deconstructive analysis.

We argue that this perspective could be adopted in relation to mathematical modelling, which might involve three distinct, important uses of models and modelling, as in (i) generative modelling, (ii) the use of well-known ‘cultural’ models in communication and in applications, and (iii) deconstruction of dead, crystallised models. In the next section, we will look at a case of breakdown in which an appeal to a well-known cultural ‘mathematical’ model – a double number line – proved generative. It is a well-known model, almost dead in some uses, but it takes on new life in the context of the workplace practice where it is fused with particular work processes.

3. An example of modelling at work: the gas estimation formula

Let us return to the complex formula used by Dan to estimate the amount of gas that his power plant would use in a "gas day" (i.e. from 6 am one morning to 6 am the next):

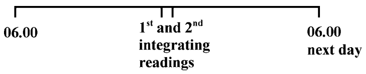
{{{{{"2nd INTEGRATING READING"-"0600 INTEGRATING READING"}
+{{{"2nd INTEGRATING READING"} – {"1st INTEGRATING READING"}}/T_2}*TIME4}}/100000}/3.6*CALCV*1000000/29.3071}

This spreadsheet formula was constructed by Dan himself, using terms from the workplace as variables, making it easier for him to recall the reasoning encapsulated by his formula. The use of so many brackets lengthens the formula and makes it look very complex but perhaps this arose because Dan has built up the formula by constructing it in stages. The brackets may well be useful in emphasising this and help him to 'unpack' the logic in the construction of the formula when necessary. However, although Dan confidently reconstructs the formula's meaning for the mystified researcher in the following transcript, there are indications that there are elements that have been forgotten by him; (we notice his hesitation over the unit of time in use, for instance).

In the following dialogue Dan, the power room engineer and guardian of the spreadsheet, explains what the spreadsheet is doing to Kate, a visiting teacher-researcher. Two students, Ben and Adam are present also but mainly silent. Kate is trying to work out what the formula does, and gets confused about the various time variables and what they mean (section 1.1–1.3). In an effort to clarify this, Dan draws a timeline model (middle of section 1.3) which he finds useful in marking out times and intervals for Kate.

1.1	Kate, our teacher-researcher inquires into the formula	Notes
Kate	Right, what's it doing? Look at that bit (*), where it's got a minus, it's just taking the two readings away from each other, so that's just how much it's used between. . ..	(*) Pointing to **minus** sign in {"2nd INTEGRATING READING"-"0600 INTEGRATING READING"}

Dan	Between . . .	space-time metaphor signals a possibility of the time-line model to come
Kate	. . . the last reading that you took, and the very first one.	readings "2nd INTEGRATING READING" and "0600"
	Yes. And then it's added on the difference between. . .	
Dan	The difference between the last two readings..	Referring to **difference:** $+\{\{\{$"2nd INTEGRATING READING"$\}$ – $\{$"1st INTEGRATING READING"$\}\}/T_2\}$
Kate	Yes.	
Dan	That gives you your rate, over a time, .. Doesn't it?	This part of formula calculates **the rate** of use of gas between the 1st and 2nd Integrating Readings: $\frac{C-B}{t}$
1.2	Kate doesn't follow and is provoked to inquire again: the breakdown moment	
Kate	Oh, hold on, hold on. . .. I don't. . .	breakdown
	Right. So we've got three readings: reading 1, reading 2, reading 3,. . .. and the first bit gives *that* minus *that*.	Kate struggling to understand the formula, students silent. Three readings. . .0600, 1st and 2nd INTEGRATING READINGS
Dan	Yes. So you know how much gas you're taking between them.	he means how much gas is consumed between readings
Kate	That's right. . .. now the second bit. . .	Points to next chunk of formula
Dan	So you add to it. . .	
Kate	*That* (1) minus *that* (2).	(1) i.e. 2nd integrating reading, minus (2) i.e. 1st integrating reading
Dan	*That* (1) minus *that* (2).	Refers to time T2
	But then time comes into it.	
Kate	Yes.	
Dan	So you get a rate, . . . per second or per minute, I forget what.	**Rate** per sec/min. Dan seems unsure of the details – once the formula has been constructed it is not necessary to remember the details –
1.3	Inquiry about T2: the source of confusion	
Kate	Yes.	**Time** T_2 an interval not a point in time: this seems an obstacle
	Oh so that *time* that you've got there, that. . .?	
	Is it T_2?	But also there is TIME4 in the formula
Dan	Yes, there's another calculation in there, it gives you T_2	T2 is calculated from other data not visible in this formula.
Kate	Oh I see, right. So T_2: that's the *time*?	recognising that 'time' is an interval, not a reading- inquiry about T2's extent

