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EQUIVALENT STRUCTURES ON SETS: EQUIVALENCE CLASSES, PARTITIONS AND FIBER STRUCTURES OF FUNCTIONS

ABSTRACT. This study reports on how students can be led to make meaningful connections between such structures on a set as a partition, the set of equivalence classes determined by an equivalence relation and the fiber structure of a function on that set (i.e., the set of preimages of all sets $\{b\}$ for b in the range of the function). In this paper, I first present an initial genetic decomposition, in the sense of APOS theory, for the concepts of equivalence relation and function in the context of the structures that they determine on a set. This genetic decomposition is primarily based on my own mathematical knowledge as well as on my observations of students' learning processes. Based on this analysis, I then suggest instructional procedures that motivate the mental activities described in the genetic decomposition. I finally present empirical data from informal interviews with students at different stages of learning. My goal was to guide students to become aware of the close conceptual correspondence and connections among the aforementioned structures. One theorem that captures such connections is the following: a relation R on a set A is an equivalence relation if and only if there exists a function f defined on A such that elements related via R (and only those) have the same image under f.

KEY WORDS: APOS theory, equivalence classes, equivalence relations, functions, genetic decomposition, partitions, set structures

1. INTRODUCTION

This study reports on how students can be led to make meaningful connections between such structures on a set as a partition, the set of equivalence classes determined by an equivalence relation, and the fiber structure of a function on that set (i.e., the set of preimages of all sets $\{b\}$ for b in the range of the function). In this paper, I first present an initial genetic decomposition, in the sense of APOS theory, based on the Piagetian model of epistemology, for the concepts of equivalence relation and function in the context of the structures that they determine on a set. This genetic decomposition is primarily based on my own mathematical knowledge as well as on my observations of students' learning processes. Based on this analysis, I then suggest instructional procedures that motivate the mental activities described in the genetic decomposition. I finally present empirical data from informal interviews with students at different stages of learning. My goal was to guide students to become aware of a correspondence between the aforementioned structures. The interviews are analyzed with the aim of

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evaluating the theoretical genetic decomposition and the extent to which the hypothesized cognitive tools were indeed constructed by the students. The analysis also evaluates the instructional procedures.

2. Theoretical framework

This study is based on APOS theory interpretation of Piagetian constructivism (Dubinsky, 1991) and the idea of "genetic decomposition" of mathematical knowledge into an interrelated group of mental schemas (Piaget and Garcia, 1989). Schemas are sets of cognitive elements and relations between these elements. Understanding or mastering a concept would require the construction of a schema corresponding to it. The construction procedure is referred to as reflective abstraction, and involves the following mental activities: interiorization (the construction of an internal process corresponding to some mathematical transformation), generalization (assimilation of a new phenomenon to an already existing schema), coordination (linking together more than one schema), encapsulation (solidifying into a mathematical object a certain cognitive process), and finally reversal (forming a new process by reversing an existing internalized process). Briefly, reflective abstraction is the general coordination of mental constructions through the mental activities above for the purpose of building more complete structures. In the framework I adopted, mental constructions in the genetic decomposition are known as Action, Process, Object and Schema, whence the acronym "APOS theory" (Asiala et al., 1996). An *action* conception of a mathematical idea is a description of an understanding that is limited to performing an action on that concept. When an action is repeated and reflected upon, it may be interiorized into a *process*. A learner has a *process conception* of a concept when his or her depth of understanding is limited to thinking about the idea as a process without being able to execute an action on this process. A process is said to be encapsulated into an *object* when the individual can perform actions on it, and decompose it back to the matter from which it was initially formed. Finally a *schema* of a piece of mathematics is an individual's collection of actions, processes, objects, and other schemas, linked in a coherent framework in the person's mind (McDonald et al., 2000).

According to Piaget, learning requires being cognitively active, responding to mathematical situations by trying to assimilate them to one's existing schemas. A key assumption to this theory is that the student must be aware of a certain conflict or contradiction between the problem and the offered solution, or the unavailability of a solution. Obviously, this theory of learning is not in agreement with the "transmission" view of teaching, based on the belief that knowledge may be transmitted from one who knows to one who may or not be eager to learn by way of verbal transmission and providing many examples.

3. Motives and general planning of the research

This study was triggered by my dissatisfaction, as a teacher, with students' general tendency to overlook similarities between the effects of partitions, equivalence relations and functions on a set as far as setting a structure on that set. I then tried to reflect on the situation as a researcher, from the point of view of APOS theory.

Normally, students are subjected to information about functions, partitions and equivalence relations as separate fields; it is not obvious to them that they can be linked. In general, any theorem that brings together several concepts that are normally unrelated can be a source of difficulty to students. This difficulty will not have the structural aspect of a contradiction, but will cause instability (Piaget, 1980). Such instability becomes structural only when the learner tries to compare the different practices leading, in our context, to similar structures. It is the instructor's responsibility to guide learners toward finding links and connections between such concepts.

