ADALIRA SÁENZ-LUDLOW

CLASSROOM INTERPRETING GAMES WITH AN ILLUSTRATION

ABSTRACT. Classroom communication has been recognized as a process in which ideas become objects of reflection, discussion, and amendments affording the construction of private mathematical meanings that in the process become public and exposed to justification and validation. This paper describes an explanatory model named “interpreting games”, based on the semiotics of Charles Sanders Peirce, that accounts for the interdependence between thought and communication and the interpretation of signs in which teacher and students engage in mathematics classrooms. Interpreting games account both for the process of transformation (in the mind of the learner) of written marks into mathematical signs that stand for mathematical concepts and for the continuous and converging private construction of mathematical concepts. Teacher–student and student–student collaborative interactions establish a mathematical communication that shapes and is also shaped by the conceptual domains and the domains of intentions and interpretations of the participants. A teaching episode with third graders is analyzed as an example of a classroom interpreting game.

KEY WORDS: acts of communication, acts of interpretation, communication, equal sign, interpreting games, interpreting process, Peirce, semiosis

1. Introduction

The immaterial nature of mathematical objects requires that they be expressed through a variety of physical representations (i.e., mathematical signs) in order to be socially shared. Since antiquity, symbolization of mathematical entities has been both a personal and a social experience influenced by cultural factors, conditioned by contemporary requirements of society, and directed by the spur of non-mathematical events (Cajori, 1928–1974; Wilder, 1968; Menninger, 1969). Therefore, the learning of mathematics entails both the interpretation of mathematical signs and the construction of mathematical meanings through communication with others. These interpretations and these meanings are not constructed on the spot. Rather, they evolve in a continuous manner, a manner resulting from the individual’s exposure to a variety of closely interrelated experiences within different mathematical, social, and physical contexts. In such experiences, multiple semiotic systems combine (e.g., language, mathematical signs, and gestures) to ground a continuous and evolving interpretation of mathematical meanings. Since communication is possible only through signs (Peirce, CP 4.7), acts of communication are in themselves acts of interpretation.

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Classroom interactions between teacher and students and among students themselves are acts of communication at two levels: acts of communication with *oneself* and acts of communication with *others*. I argue here that *interpreting games* take into account the interdependence between thought and communication while explaining meaning-making processes from a semiotic perspective – the Peircean perspective. I will illustrate this notion of interpreting games by analyzing a 50-min classroom teaching episode with third graders as an interpreting game.

2. Communication and Language

The importance of communication in the classroom through linguistic and nonlinguistic signs has been emphasized in research (e.g., Barnes, 1992/1976; Edwards and Mercer, 1987; Ellerton and Clements, 1991; Steinbring et al., 1998; Sfard, 2001) as well as in curriculum documents. The Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000) states that "communication can support students' learning of new mathematical concepts as they act out a situation, draw, use objects, give verbal accounts and explanations, use diagrams, write and use mathematical symbols" (p. 61). The document also recognizes communication as a process in which ideas become objects of reflection, discussion, and amendments affording the construction of private mathematical meanings that in the process become public and exposed to justification and validation. In order for this to happen, the teacher needs gradually to build up the classroom as a community in which inquiry is advocated and students feel free to voice their thinking and interpretations.

The importance of language in communication is unchallenged. The sociocultural and linguistic aspects of communication have been profusely studied. See the works of Saussure (1959), Wittgenstein (1991), Foucault (1972), Habermas (1984), Vygotsky (1934/1986), Bakhtin (1986), and Bourdieu (1981/1999), among others. Classroom mathematical communication brings about more than just linguistic aspects. It entails, among other things, mathematical notation, gestures, body language, and dispositions which influence how classroom participants act and react. Natural language as a means of mediating communication is not "*neutral*" (Kanes, 1998) but neither is the system of mathematical notations. Kanes observes that natural language "at worst distorts the formation of ideas" and "at best it makes the construction of concepts and communication possible" (p. 135). By the same token, mathematical notations at worst distort mathematical meanings if they are not interpreted in context and unfamiliar distinctions are not taken into account; and at best they enable us to state explicitly what we want to state circumventing the ambiguities which the structure of natural language is not equipped to avoid (Nagel, 1956).