	So is it the time from first to the last, or is it a combined. . .	
Dan	Because you've already got. . .	Dan seems to recognise a need to help his explanation.
	All I'm interested in, is. . .	Dan drew the following time-line sketch: from left to right: first the unlabelled line. . .
	Let me draw it out. . .	then marks points on the line and labels them
		
Dan	The gas day: 0600. . . reading 1,. . . reading 2, . . . end of gas day.	'gas day' is 0600 to 0600 next day
Kate	Right.	
Dan	You've got a reading *there (1),* and you've got a reading *there (2)*. . .	(1) points to. first 06.00 reading
	so subtract one from the other, and you know how much you've used *there (3)*.	(2) points to. 2nd integrating reading i.e. 'how much gas' you've used: (3) i.e. between first 06.00 reading and 2nd integrating reading
	Reading 1 and reading 2, subtract one from the other, you know how much you've used *there (4)*;	'there' indicates points on line, in time, and readings
	but you also know the time difference between *there* and there (4).	(4) i.e. between 1st and 2nd integrating readings, the interval of time (T2)
Kate	Yes.	
Dan	So if you say. . .. If that was one minute and you took two units, it would be two per minute.	Dan refers to easy values of times in his attempt to explain "rate".
Kate	Yes. (*to students*) Are you with this, alright?	Checks students
Ben	I'm all right. . ..	One student, Ben, claims to follow . . .
Kate	Sort of?	Kate not convinced that students have followed this.
Dan	Because these lads have had to input a *time* as well as a number.	'these lads' who work for him put readings and times into a worksheet. . . here, 'number' is the measure of gas used
Kate	So you know how long it's taken.	
Dan	So now you know the *rate* between *there (1)* and *there (2)*.	(1) i.e. 1st integrating reading (2) i.e. 2nd integrating reading
Kate	Yes.	(3) i.e. 06.00 reading next day
Dan	You also know the time difference between *there (2)* and *there (3)*.	
Kate	Yes.	

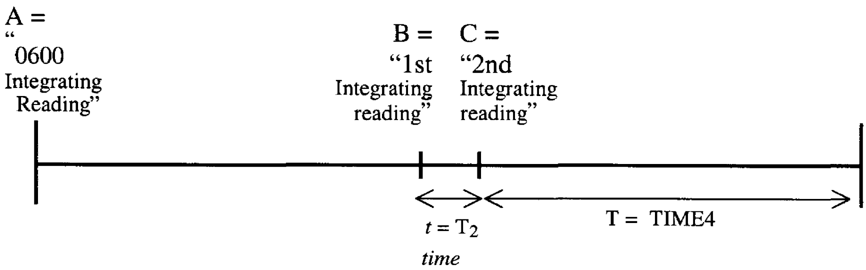


Figure 2. A double number line representation of gas used and time elapsed.

The function of the drawing of the timeline is to afford explanatory discourse that is indexed by gestures pointing to the points and segments of the line, and simultaneously by the verbal pronouns “there” and “between there and there”. Each point (interval) can reference a moment in time (time-interval), a gas reading (amount of gas consumed), or even the variables (and expressions) in the formula that signify these.

Thus we claim that the number line serves as a powerful semiotic support for the articulation of expressions relating to the times, intervals, and readings and their connection with the formula. A more powerful and complete representation such as a graph or a double number line separately representing time and gas-reading could help disentangle these references even more efficiently (see Figure 2 below and later).

Having established the rate of gas use, Dan now explains the ‘rate’ method for estimating gas used:

Dan	So if you transpose *that* rate into *that* time, and then. . . we get the volume that you’re going to use between *there* and *there* (*pointing to C and D*) add it to that.	Unusual use of the mathematical term ‘transpose. . . into’. . . he might mean he assumes you can ‘use’ this rate over the longer time?
	That’s why the first part of the sum is so easy: subtract *that* from *that*. You’ve already got the bulk of your day. . .	Pointing to terms in the formula: ‘that’ and ‘that’
Kate	Ah. . . (to students) So do you follow what’s happening? It’s only that bit that you’re using to do the rate. It’s not the whole lot. That was where I was thrown, because I thought you were trying to bring this in. Right.	‘That bit’ between 1st and 2nd integrating readings. . . rather than the average rate over the whole day ‘this’ referring to the first 6 am reading, 0600 is a number

Kate has understood the formula now and the number line has served Dan's explanation: let us examine the affordances of this model as an explanatory device here.