According to the adopted theory, to find ways of fostering conceptual thinking in mathematics, the researcher must follow several steps: first observe the students in the process of learning, and then analyze these observations in the light of the theory. These observations, together with the researcher's own mathematical understanding, help hypothesize the genetic decomposition for the concept involved. Instructional methods are then designed for the purpose of coaching students along the cognitive steps in the proposed genetic decomposition. Following that, empirical data are once again collected, this time focusing on the mental constructions that are made while students are subjected to the planned learning procedures (Dubinsky, 1991). The purpose this time is to verify if these actual mental constructions coincide with the theoretical genetic decomposition. The conclusion would either be that mathematics has indeed been learned, or that the initial analysis needs revision. As an outcome of this analysis, more light is shed on the epistemology of the concept; at the same time, pedagogical strategies are designed in agreement with the way we believe that students can actually learn.

4. The context of the study and its guiding questions

4.1. *The course content and students' background*

I conducted this study in a discrete mathematics course during two semesters, once in a forty-student-class in the fall, and another time in a twenty-five-student class in the summer. The fall class met three times a week for 14 weeks, for 50 min each time, while the summer class met daily for 5 weeks for 100 min each time. The students were either computer science or business-computer majors. The prerequisite for the course is either one course in calculus or one course in basic mathematics, a terminal course for non-science-majors that includes the topics of basic set theory, logic, linear programming, number theory, binary systems, and basic information on functions and graphs. In discrete mathematics the topics are fundamentals of set theory, functions, logic, induction, relations, introduction to graph theory, and basic probability.

When they reach this part of the course, it is safe to assume that students have an acceptable knowledge of functions: they think of a function f defined on a set A called its domain as a machine that takes input (usually called x) from the set A and transforms it or maps it to a unique image, called $f(x)$, in the set $f(A)$, the range of f. As far as representations are concerned, they are familiar at least with the "formula" representation, which they call the "equation" of the function. They are able to graph simple functions: linear, quadratic, and exponential functions. They know how to compare the growth of functions and they can compose functions. They can check if a function is increasing or decreasing, if it is an injection, a surjection or a bijection. They know that bijections are invertible, and they have learned some techniques for finding the inverse of a function.

4.2. *Description of my teaching approach*

When teaching the course, I like to promote an unusual perspective, which I called "grouping like-elements." As part of my teaching experiment, I guide students through class activities and interactive class discussions into thinking of a function and a relation in a new perspective. This perspective primarily involves shifting the emphasis from the effect of the function or relation on the elements on which they act, to grouping those elements in the domain of the function or the relation into categories depending on the way they are being acted upon by those operators.

4.2.1. *Rethinking equivalence relations: Change of emphasis*

Equivalence relations are normally introduced as special relations with the three properties: reflexivity, symmetry and transitivity. Equivalence classes come forward as a typical structure on a set, hard to identify just by "contemplating" the relation. When teaching this section, I guide students into *envisioning* the equivalence classes corresponding to an equivalence relation, which requires going beyond the element-wise conception of a relation, to a more global perception of it. This is not a straightforward activity, especially that there is no definite pattern that can always be applied.

4.2.2. *Rethinking functions: Change of emphasis*

Just as an equivalence relation must be rethought in order for its equivalence classes to be visualized, a function must also be rethought through its effect on its elements to attend to and visualize the "fiber structure"[1] of the function, i.e. the set of all subsets of the domain that are made of elements mapped on a single element of the range. While the main emphasis in teaching functions is normally put on the images of the elements in the domain, the emphasis in this case is shifted toward regrouping the elements in the domain according to their images. In other words, a function will be characterized by the manner according to which it clusters elements in its domain, sending all elements in one group to a common image; in that sense, less attention is awarded to what is that image, with more emphasis on the clusters themselves.

4.2.3. *Good use of partitions*

An extensive work on subsets, complements, power sets, and cardinality always precedes the introduction of partitions on a set, which is in general a subject whose usefulness is hard to show. In teaching the section on partitions, I make it a point to illustrate their effectiveness in proofs that use Venn diagrams, as in showing that: $U-(A\cup B) = (U-A)\cap(U-B)$. In this case I use a partition determined on U by the two subsets A and B, consisting of the following four (non-overlapping) subsets: $U - (A \cup B)$, $A - B$, $B - A$, *and* $A \cap B$. Students see that each element of U must fall in exactly one of these subsets. Just like when entering a four-room house (with no corridors), one must be in exactly one room at any given time. With this understanding, it suffices to verify that the subsets of the partition making up the left side are themselves the subsets making up the right side; and that would be a rigorous mathematical proof, to the surprise of most students.