3. Communication as a semiotic activity

Mathematical communication entails natural language, mathematical notation, gestures and body language among other semiotic systems. Thus, classroom communication exhibits a semiotic nature which has been acknowledged, directly or indirectly, by mathematicians, mathematics educators, and psychologists (e.g., Wilder, 1968; Piaget, 1970; Vygotsky, 1978; Skemp, 1987; Pimm, 1987; Walkerdine, 1988; Vile and Lerman, 1996; Vile, 1997; Whitson, 1997; Sáenz-Ludlow, 1998; Otte, 1998, this issue; Steinbring, this issue; Duval, 1999, this issue; Cobb, 2000; Dörfler, 2000; Van Oers, 2001; Ernest, 2003; Radford, 2003; Hoffmann, in press a, in press b).

Any semiotic activity of the individual assumes the existence of social interaction (Peirce, 1906; Whitehead, 1927/1985; Vygotsky, 1934/1986; Dewey, 1938/1963), but the debate continues as to whether the role of social interaction in thought processes is primary or secondary. For Piaget, the cognitive activity of the individual is primary in the construction of knowledge, while social interaction remains in an irreplaceable but secondary place (Piaget, 1970, 1973). In contrast, for Vygotsky (1934/1986), social interaction, mediated by symbolic tools, plays a fundamental first role in the cognitive activity of the individual. For symbolic interactionists and socioconstructivists, knowledge is a social product constructed as people interact (Blumer, 1995; Bauersfeld, 1995; Cobb and Yackel, 1995).

From a semiotic perspective neither the cognitive activity of the individual nor his social interaction is primary; both co-exist and co-act in a synergistic manner to support the evolving process of sign interpretation and meaning-making. Thought and communication (taken in its broadest sense as social interaction) both appear to be parallel and interrelated at the same time. They co-exist like the two sides of the same coin and although one can talk of each side of the coin as an entity in its own right, they cannot be separated from each other (cf. Sfard, 2001). I interpret communication in its broadest sense and use it interchangeably with social interaction. On the one hand, both comprise not only communication of a message but also, an utterer or intender, an interpreter, and a plurality of semiotic systems. On the other, both comprise a conceptual and a social context, doing (conceptually or physically), observing, speaking, listening, reading, writing, interpreting, and thinking.

Peircean semiotics provides unique elements to understand the interdependence between thought and communication when signs are interpreted and transformed into new signs. The development of mathematical meanings in the classroom emerges in sign interpretation (i.e., semiosis) when teachers and students cross the threshold of their own individuality in social interaction. I propose that sign interpretation and sign use are in essence continuous and evolving *interpreting games* in which teacher and students constitute themselves as intentional subjects capable of interpreting linguistic, mathematical, and other kinds of signs. This process of interpretation gives rise to private meanings subject to modification and refinement through a succession of collaborative acts of interpretation and communication.

Dewey notes the intertwining relationship between individuals who engage in communication and the positive effect of communication in the process of education.

> Not only is social life identical with communication, but all communication (and hence all genuine social life) is educative. To be a recipient of a communication is to have an enlarged and changed experience. One *shares* in what another has thought and felt and in so far, meagerly or amply, has his own attitude modified. Nor is the one who communicates left unaffected (Dewey, 1916/1997, p. 5; emphasis added).

Exactly how processes of communication constitute processes of meaning-making is synthesized by Liszka from the works of Peirce.

> As a process, meaning is *communication* (cf. LW 196 f.), either among sign-interpreting agents (such as human communication) or among thoughts within the same agent (intra-agency communication) . . . . The *product* of communication is *information*; the *effect* of communication is *understanding*, in the sense of a shared common understanding, in the case of inter-human communication (cf. LW 197). (Liszka, 1996, p. 81; emphasis in the original)

Thus, for Peirce, thought, sign, communication, and meaning-making are inherently correlated. The result of individual and collective processes of sign use and sign interpretation (i.e., semiotic activity or semiosis) rooted in social interaction is the construction of private meanings. These private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established *in* the community and *by* the community. The constitution of these private and subjective meanings and their convergence towards more objective meanings is what *interpreting games* endeavors to address using Peirce's triadic sign relation.