This time-line model is a special case of the number-line model that Lakoff and Nunez (2000) analysed and related to the grounding metaphors of arithmetic. It 'blends' the 'source-path-goal' metaphor (hence space and time), with the metaphors of numbers as 'collections/sets', 'objects to build with' and 'measuring sticks'. The line as a collection of points metaphor affords ordinal and cardinal properties of whole number, while the line segments, intervals or 'sticks' afford measurement. Thus far in the discourse above, we can see the metaphorical use of intervals as measures, and as arithmetical differences between values at points. Ratios between measures were also implicit in Dan's appeal to gas used as a function of time: 'if that was one minute and you had two units...' An important pedagogical feature of the number line is that numbers are represented *both* by points on the line and by segments or intervals between points: thus it is ideal for exposing and working with the arithmetic of addition and subtraction, as Gravemeijer et al. (1999) have shown in work on children's strategies.

In this case, the points of the time-line, signify both instants in time (e.g. 0600) and gas readings (e.g. "0600 Integrating reading"); while the intervals between them signify both intervals of time (e.g. T2 and TIME4) or gas consumed e.g. {"2nd integrating reading" – "1st integrating reading"} etc. Thus, there is here the double metaphorical blend of the number line itself with time and gas consumption, which makes it a particularly apt representation for this situation. In this 'applied' version of the number line, we see a conceptual blend of the gas day and gas consumed, but to this must be added:

- The number line metaphor itself: i.e. a blend of the grounding number metaphors on a source-path-goal metaphor;
- Symbols in the spreadsheet are also points and intervals on the line, and *hence* instants and intervals in time, or gas readings and quantities of gas consumed.

Consequently, every point on the line is associated or even semiotically 'fused' with (i) an instant in time and a gas reading, and (ii) the (pairs of) numbers or algebraic symbols that represent these. The pair of numbers involved suggests the need for a coordinate pair, i.e. a graph rather than a line. This will be discussed later, as it was in fact introduced as an explanatory model by the teacher – researcher working with the College students who accompanied her on the workplace visit.

In the above discourse, we claim that indexical gestures (pointing to 'points', waving back and forth at 'intervals') and indexical pronouns in the discourse (*it, here, there, between*) indicate – albeit ambiguously – the concepts or signs with which the points or intervals are fused, including implicitly expressions in the spreadsheet formula. Thus the timeline affords an 'embodied' sensori-motor and associated multiple discursive world of engagement, grounded in the space-time image-schema and a narrative of passing through time. This obviates the need to call on formal language such as 'time interval' and 'instant in time', 'gas reading' etc. in favour of pointing gestures which can convey the relevant meanings. Indeed the verbal equivalent of 'between here' – point 1 – 'and there', point-2 – might be as complex as "from the time of the first integrating reading to the time of the second integrating reading", or "the gas consumed between the first and second integrating reading" and indeed the non-verbal gesture may be read in either manner. Thus gestures make communication both easier, more fluent, and also more ambiguous, allowing the interpreter to generate meaning, or several meanings. Initially lacking precision, gestures indexing points and intervals afford negotiation and elaboration, and hence progressive refinement and precision to emerge inter-subjectively. For instance, when Dan first points to *there* and *there* on his line, it is unclear whether this refers to gas consumed or time elapsed, but meanings emerge in subsequent dialogue.

According to Roth (2001) 'gestures constitute a central feature of human development, knowing and learning across cultures' (p. 365). Roth shows how expositions with graphs in a science education context can 'have both narrative (iconic gesture) and grounding functions (deitic gestures) connecting the gestural and verbal narratives to the pictorial background' (p. 366). But Roth's review suggests the significance of gesture is even deeper than this: there are suggestions in the literature that gestures provide access to another dimension of communication (McNeill, 1992). For instance, when gesture conflicts with the verbal, it usually signifies a transition in meaning or development of understanding, and gesture leads the verbal development. In Roth's own studies, (op. cit.) the emergence of coherence from 'muddled' verbiage in children's explanations is accompanied by gestural embodiment of relations in advance of their formal, verbal articulation. In sum, gesture can provide a midwife for the birth of understanding.