4.3. *General questions guiding the study*

In this study, I am interested in answering the following general questions:

- How do students understand partitions?
- How do students understand partitions determined by equivalence relation?
- How do students understand partitions determined by functions?

And most importantly

- Can students surf between structures determined on a set once by equivalence relations and another time by functions? And if so, how?

5. Theoretical analysis (genetic decomposition)

The purpose of the theoretical analysis is to propose a genetic decomposition of the concepts of equivalence relation and function in the context of the structures that they determine on a set. At least two major cognitive structures must be present in order to begin the construction of equivalence classes: that of a function and that of an equivalence relation.

In this section I will analyze the cognitive tools and mental activities involved in the constructions stated below:

1. Finding the partition on the set A determined by an equivalence relation R on the set A. (In other words, given R, find the corresponding partition structure on A.)
2. Finding the partition on the set A determined by a function f with domain A. (In other words, given f, find the corresponding partition structure on A.)

In the following, the procedure of deducing the equivalence classes determined by an equivalence relation will be referred to by "(M1)". The procedure of identifying the fiber structure of a given function f will be labeled "(M2)".

3. Finding a function that would determine the same partition on A as a given equivalence relation R on A. (In other words, given R, find the corresponding f.)
4. Finding an equivalence relation R on a set A that would determine the same partition on A as a given function f with domain A. (In other words, given f, find the corresponding R.)
5. Coordinating the constructions in 3 and 4 above in order to establish theorem EQF: a relation R is an equivalence relation if and only if there exists a function f such that elements related via R (and only those) have a common image under f.

The following is an analysis of constructions (1–5). The terms in italics represent the reflective abstraction mental activities, or cognitive tools.

5.1. *Analysis of construction 1*

Finding the partition of the set A determined by an equivalence relation R on A requires different tools depending on many factors. When the set A is finite and relatively small, or the relation is explicitly described as a subset of $A \times A$, the task of identifying the elements that are related to a fixed element *m* in *A* is a mere *action*. However, if the set *A* is infinite (or large) or the relation *R* is represented by a formula, then, given an element *m* in *A*, one needs to *interiorize* (into a process) the action of checking for all the elements related to *m*. In other words, one needs to have a practical inner sense of the effect of *R*. This makes it possible to identify all those elements related to *m* without having to go through all the elements of *A*, (mechanically) checking whether or not each one of them is related to *m*. Reaching this inner sense requires a process conception of the relation, which is the result of building an internal construction that does the same thing as the action of checking for the elements related to *m*. Once this is done, one needs to *encapsulate* the process of finding the many elements related to *m* into an object, which is nothing but the equivalence class [*m*]. This would correspond to grouping elements in *A* in a many-to-one mode (Piaget, 1977).

5.2. *Analysis of construction 2*

Given a function *f* defined on a domain *A*, finding the partition of the set *A* determined by *f* requires the *reversal* of the action of mapping *x* in *A* to *y* in *f*(*A*) into the action of, given *y* in *f*(*A*), finding the cluster of elements *x* in *A* such that $f(x) = y$. This corresponds to grouping elements in *A* in a many-to-one mode (Piaget, 1977).

5.3. *Analysis of construction 3*

Given an equivalence relation *R* on a set *A*, finding a function that would determine the same partition on *A* as *R* requires a *generalization* of the schema of function construction. In this case, the function is constructed based on its fiber structure. Thus the required function at this step maps elements in a class into a single image. Less importance is given to what that image might be. There are two alternatives for constructing such a function:

(a) One possibility is to take, for *f*, the canonical mapping from A onto the quotient structure A/*R*,[2] whereby each element *m* in *A is mapped* onto its equivalence class [*m*]. Now that the classes are identified, their elements can be acted upon by a different and higher level structure, namely that of a function, as in $f(m) = [m]$. Note that although an element *m* appears to be associated with a single element (otherwise the function *f* would not be well-defined), yet, in reality, *m* is associated with all those elements *n* in that class, as a collection rather than as separate elements. Hence, grouping elements in *A* are now performed in a one-to-many mode. One can describe this process of function construction as follows: the classes [m] that became concrete impose a certain partition on A. Based on that partition, the function f is defined. So this process can be summarized by: Equivalence relation, hence partition, hence function (whose fibers constitute that partition.)

(b) The other possibility for constructing a function f that agrees with R runs as follows. R classifies the elements of A into equivalence classes according to certain characteristics that they have in common (e.g. integers are classified into even and odd numbers). The function f is then constructed based on these characteristics, in the sense that it will map an element to that identified characteristic (continuing the example: f maps numbers divisible by 2 onto "even" and numbers not divisible by 2 onto "odd"). Consequently, it will naturally follow that elements in A that are in the same class are mapped to the same image. In a way, this construction relies on the range since we are considering that f^{-1} (characteristic x) = the set of those elements of A which possess the characteristic x; hence the one-to-many grouping mode.