Individuals interpret signs according to their own conceptual webs and this interpretation carries with it epistemological consequences that are latent or manifest. Otte (1998) argues that "A sign is a general, a *type*, whose mode of being consists in that it has a great variety of *tokens* or *replicas*, which may influence or determine the interpreter in very different ways, such as to lead to different developments" (p. 447). The continuous construction of mathematical meanings through the interpretation of mathematical signs in acts of communication with oneself and with others appears to be the essence of teaching and learning. This continuity in meaning-making leads toward abstraction and generalization which Peirce (1956) considers the most important characteristics of mathematical thought. The question is how individuals construct these meanings in a continuous manner through a process of interpretation. Peirce's semiotics provides an answer in his *principle of continuity* and his triadic sign relation.

In the following section I shall elaborate on Peirce's notion of sign, meaning, and the principle of continuity insofar as these notions are needed to formulate the *interpreting games.* It is well known that Peirce's conceptualization of sign evolved during the years into a comprehensive doctrine. He has a spectrum of definitions of signs that are complementary and clarify each other. These definitions exemplify a profound sense of the difficulty of expressing in a single and comprehensive definition the nature of signs.

4. Peirce's triadic sign relation

Peirce locates sign activity both within the self and its communities which, for him, are located within an evolving universe that is profuse with signs. He defines his triadic sign relation as constituted by *object*, *sign*, and *interpretant*.

> I define a *Sign* as anything which on the one hand is so determined by an *Object* and on the other hand so determines an idea in a person's mind, this later determination, which I term the *Interpretant* of the Sign, is thereby mediately determined by that Object. A Sign, therefore, has a triadic relation to its Object and to its Interpretant. (Peirce, 1906, p. 276; emphasis in the original)

In addition, Peirce considers that a sign is not a sign unless it is interpreted or translated into another sign that is more fully developed (CP 5.594). Signs, in his triadic sign relation, have the epistemological function to represent *objects* for an *interpretant* and to mediate between *object* and *interpretant* to make objects accessible to the mind. That is, the sign is something irreducible between its object and its interpretant. In this translation signs are interpreted into new signs and concomitantly more sophisticated interpretants and objects emerge in the mind of the interpreter (see Figure 1). He also considers that meaning emerges in such a translation "the meaning of a sign is the sign it has to be translated into" (CP 4.132) and "meaning ... [is] in its primary acceptation the translation of a sign into another system

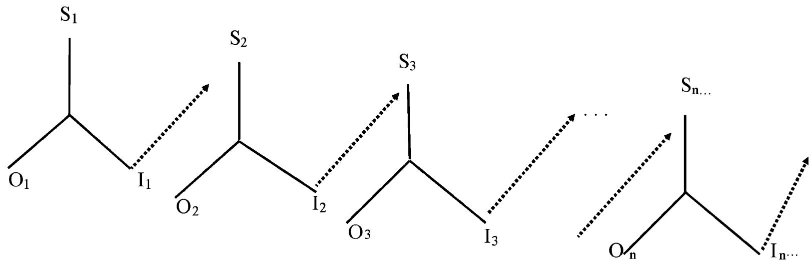

Figure 1. Meaning emerging in the translation of signs into new signs.

of signs" (CP 4.132). Thus meaning emerges as signs are translated into new signs.

Since, for Peirce, the triadic sign relation is *genuine* and "its three members are bound together by it in a way that does not consist in any complexus of dyadic relations" (CP 2.274), then the interpretant (in its broadest sense) can be understood as the *translation* of a sign into a new sign (see Figure 1). This translation of signs is at once a *product* which is the result of some *process* (i.e., the process of semiosis itself) and which has some *effect* on the translator or sign agent (interpreter or intender). This notion of *translation* reconciles Peirce's various definitions of interpretant. He defines interpretant as:

(i) another sign which *results* from other signs (CP 5.483; 5.484) or as the *product* of semiosis (CP 4.536); or
(ii) the *process* of semiosis itself or a *rule* of sign translation (CP 5.483; 5.484); or
(iii) the essential *effect* upon the interpreter brought about by the semiosis of the sign (CP 5.473; 5.484; 8.191).

The interpretant is the sign's relationship to its semiotic object with respect to its interpreter or sign agent. The *sign agent* is the person (i.e., utterer/intender or interpreter) that takes over the *interpretant* and modifies it in such a way as to create yet new interpretants. There is no end to the number of actual or possible interpretants created by the interpreter, and the interpretant is always in a state of becoming, in a process of modification (i.e., unlimited semiosis).