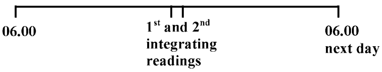
In the above example, the double number line is surely as important as the midwife's obstetric instruments. It is a particularly apt tool for the purpose, affording gestures the precision required to associate the context with the mathematical formula with near optimal efficiency. The gestures that mark out an interval and associate it with the symbol T2, allow Kate to see that T2 *is* an interval, and, by homing in on its endpoints, the extent of that interval. Finally we note the critical role of the number line as a means of identifying points and intervals with the spreadsheet terms and workplace realities they represent.

But in the above discourse this role of the model did not appear all at once, ready-made and complete. We suggest that the affordances of the time-line emerge like that of the paint-brush metaphor in Schon's account. Dan seems to produce the unmarked line, representing the course of the gas day. This metaphorical allusion is then developed by placing labels on points to represent the day's source (0600) and goal (0600 next day). Thus the line is manifested as a line containing some special points indicating the times and integrating readings taken at these times, and connecting these to variable names in the spreadsheet. We suggest that the complex of metaphorical, or model entailments has now begun to develop. But at this point in time (!), not all the relevant metaphorical entailments of the number line have come into play: for instance the 'measuring stick' metaphor has not yet played a role (see later where this will become apparent). We suggest also the key generative property which seemed to enable Kate's understanding was not initially manifest from the moment the line was drawn. Rather it emerged through the inquiry dialogue. It seems that neither Kate nor Dan knew quite what the communication problem was, nor how or whether it would be cleared up. Perhaps Dan reached for the number line as an externalisation of an image that came to mind, a habit of engineers in supporting explanation when problem solving. The nature of the problem and its solution then emerged as the model's representational potential developed in the inquiry, as terms in the formula began to be attached by gesture and deictics to the relevant points and intervals in the model.

In the next section we show how the time-line was replicated and its entailments extended by Kate when discussing the formula and its underlying assumptions with one of the students, Adam. This will illustrate the pedagogic use of a double number line model, validating its potential as a communication and problem solving device of significance to modelling and to mathematics education.

4. THE PEDAGOGIC USE OF THE MODEL WITH THE STUDENTS

The students present on the visit said very little during the entire interchange between Dan and Kate. Kate had good reason to think that the students had found the explanation difficult to follow, as they had earlier found a much simpler formula problematic. In the follow-up interview, she reminded Adam of the situation, and drew the timeline again, and engaged him in recalling Dan's explanation of the formula:

		Notes
Kate	Right. Well, if I remind you what he did.	Student was shown the formula.
	First of all he took the reading just before they went home. Subtracted the reading at 6 o'clock, so he would actually *know*	
	how much gas he'd used then,	Kate draws out Dan's time line to help her exposition
	Because 	
	he's got them on the readings. So right, so looking at the formula, that's what this bit's doing, right. It's the second integrating reading minus the 6 o'clock reading.	Bit = segment
Adam	Right.	
Kate	So that's telling him how much he's used *there*.	
	Now the problem is, that there's the bit *after* he goes home.	'goes home': traces the work process
Adam	Yes.	
Kate	So, what he does, is he takes two readings just before he goes home, so they're quite close together and what he does, is find how much gas he's used.	1st & 2nd Integrating Readings are 'close' in time and space on the line
Adam	And then plus some on to each other.	i.e. add estimate?
Kate	Right. So when you say he pluses some on to each other. Any ideas how he'd do that? I mean, how would you do it?	check meaning? Initiate (in I-R-F)
Adam	There'd be a certain amount of time between them so he just added the times with the values on.	Indicating time-line between 2nd Integrating Reading and 6am next day.

It is not clear what Adam means, but he uses the term 'times', and Kate picks up on this as a sign of a 'rate' conception, and translates his comment into a mathematical form 'however many times' i.e. a number of times. Then Adam makes use of the number line to identify a ratio as the number of times one interval will go into another, i.e. its measuring stick metaphorical entailment:

Kate	Good. That's right, so what I'm understanding you to say, is that he found how much he'd used there and then, however many times. . .	i.e. between 1st & 2nd Integrating Readings
Adam	Times-ed it by how many there are. . .	i.e. between 2nd Integrating Reading and 6am next day. See figure. 'Times' conception of ratio, as opposed to more sophisticated 'rate' conception.
	How many time periods there are between there and there.	