Piaget (1977) uses the expression "classifying the elements according to similarities between the *elements*" to describe this transition from relating elements in A to one another via R, to relating elements in A to one another according to a "common characteristic". Applying his analysis to our situation, and given the properties that put elements into classes, it will follow that the function which maps an element to its property is a function whose inverse images form that initial partition on A.

During a class discussion, as I will later narrate in the section on empirical data, one student proposed a technique that he called the "break-up technique" for isolating that characteristic x common to elements in a class $[m]$. The technique simply involves a re-reading of the relation R in the form "mRn if and only if both elements m and n independently satisfy a property x". As a result, the properties become labels associated with elements in the set A; consequently, elements with the same label are placed in one subset of A; these subsets obviously form a partition on the set A. The function f will be defined in such a way that an element m is mapped on the label that was attached to it. The cognitive tool used here, as far as reflective abstraction is concerned, is *reversal* since the function f is constructed based on its inverse images: all those elements carrying that label. Needless to say that it is possible to separate m from n in the expression for R only if R is an equivalence relation.

5.4. *Analysis of construction 4*

Given a function f defined on a set A, finding an equivalence relation R on A that would determine the same partition on A as f requires a *generalization* of the schema of relation construction so as to include the construction of an equivalence relation based on an existing partition on A. The partition in this context consists of the fiber structure of *f*. The equivalence relation *R* is defined by: *mRn* if and only if $f(m) = f(n)$. Moreover, the "many-to-one mode" grouping mode referred to earlier has been *reversed* into a "one-to-many" grouping mode in the following sense: (many) elements in (one) single fiber of *f* are grouped in [*m*], whereas (one element) *m* is related to (many elements) in [*m*] via *R*.

5.5. *Analysis of construction 5*

The coordination between the processes described above culminates in the following theorem, labeled *EQF*: if the relation *R* is such that there exists a function *f* with the property that for every (*m*, *n*) in $A \times A$ we have $f(m) = f(n)$ if and only if *mRn*, then *R* is reflexive, symmetric and transitive.

In what follows I will refer by DEQ to the definition: a relation *R* is an equivalence relation if and only if it is reflexive, symmetric and transitive.

The difficulty of understanding EQF can be explained by the presence of a double level quantification, and the fact that two different types of quantifiers are applied in succession to a single proposition (Dubinsky, 1997). Understanding this theorem requires the following steps of reflective abstraction: first, the *encapsulation* of the process of double implication in "$f(m) = f(n)$ if and only if *mRn*" into an object, one for each element (*m*, *n*) in $A \times A$, which entails browsing through $A \times A$, checking the validity of the proposition "$f(m) = f(n)$ if and only if *mRn*" for all elements of $A \times A$ (Dubinsky, 1997). Notice here the use of existential quantifiers with which students have trouble in general.

Alternatively, instead of iterating through $A \times A$, one may allow the object which is the double implication "$f(m) = f(n)$ if and only if *mRn*" to be the output of a function *Q* with domain $A \times A$, namely,

$$Q(m, n) = \text{“}f(m) = f(n) \text{ if and only if } mRn\text{”}.$$

This function is a great example of a cognitive step, which is not a mathematical step, since it is not mathematically necessary (Dubinsky, 1997). The use of this function requires the *generalization* of the schema of functions so as to include double-implication-valued-functions.

Finally what yields the second level quantification is applying the existence of the function *f* to the previous proposition *Q*.

In conclusion, one can think of (M1) and (M2) as objects in the general schema for proofs of the existence of partitions on a set. This means that the two practices must have been *encapsulated* so that the student can reflect on them along with other similar methods, when faced with a situation that requires proving the existence of a partition on a set (Dubinsky, 1991).

6. Instructional procedures

I assume that any successful instruction of mathematical constructions would take into consideration the cognitive structures, as well as the mechanism (reflective abstraction) on which these constructions are built. The preceding epistemological analysis serves as a guideline for planning instruction. In the following I present a selection of activities that were designed to help students along the cognitive steps in the genetic decomposition.

The classroom treatment of partitions, and set structures determined by functions and equivalence relations, takes about a week and a half. It consists of a relatively traditional treatment of the topic, in which a general description of the different concepts is presented in lecture form, interspersed with the working of an example, followed by the students working on problems in class and receiving immediate feedback in the form of a subsequent explanation of each problem and a discussion of errors that may have been made.