From the continuous translation of signs into new signs a concomitant sequence of interpretants evolves giving rise to the emergence of meaning. This process of translation is in itself a process of interpretation through which signs become translated into less contextualized and therefore more generalized signs in the minds of the sign agents (interpreter or intender). For Peirce "the interpretant of the sign is all that is explicit in the sign itself apart from its context and circumstances or utterance" (1906, p. 276). He considers various kinds of interpretants, each with different functions. Moreover, he recognizes that interpretants can not only be of different degrees of complexity, but they can also be of different kind. Here we consider the intentional, effectual, communicational, and logical interpretants. See Hoffmann (this issue) and Buchler (1955) for other intepretants and their interrelationships.

> There is the *Intentional* Interpretant, which is a determination of the mind of the utterer; the *Effectual* Interpretant, which is a determination of the mind of the interpreter; and the *Communicational* Interpretant, or say the *Cominterpretant*, which is a determination of that mind into which the minds of utterer and interpreter have to be fused in order that any communication should take place. This mind may be called the *commens*. [The *commens*] is all that is, and must be, well understood between utterer and interpreter, at the outset, in order [for] the sign in question [to] fulfill its function (Peirce, 1908, p. 478; emphasis in the original).

The *intentional* interpretant prompts the utterer (i.e., intender) to select a sign for the purpose of sending a message. While the utterer (i.e., intender) produces an intentional interpretant, the interpreter interprets and produces, according to Peirce, an *effectual* interpretant. The effectual interpretant produces an action upon the inner world of the interpreter (a mental effort) that may or may not produce a mental or a physical action. Finally, the *communicational* interpretant is the thought co-produced and *shared in communication* by the utterer/intender and interpreter of the sign or what Peirce calls *cominterpretant* of the sign agents. In other words, the cominterpretant can be understood as the process of, the product of, and the effect on the individual minds of the sign agents (Liszka, 1996, p. 81). The sign agents also construct logical interpretants; these interpretants are called logical because they forward conjectures and plans of action that make sense to the sign agents within their own conceptual webs. These logical interpretants are further subjected to modifications and corrections in subsequent communications.

> In the next step of thought, those first *Logical* Interpretants stimulate us to various voluntary performances in the inner world and we proceed to trace out the alternative lines of conduct which the conjectures would have opened to us. Moreover, we are led by the same inward activity, to remark different ways in which our conjectures could be slightly modified. The logical interpretants must, therefore, be in a relatively future tense (Peirce, 1906, p. 280).

Peirce contends that although the existence of logical interpretants depends on the existence of intentional and effectual interpretants, logical interpretants are more dynamic and bring forward creativity in the inner world. The past, present, and future time element in the network of interpretants speaks directly about the successive refinements of existing interpretants, about the potential construction of new ones, and about the continuous nature of meaning-making through the sign interpreting process. From the Peircean semiotic perspective, the meaning of a mathematical concept is accounted for by the network of interpretants. "The problem of what the 'meaning' of an intellectual concept is can only be solved by the study of interpretants or proper significate effect of the signs" (CP 5.475).

These networks of interpretants constitute the backbone of the interpreting and knowing activity of the classroom participants playing the role of utterer/intender or interpreter in the process of interpreting mathematical signs. In this activity, both teacher and students understand each other as intentional subjects as well as interpreting subjects whose concomitant individual activities are essential in their meaning-making processes. Such activity is primarily concerned not only with the translation of signs into more developed signs but also with the significant effects that such signs have on the sign interpreting agents (teacher and students). That is, constructing interpretants means structuring mathematical meanings through subjective acts of interpretation that are, *by no means*, simplistic transferals of information from the teacher to the student.

Mathematical concepts are one of the main components of the universe of classroom discourse. These concepts, over time, have been validated by different communities of mathematicians across different cultures and therefore they have reached the status of *commens*. The final goal of sign interpretation in the classroom is to approximate the meanings of these mathematical concepts as much as possible by means of successive networks of interpretants individually generated with the collaboration of the classroom participants. In this process, teacher and students interchange their roles of intenders and interpreters. This interchangeability implies a corrective effect geared towards the determination of consensus which is nothing more than the constitution of *cominterpretants*. The approximation of private mathematical meanings toward the conventional mathematical meanings is a process that carries on in the mathematical careers of students of mathematics.