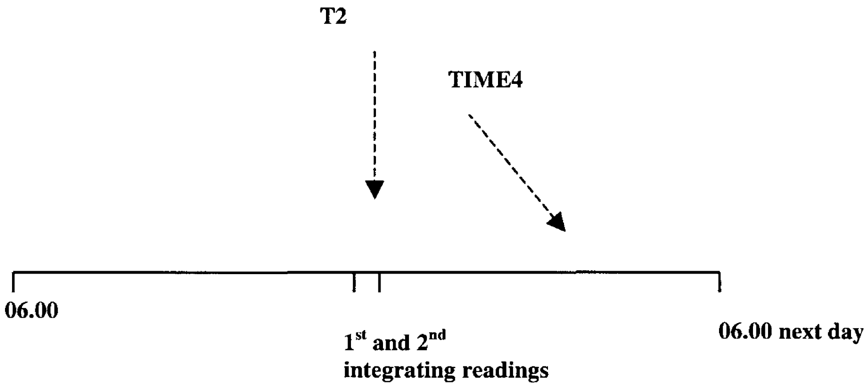


Figure 3. Time line used as a measuring stick to indicate how many 'times' T2 goes in TIME4.

By pointing to the segments (or 'sticks') "between first and second integrating reading" (T2) and "between the second integrating reading and the next day's 0600" (TIME4) he makes use of these segments as measures to access a 'number of times' conception of rate here, i.e. he uses the measuring stick property of the line. Where Dan had used the rate in the formula formally, the student accesses the same idea through a more concrete and 'embedded' number of times conception.

Thus we see one more metaphorical affordance of the number line becoming energised and providing a resource for problem solving and communication. As in the 'paint brush' metaphor, we claim that the number line has a generative character here emerging in pedagogic discourse, relevant because of the conceptions accessible to the student.

In the next part of the conversation, Adam reveals that he has deconstructed the formula sufficiently to appreciate its underlying, linear assumption, and questions this as a surprising feature of 'gas consumption' during a 24 hour period. His teacher imagines work process conditions under which this assumption might seem reasonable.

Kate	Good. That's right. That would be one way of doing it.	recognises this was not Dan's rate method
Adam	It's just that I thought, before he'd go home people would still be using it quite regularly. It'd be different between that time and say, 3 o'clock in the morning.	Adam questions the assumption that the rate of use of gas will remain constant overnight. . . his analogy is with 'people using gas'. . . which does not reflect this industrial context
Kate	Excellent, yes. So, *you* are realising that *he's* using that last bit of time, and *assuming*. . .	Kate says 'assuming. . ..' Making model assumptions explicit
Adam	That it's going to be the same throughout the night.	. . . i.e. 'same' rate of consumption of gas all night: seems counterintuitive
Kate	Now, I don't think he actually said anything about what he was assuming, but you're right, he is. So, because these are quite close together, I think possibly, I don't think he said this, but possibly from that little bit before he's going home, perhaps the rest of the plants all go home at 4 o'clock so he knows that it's down to the right sort of level. Perhaps that's it.	Kate constructs Work Process Knowledge that could justify the linear model

Adam shows a really significant insight here, all the more remarkable because linearity is one property of the model which is not iconically represented in the number line. Kate later is able to bring this out more clearly in a graph, where the gas consumed as a function of time is shown clearly as a straight line (its constant slope indicating the linearity in the model!).

Examining the two discourses in the case study above we see patterns of inquiry and pedagogic discourse genres. An inquiry genre is characterised by questions which seek information and clarification (as in Kate's inquiring of Dan), while the pedagogic genre is characterised by Kate's questions for which she already knows the answer. The latter are motivated pedagogically and initiate a response and is followed by assessment feedback ('good- yes'). Elsewhere these have been characterised as Inquiry (Wells, 1999) in contrast to 'Initiate-respond-feedback' I-R-F (Sinclair and Coulthard, 1975). Clearly in these two discourses above we see Kate in mathematical 'inquiry' and then in 'pedagogic' dialogues, but the timeline model plays a mediational role in both styles of communication. Thus we conclude that, while there are evident differences between inquiry and pedagogic discourses, mathematical models can play an effective role in both, making communication about complex ideas more accessible, particularly drawing on gesture and informal language referencing workplace variables and quantities on the one hand, and spreadsheet symbols and mathematical signs on the other.