In all my classes, I usually try to periodically guide students toward connecting concepts. Many times students have pleasantly surprised me by uncovering hidden unintended relationships and connections between concepts, such as looking for a formula for the total number of possible partitions within the power set of A, or finding a practical technique for isolating common properties in an equivalence class, as mentioned in the previous section.

Below, I present some of the in-class activities with some comments and relate them with the respective mental constructions mentioned in the proposed genetic decomposition.

6.1. *Warm-up activities on partitions and relations*

The following three activities are meant to merely reinforce what is supposed to be prerequisite cognitive structures to the construction of the involved concepts: they represent a revision for the foundation of the construction.

Activity 1. Given a set A, select as many partitions of A as possible from a given collection V of subsets in the power set $P(A)$ of A. For instance, $A = \{a, b, c, d, e, f\}$ and $V = \{\{a, b\}, \{a, c\}, \{c, f\}, \{a\}, \{e, f\}, \{d\}, \{e\}, \{c\}, \{c, d\}, \{b\}\}$.

Even a student with only an *action* conception of partition would be able to complete such an activity. One student asked whether there was a formula that gives the number of partitions of A in terms of the cardinality $|A|$ of A, since there is already a similar formula that gives the cardinality of the power set in terms of the cardinality n of A. I included this exercise as a take-home project for $n = 2, 3, 4$, and received interesting responses.

Activity 2. Extend to a partition a certain subset V of $P(A)$ in as many ways as possible. For instance, $A = \{a, b, c, d, e, f\}$ and $V = \{\{a, b\}, \{c\}\}$.

Note that in this type of activities, it is enough to focus on isolated pieces of the material without necessarily noticing the cognitive connections between similar situations.

Activity 3. Given a set, to construct a relation with a given set of properties (e.g., a relation that would be reflexive and transitive but not symmetric).

This class exercise turned into a fun and challenging game. Students decided to split into two opponent teams: one would impose the required properties, while the other would construct the appropriate relation. It was added later that the team that proposed inconsistent conditions would lose the game.

6.1.1. *Activities promoting M1*

Activity 4 below requires the mental constructions exhibited in step 1 of the proposed genetic decomposition.

Activity 4. Given a relation R on a set A, first verify that R is an equivalence relation and then find the equivalence classes relative to R (e.g., as in the case of congruence mod p).

The next activity aims at preparing students for the mental constructions exhibited in step 4 of the genetic decomposition since here, given a partition, they need to come up with the corresponding equivalence relation; whereas in step 4, given a function, they need to come up with the equivalence relation.

Activity 5. Given a set A and a certain subset V of the power set $P(A)$, first verify that V is a partition of A, and then recognize a "known" equivalence relation R that would correspond to that partition (e.g., as in the case of $A = \mathbf{Z}$ and $\{\{3k\}, \{3k + 1\}, \{3k + 2\} : k \in \mathbf{Z}\}$ or $\{\{k, (-1)k\}, k \in \mathbf{N}\}$.

6.1.2. *Activities promoting M2*
The following activity requires the mental constructions exhibited in step 2 of the proposed genetic decomposition.

Activity 6. Given a function *f* with domain *A* and range *B*, find the inverse images, under *f*, of all $\{b\}$, where $b \in B$. In other words, identify elements in *A* with a common image under *f* (e.g., as in the case of the functions $f(m, n) = m/n$ in $\mathbb{Z} \times \mathbb{Z}^*$ or $f(m) = m^2$ on $\mathbb{Z}$.

Just like in Activity 5 above, the following activity (Activity 7) prepares students for the mental constructions exhibited in step 3 of the analysis since here, given a partition, they need to come up with the corresponding function that puts the same structure on its domain as the preset partition.

Activity 7. Recognize a known function *f* on a set *A* whose inverse images are a preset partition *V* on *A* (e.g., as in $V = \{\{3k\}, \{3k+1\}, \{3k+2\} : k \in \mathbf{Z}\}$ or $V = \{\{k, (-1)k\}, k \in \mathbf{N}\}$, where in both cases A is $\mathbf{Z}$).

6.1.3. *Activities promoting coordination between M1 and M2*
The following activity requires the mental constructions exhibited in step 4 of the proposed genetic decomposition.

Activity 8. Given a function *f* on a set *A*, find a relation *R* on *A* such that *f* and *R* would give the same partition on *A* (e.g., *A* is the set of rational numbers and $f(m) = m^2$ or $f(m) = \lfloor m \rfloor$, the greatest integer in *m*. Or $A = \mathbb{Z} \times \mathbb{Z}^*$ and $f(m, n) = m/n$.

6.1.4. *Activities promoting EQF*
The following activity requires the mental constructions exhibited in step 3 of the proposed genetic decomposition.

Activity 9. Given a relation *R* on a set *A*, construct a function *f* on *A* whose fiber structure is the same as the structure of equivalence classes determined by *R*, that is, such that $f(m) = f(n)$ if and only if mRn, for all (m, n) in $A \times A$. (For instance, $A = \mathbf{Z} \times \mathbf{Z}$, and $(m, n)R(p, q)$ if and only if $mq = np$, or $m + q = n + p$.

Finally, activity 10 requires the mental constructions exhibited in step 5 of the proposed genetic decomposition.

Activity 10. Given a set *A*, a collection of functions with domain *A*, a collection of relations on *A*, and a collection of partitions of *A*, match them appropriately in order to end up with triplets: (relation, function, partition). In case you matched a relation with a partition but you are missing the corresponding function, find it. In each case find the corresponding classifying set of properties.

As a whole, these activities serve as motivating students to surf between equivalence relations, functions, fiber structures, and partitions, finding the corresponding components.

7. Evaluation of the genetic decomposition: Excerpts from interviews with students

Students' performance on assignments and their responses to the informal interviews have served, in this research, as instruments of evaluation of the proposed genetic decomposition. In this section, I will present excerpts from interviews and discussions with students, together with a short analysis. The interviews are supposed to show the extent to which the student made the previously listed required mental constructions posited in the genetic decomposition. In successful cases (where progress was made and where mathematics was learned), I will try to determine those activities that contributed to this progress. Another purpose of the analysis is to evaluate the instructional procedures. Finally I hope to reach an evaluation of the entire approach both in terms of a supported explanation of *how* one might learn the involved mathematical concepts, and *a pedagogical* strategy for enhancing that learning.

The interviews were conducted with twenty-three students who came to my office to collect their graded homework. The chosen interviews present some common trends among the participants. Not all the questions were determined beforehand, and the students did not always know what was going to happen. They were coming for a discussion of the material, but knew that they were not going to be graded. Most of the interviews lasted for less than 10 min. Only notes were taken, and no recording was done.

7.1. *Construction of equivalence classes*

When the following question was asked in an interview, students had just been exposed to the procedure M1, which consists in deducing the equivalence classes determined by an equivalence relation. At the time of the interview, type 4 and 5 activities had been completed in class, and problems similar to those activities had been assigned as homework.

Interviewer: Given an equivalence relation *R*, how do you find the equivalence classes [*m*]?

Student A: I take an element *m* in *A*, and I check with all the other elements in the set *A* for the ones that are related to *m*. If they are, then I just place them in [*m*].

Interviewer: You would do this how many times?

Student A: Till I've finished [checking] all the elements in *A*.

When asked about what to do in case the set was infinite, the student did not know what to say.

This student needs to interiorize the relation *R*. He was merely responding to an external stimulus, namely checking if an element *m* is related to an element *n*. Had he mastered type 4 and 5 activities he would have been able to answer the question. We also note that the student did not acquire the complete mental constructions described in step 1 of the genetic decomposition. He had only acquired an *action conception*. The "practical inner sense" of what R really does was not depicted.

In contrast, Student B below had a better mastery over type 4 and 5 activities, which allowed her to "guess", without prompting, which pairs of elements were related to each other.

Student B: I need to really know what *R* does. For that, I should see lots of examples of elements that are related via *R*, and observe what makes them related. Once I really know *R*, I should be able to guess what are the next elements that are related to each other without having to check every time if *mRn* is satisfied.

One can easily see that Student B interiorized, into a process, the action of checking for all the elements related to a given element *m* in *A*.

7.2. *Understanding M2*

The interview cited below was done in my office when the student came to pick up his homework involving questions similar to type 8 activities.

Interviewer: In your opinion, how does a function *f* defined on *A* determine a partition on that set?

Student C: First I have to find an equivalence relation *R* that does the same thing as *f*.

Interviewer: What do you mean by that?

Student C: I mean that if the function does not distinguish between some elements, then they must be related via *R*. As if both *f* and *R* see those same elements as one entity. Then after I show R is an equivalence relation, I will find its corresponding classes. Those classes will be the partition on A, relative to R, and consequently relative to f.

To this student, it is more natural for an equivalence relation than it is for a function to set a partition structure on a given set. Student C may have seen this approach as a shortcut and, in fact, may have already encapsulated the equivalence between partitions, equivalence relations and fiber structures. So, one can say that in a way, the student went too fast through the APOS process. When I first heard his comment, my initial interpretation was that he ignored or overlooked the class effort in activity 6 (given a function, to find the corresponding partition), thus refuting the seemingly more direct procedure, and remembered only the procedures in type 8 activities (given a function, to find the corresponding equivalence relation) just because they were more recently learned procedures. But later I was more inclined to believe that it was more natural and straightforward for him to go through equivalence relations because to him, a partition is more naturally connected with equivalence relations than with functions. That procedure, although takes a longer time, is more elementary and requires less proven facts.