Finally, we draw attention to other relevant models and translations. Our translation of Dan's spreadsheet formula led to an academic form in Figure 1. This was our attempt to prepare a mathematical reader for an understanding of Dan's formula, at least sufficient to follow the transcript. This 'translation' involves a 'hiding' of workplace/related details, (names of readings, conversion coefficients, bracketing, obscurities such as T2 and TIME4) which expose other, structural features of the formula (difference, rate, linearity). The translation into 'College mathematics' strips it of vestiges of the workplace practice, and a framing text has to be attached so that the situated sense of the formula can be re-constructed. Although we saw significantly different, potentially generative entailments of each mathematical model, the two formulae (spreadsheet and pure-mathematical), the double – number line and the Cartesian-graph model. Each model highlights different aspects of the practice of estimating gas, and each affords different connections and insights.

The use of the models in the two discourses revealed (i) the significance of the use of well known aspects of a well known model in supporting communication and problem solving in an applied context, but also (ii) the significance of 'generative' modelling, in the sense of the generative use of affordances of a model initially implicit.

5. Discussion

In the previous paper (Williams and Wake, this issue) we drew on Cultural Historical Activity Theory (CHAT) to show how context structures mathematical practices in work and College, (especially the former) and to illuminate contradictions between them. We also developed a new methodology to examine these: we study the contradictions as 'lived' in communication between workers and outsiders.

Any community engaged in a collective activity develops its own microculture, including its own discourse genres. We note this for situated mathematical practices too: they are mediated by their Activity System and tend to develop a genre of their own. In workplaces we suggested that this mediation involved workplace instruments, technical language and work-process knowledge. Outsiders have to learn something of the work process (norms, rules and division of labour) and workplace technology and jargon if they are to make sense of these practices, and this demands inquiry skills-and-predispositions as well as social confidence: many of our students seemed to lack these. Focusing on the different genres of mathematical language involved, the concept of translation or re-interpretation has attractions here. In the process of translation across mathematical genres, worker and outsider have to engage in new interpretations.

Productive explanations by workers sometimes draw on cultural models including metaphors and mathematical models in ways that serve the translation by making connections with relatively more concrete, relatively 'universal' cultural resources: the metaphors of communication and time-line and the model of the double number line are examples of these. In particular in the case of the time-line model, this embodiment affords a powerful combination of gestures and pronouns that semiotically link points and intervals with numbers or formulae and their workplace objects. Modelling can then be conceived of as the process of introducing a powerful model into a situation. A mathematical model may be powerful by virtue of the wealth of its potential mathematical, metaphorical entailments. Thus an initial insight to attach a model to a problem can be generative of new semiotic connections, as with a living metaphor.

The perspectives on metaphor offered by Black (interactive, living, dead, and resuscitated metaphor) Lakoff, Nunez and Johnson, (deconstruction of metaphor as embodied, ultimately grounded in sensori-motor action), and Schon (metaphor as generative) offered us a helpful, new perspective on mathematical models and modelling. We saw that tacit or dead mathematics can be deconstructed, or resuscitated. We saw that models can offer support for communication in inquiry and pedagogy, especially in offering an embodied, spatio-temporal world of sensori-motor action and communication through gesture and deictics. But finally we found that models can become generative, they may begin with an insight, the briefest of metaphorical connections, and become generative as the full entailments of the model emerge in its interaction with the context.

We argue that this perspective helps reveal the significance of mathematical modelling. Models do not only provide routine (dead) methods of solution, they also can be used to facilitate communication and thought in new applications and contexts, and they can interact with their field of application, or 'target domain' in creative, generative ways. This offers the possibility of solving genuinely new problems with a known model. It also accounts for the applied mathematician's view of mathematical modelling, in which mathematics is a model and representation of an external reality, and modelling strategies are described as metacognitive problem solving heuristics.

But recall also that the interaction between target and source domain can lead to new insights in the *source* domain. Just as the computer-is-a-brain metaphor leads computer scientists to insights into what a computer might be and do, so also the brain-is-a-computer metaphor led cognitive psychologists to new ways to model mind and thought. Similarly therefore, we suggest that the application of a model in a creative way to a new problem can involve some generative work on the mathematics of the model itself. This is the basis for the Freudenthal design of productive problems for mathematisation. In the example to hand above, for instance, we suggest that the pedagogical discourse between Kate and her student Adam above may have led to a new appreciation, at least for Adam, of the significance of 'the linear function' and its representations. We therefore argue that the role of real and authentic problems in developing mathematics can best be thought of as a metaphorical recursion or dialectic between mathematics and problem context. We find it elegant and seductive to tie together these two roles - modelling of mathematics and modelling with mathematics – as a dialectical, recursively-metaphorical interaction.

We used the term cultural model for such mathematical models because of their accessibility across a 'common culture'. There is a sense in which the College and workplace are sub-cultures of this common culture. When breakdowns in communication occur then explanations fall back on this wider culture, a common knowledge involving cultural resources incuding models. But these are surely by no means universal, even if they are often considered to be shared among educated people. Indeed they may be thought of as instantiating what it might mean to be 'educated' mathematically. We therefore infer that the discovery and validation of these cultural models *in practice* can and should inform our curriculum and pedagogy: both in the particulars of working out a workplace-relevant curriculum, and in general in terms of developing pedagogical strategies for making formal and abstract mathematics more accessible. We argue, again, that the research strategy we employed might provide an example for future curriculum research in relation to prevocational mathematics. By exposing the College maths curriculum to such distress, we may generate deeper development of it as a cultural resource.

NOTES

1. The Project 'Using College mathematicxs in understanding workplace practice' was funded by a grant from the Leverhulme Trust to the University of Manchester.

REFERENCES

Bakhtin, M., Emerson, C. and Holquist, M.: 1986, *Speech genres and other late essays* (Transl. C. Emerson and M. Holquist), University of Texas Press, Austin.

Black, M.: 1962, *Models and metaphors: studies in language and philosophy*, Cornell UP, Ithaka, NY.

Black, M. (ed.): 1993, *More about metaphor*, Cambridge University Press, Cambridge.

Blum, W. and Niss, M.: 1991, 'Applied mathematical problem solving, modelling applications and links to other subjects – state, trends and issues in mathematics instruction', *Educational Studies in Mathematics* 22, 37–68.

Cobb, P., Yackel, E. and McClain, K. (eds.): 2000, *Symbolizing and communicating in mathematics classrooms: perspectives on discourse, tools, and instructional design*, Lawrence Erlbaum Associates, Mahwah, N.J.

English, L.D.: 1997, *Mathematical reasoning: analogies, metaphors and images*, Lawrence Erlbaum Associates, London.

Freudenthal, H.: 1983, *Didactical phenomenology of mathematical structures*, D. Reidel, Dordrecht.

Gee, J.P.: 1996, *Social linguistics and literacies: ideology in discourses*, Taylor & Francis, London.

Gee, J.P. and Green, J.L.: 1998, 'Discourse analysis, learning, and social practice: A methodological study', *Review of Research in Education* 23, 119–169.

Goldin, G.: 2001, 'Countinng on the metaphorical. Where mathematics comes from: how the embodied mind brings mathematics into being', Nature, 413(6851), 18–19., 413(6851), 18–19.

Gravemeijer, K., McClain, K. and Stephan, M.: 1999, 'Supporting students' construction of increasingly sophisticated ways of reasoning through problem solving', in A. Olivier and K. Newstead (eds.), *Proceedings of the twenty-second annual meeting of the International Group for the Psychology of Mathematics Education*, Vol. 1, Stellenbosch, South Africa, pp. 194–209.

Halliday, M.A.K.: 1978, *Language as social semiotic: the social interpretation of language and meaning*, Edward Arnold, London.

Halliday, M.A.K. and Hasan, R.: 1985, *Language, concept and text: aspects of language in a social-semiotic perspective*, Deakin University Press, Victoria, Australia.

Holland, D. and Quinn, N. (Eds.): 1987, *Cultural Models in Language and Thought*, Cambridge University Press, Cambridge.

Hutchins, E.: 1995, *Cognition in the wild*, MIT Press, Cambridge, Mass.

Lakoff, G.: 1987, *Women, fire, and dangerous things: what categories reveal about the mind*, University of Chicago Press, Chicago.

Lakoff, G. and Johnson, M.: 1980, *Metaphors we live by*, University of Chicago Press, Chicago.