7.3. *Constructing the function f given an equivalence relation R*

When the interview quoted below was done, procedures M1 and M2 had both been practiced in class, but theorem EQF was not discussed yet. Students were able to surf between an equivalence relation and a partition, or between a function and a partition. Also type 8 and 9 activities had been extensively practiced in class, and so students were trained on how to find R given f, and vice versa, yet, they could not transform this ability into a theorem that could be used to determine whether a certain relation is an equivalence relation (as in EQF).

Interviewer: So far we know that an equivalence relation and a function both determine a partition on a set A. Now given a partition determined by an equivalence relation, can we find a function that would correspond to that same partition?

Student D: First I need to have a picture of the set A as a whole so that I can see those elements in A related to each other. Then I collect them in groups. Then I need to make sure that f sends elements in one group to one single image.

This is a clear reference to a many-to-one grouping mode.

Student D: OK. So I must say that this must eventually happen, that *f* should send a whole group to the same image. So, remind me what am I given as a start?
Interviewer: You are given a partition say $\{A_i\}$ determined by an equivalence relation *R*.
Student D: OK. This means that each of these subsets in the partition must be some class $[m]$. But let me see, how to insert a function in here?
Interviewer: Why don't you start from the end? If a function *f* were to determine that partition, what would those sets $\{A_i\}$ be with respect to *f*?
Student D: I guess A_i are the preimages[3] of *f*. . . I am sure.
Interviewer: OK, so now can you say what *f* exactly is?
Student D: Since the preimages of the function *f* are obviously those sets $\{A_i\}$, I could crush the whole set A_i to one element, so to speak.

The term "crush" above is used in reference to the "encapsulation" into an object of those elements related via *R*, namely the encapsulation into an equivalence class. However, it was not clear to that student whether *f* acts on the elements that make up those subsets in the partition, or whether it acts on the partitions as elements in its domain. She still needed to "loosen" the elements inside every subset. I can say that student D knew how to go from a relation to the corresponding function, that is, she had the mental constructions described in step 3 (and 4) of the genetic decomposition, yet, she needed the stronger mental construction in step 5 to answer that question. In conclusion, my interpretation is that types 1 through 9 activities handle at most two out of three components at a time; (the three components being functions, equivalence relations and partitions). It is not till activity 10 that the three components are explicitly coordinated. The needed mental construction for such a complete coordination is described in step 5 of the genetic decomposition.

7.4. *A technique for isolating the common properties of elements in one equivalence class*

The mental construction described in the alternative 3b of the genetic decomposition was hypothesized right after the in-class discovery made by student E quoted below. When this class revelation occurred, students were very familiar with procedures M1 and M2, and had extensively practiced with type 1 through 9 activities. No coordination between the two methods had yet been explicitly stated. Up until this class discovery led by Student E, the mental constructions applied to find *f* given *R* were the ones described in point 3a of the genetic decomposition.

This discussion came right after we had covered in class a number of examples of equivalence relations, and in each case we were collectively coming up with the corresponding function (that would determine the same partition).

Student E started the discussion by making a very wise observation:

Student E: We can imagine that we are coloring the elements that are related to each other in one color.

Interviewer: What guarantees that no elements can have more than one color?

Another student: R is an equivalence relation.

Student E: Yes, yes. The function we are looking for must then send those like color elements to the same image. That way, elements will be in a subset of that partition in case they have the same color. We are interested in deciding on the color categories, that's all. We need to know what it is that places an element in a certain class.

Another student: What do you mean by we need to know what places an element in a class? It is R, since elements that are related via R are automatically in one class.

Student E: Yes, yes, of course. But there is a certain reason, a formula, a condition that makes two elements related; you see what I mean? Otherwise the relation will have no meaning. Just like a random list of elements connected to each other.

Interviewer: And we are now treating, I assume, relations that are given by a formula and where the set A is infinite. Great strategy. You mean to say related elements must have a common property?

Student E: Yes, yes, of course. This is it. And this is what gives them the same color. In fact it is as if the color is the property.

Another student: So when we are given the relation in the beginning, it is as if we are given a certain number of colors and using them to color the elements.

Student E: Yes, and then the function will be simply the map that assigns to each element that very color!

Interviewer: Great strategy. Now how to find those properties? Consider the relation on the set of lines in the plane or space: $l_1 R l_2$ if and only if l_1 is parallel to l_2.

When there was no response, I had to give some help.

Interviewer: Try to find what those lines that are related to one another have in common.

Student E: The direction, of course, or slope. So the common property will be common slope.

Interviewer: And in general, any idea of how to separate those common properties?

Student E: I guess what should be done is have the relation specify that m and n must both do something, *but separately*.

Student E: We need to rewrite whatever equation or expression of R so that m and n are each on one side of the expression.

Interviewer: Well, let us take an example of a relation that is not an equivalence relation. Consider, say: mRn if and only if $|m - n| = 4$.

Student E: Well, suppose I rewrite it as $m = n \pm 4$. But I cannot get similar expressions on the two sides, no matter what. I guess that implies that R cannot be an equivalence relation.

The class decided to call this method the break up technique, and they decided that it works only in the case of an equivalence relation.

7.5. *Formulating EQF*

The following discussion occurred during the next semester. At that time coordination between procedures M1 and M2 had already been solicited but theorem EQF had just been stated without application. All except type 10 activities had been practiced in class. Student F came to my office at the end of the lecture where EQF was stated, asking for some clarification.

Student F: When we say that we must have $f(m) = f(n)$ if and only if mRn, I first thought you meant that both the relation and the function must be given to us?

Interviewer: And now you know that you are only given the equivalence relation?

Student F: Yes, but, this f, I am not going to use it exactly. I will not have to evaluate the value of f at any point. I should just find it and that is it? It looks kind of tentative [undefined] to me. Do I need f for any other purpose?

Interviewer: Not exactly. You are right. You need to find it for reassurance? Is that what you meant to say?

Student F: Yes, and I don't know if I need it to check if R is an equivalence relation, since this is what I need, right?

Interviewer: Yes.

Student F: I only need to check if R separates the elements of A into subsets that are not overlapping and if they fill A.

Interviewer: Do you know another name for those subsets?

Student F: A partition, yes. At the same time, they are the equivalence classes, of course. So, if R creates such equivalence classes, then it must be an equivalence relation. It is a feeling only. I know that I cannot present the proof this way; it is not formal yet, if you know what I mean.

Interviewer: Yes, but you have not used *f* yet.

Student F: Exactly. Those elements that are in the same subset (of the *R*-created partition), *f* must send them to the same image. But in order to find *f*, I start from the end exactly as in a backward proof: I need to have a feel for what makes the partition the way it is, and what the relation is exactly doing, and why the elements of *A* are grouped in a certain way in those subsets (which will later be equivalence classes). When I figure this out, I can form *f*, as if I am guessing and checking and all that.

The mental construction witnessed in this excerpt is the implicit interiorization into a process of the action of checking for the elements that are related to each other. This corresponds to the cognitive tools described in step 1 of the genetic decomposition. Also, the mental construction in step 3 is demonstrated since the student was able to describe although he did not exactly execute the mentioned steps. The activities that promoted such a construction are of type 4 and 9.

7.6. *A comparison between DEQ and EQF*

The following is a very interesting comment as far as the different roles that an equivalence relation and a function play.

Interviewer: To show that a relation *R* is an equivalence relation, when would it be easier for you, if at all, to come up with a function *f* that maps elements related via *R* to a common image, rather than to verify that *R* is reflexive, symmetric and transitive?

Student I: Rarely. But I can say that theorem EQF is used when we are able to visualize *R* through its classes. In that case, constructing a function that maps elements in one class to a common image is easier or more straightforward than showing that *R* is reflexive, symmetric and transitive. In a way *f* and *R* do the same thing. The relation would say elements are related to each other, whereas the function would say they are mapped to the same place.

What paved the way to such an insightful observation is the practice of activities of type 10 that promote the sought coordination between the different set structures. We can see that in order for a student to make such an observation he must have had a deep understanding of the material that allowed him to express such a clear comparison between cases that involve a visualization of *R* and *f*.

8. CONCLUSION

In an attempt to evaluate the genetic decomposition, I demonstrated to some extent, how in successful cases of learning, the students made the mental constructions listed in the genetic decomposition. At the same time, in an attempt to evaluate the instructional procedures, I pointed out to the activities that contributed to this progress.

Yet, I would still consider the genetic decomposition presented above only as a preliminary decomposition, subject to adjustment and modification depending on further interviews with students. Ideally, one should repeat and revise both the genetic decomposition and the instructional procedures until a balance is reached.

Overall, I can say that the activities and instructional procedures that I designed and applied helped in building the mental tools listed the genetic decomposition at the different stages of learning.

In the future I intend to pursue this interest in observing how students perceive equivalent definitions in general, using the same theoretical framework.

NOTES

1. The fiber structure of a function on that set is the set of preimages of all sets $\{b\}$ for b in the range of the function.
2. Understood, by students, as a collection of equivalence classes; students don't have any formal notion of quotient structure. The language in which this theoretical analysis is conducted is not the language of students' understanding.
3. The term preimages was used in class to mean fiber.

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