Lakoff, K. and Johnson, M.: 1999, *Philosophy in the Flesh*, Basic Books, NY.

Lakoff, G. and Nunez, R.E.: 2000, *Where mathematics comes from: how the embodied mind brings mathematics into being*, Basic Books, New York, NY.

Lamon, S.J., Parker, W.A. and Houston, S.K. (eds.): 2003, *Mathematical Modelling: A Way of Life* (*ICTMA 11*), Horwood, Chichester.

McNeill, D.: 1992, *Hand and mind: what gestures reveal about thought*, University of Chicago Press, Chicago.

Niss, M.: 1996, 'Goals of Mathematics Teaching', in A.J. Bishop (ed.), *International handbook of mathematics education*, Kluwer Academic Publishers, Dordrecht, London, pp. 11–47.

Nunez, R.E.: 2000, 'Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics', *Proceedings of 24th Conference of International Group for the Psychology of Mathematics Education (PME)*, Hiroshima University, Nishiki Print Co. Hiroshima, Japan 1, pp. 3–22.

Pimm, D.: 1987, *Speaking mathematically*, Routledge & Kegan, NY.

Pozzi, S., Noss, R. and Hoyles, C.: 1998, 'Tools in practice, mathematics in use', *Educational Studies in Mathematics* 36, 105–122.

Reddy, M. (ed.): 1993, *The conduit metaphor*, Cambridge University Press, Cambridge.

Roth, W-M.: 2001, 'Gestures: their role in teaching and learning', *Review of Educational Research* 71(3), 365–392.

Schiralli, M. and Sinclair, N.: 2003, A constructive response to 'Where Mathematics Comes From', *Educational Studies in Mathematics* 52, 79–91.

Schon, D.A.: 1995, 'Generative metaphor: A perspective on problem-setting in social policy', in A. Ortony (ed.), *Metaphor and Thought:* 2nd Edition, Cambridge University Press, Cambridge, pp. 137–163.

Sfard, A.: 1994, 'Reification as the birth of metaphor', *For the Learnng of Mathematics* 14(1), 44–55.

Sinclair, J. and Coulthard, M.: 1975, *Towards an analysis of discourse: The english used by teachers and pupils.* Oxford University Press, London.

Spiro, R.J., Feltovich, P.J., Coulson, R.L. and Anderson, D.K.: 1989, 'Multiple analogies for complex concepts: antidotes for analogy-induced misconception in advanced knowledge acquisition', in S. Vosniadou and A. Ortony (eds.), *Similarity and Analogical Reasoning*, Cambridge University Press, Cambridge, pp. 498–531.

Streefland, L.: 1991, *Fractions in realistic mathematics education: a paradigm of developmental research*, Kluwer Academic Publishers, Dordrecht.

Treffers, A.: 1987, *Three dimensions: a model of goal and theory description in mathematics instruction–the Wiskobas Project*, D. Reidel, Dordrecht.

van Oers, B.: 1998, 'From context to contextualising', *Learning and Instruction* 8(6), 473–488.

Wake, G.D. and Williams, J.S.: 2001, *Using College Mathematics in Understanding Workplace practice: Summative Report of the Research Project Funded by the Leverhulme Trust*, Manchester, University of Manchester, http://www.education.man.ac.uk/lta/publications/leverhulme_summary.htm

Wartofsky, M.W.: 1979, *Models: representation and the scientific understanding*, Reidel, Dordrecht.

Wells, C.G.: 1999, *Dialogic inquiry: towards a socio-cultural practice and theory of education*, Cambridge University Press, Cambridge.

Werner, H. and Kaplan, B.: 1963, *Symbol formation: an organismic-developmental approach to language and the expression of thought*, John Wiley and Sons, New York.

Wertsch, J.V.: 1991, *Voices of the mind: a sociocultural approach to mediated action*, Harvester, London.

Williams, J.S. and Wake, G.D.: this issue, 'Black boxes in workplace mathematics', *Educational Studies in Mathematics*.

Williams, J.S., Wake, G.D. and Boreham, N.C.: 2001, 'College mathematics and workplace practice: an activity theory perspective', *Research in Mathematics Education* 3, 69–84.

School of Education
University of Manchester
Oxford Road, Manchester, M13 9PL
E-mail: Julian.Williams@manchester.ac.uk;
Geoff.Wake@manchester.ac.uk