5. Continuity in the Process of Interpretation

Peirce broadly conceptualizes his *principle of continuity* (*synechism*) to refute the distinction between the physical and psychical phenomena (the Cartesian perspective). He considers that both phenomena "are of one character though some are more mental and spontaneous, others more material and regular" (CP 7.570). But for Peirce continuity goes beyond the continuity between the physical and the psychical or the continuity of chronological sequences of events. It also means the continuity between the individual and the social, the continuity of the translations of signs into more developed signs, and the continuity of the interdependence between thought and communication. His principle entails that "ideas tend to spread continuously and to affect certain others that stand to them in a peculiar relation of affectability" (CP 6.104). This continuous spreading of ideas has different effects. On the positive side, "ideas gain generality and become welded with other ideas". On the less positive side, "[ideas] lose intensity and especially the power of affecting others" (CP 6.104).

Corrington summarizes Peirce's application of the principle of continuity to communication in the following way.

> Peirce applies the principle [of continuity] to the correlation between the individual and the social order, by insisting that there is no break between the selves. Each self fully participates in the larger social self and can only separate itself from the pregiven continuum through ignorance. There is thus an unbroken connection between matter and mind, and personal and social selves (Corrington, 1993, p. 102).

This continuity between the personal and the social selves was also conceptualized by Vygotsky (1978). "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*)" (p. 57; emphasis in the original).

Otte also captures Peirce's principle of continuity in the personal development of mathematical knowledge.

> The paradox of mathematical knowledge that mathematics cannot be conceived of as completely separated from empirical experience and yet cannot be explained by empiricist epistemology... can only be resolved if one accepts that the causal interactions between knower and environment have themselves a generalizing tendency, a sort of continuity, rather than consisting just on singular events (Otte, 1998, p. 425).

Dewey also notes the latent effect of the continuity of experience (conceptual or physical) and its importance in education. "Every experience lives in future experiences. Hence the central focus of... education... is to select the kind of present experiences that live fruitfully and creatively in subsequent experiences" (1938, pp. 27–28).

Peirce's principle of continuity has implications for the teaching-learning process since mathematical concepts appear to be constructed not through one isolated interpreting act but through sequences of interpreting acts carried out in communication (social interaction). These co-constructed mathematical meanings tend to be generalized in the continuous evolution of interpretants as they translate into new signs. In the realm of this continuum, all interpretants are formed in a corrective and modifying process and then expressed in new signs (e.g., verbal and written arguments and expressions, diagrams, pictures, gestures, etc.). Thus, continuity renders the generalizing aspect of meaning-making processes as constituted in the intertwinement of the social and the personal which allows the participants to consider the meanings elaborated in collaboration as their *own* and therefore as *shared in communication*. In this continuous interpreting process, participants construct shared meanings (*cominterpretants*) that converge asymptotically towards mathematical concepts (*commens*) established *by* the mathematical community and *in* the mathematical community.

6. MATHEMATICS, THE LEARNING OF MATHEMATICS, AND COMMUNICATION

The nature of school mathematics is necessarily connected to what mathematics is. However, the nature of mathematics is not only just an educational problem; it is first and foremost a philosophical problem that philosophers and mathematicians have grappled with for a long time from different philosophical perspectives.

In general, it appears to be a consensus that mathematics is a sophisticated activity rooted in reasoning and communication. For example, Peirce (1956) considers that mathematics has some striking characteristics. Among them are the fleshless and skeletal make up of its propositions, the generality of its results, the extraordinary use of abstractions, and the stress on reasoning. Rotman (1988) argues that mathematics is an activity and a practice with formal and informal, written and verbal ways of communication. “If one observes its participants [mathematicians] it would be perverse not to infer that for large stretches of time they are engaged in the process of *communicating* with themselves and each other” (p. 6, emphasis added). Sierpinska (1994) characterizes mathematical activity “as a dialectic game between freedom and restrictions, invention and discovery; between the liberty of initial choices and the confinement within the laws of a deliberately chosen system; between the free creation of objects and the struggle of understanding their properties and significance” (p. 30).

From a didactical point of view, several notions have been advanced about the learning of mathematics. Most of the perspectives focus on the nature of the mathematical activity of the students. For example, Skemp (1987) considers that learners of mathematics should be involved not only in instrumental understanding (manipulation of marks on paper devoid of meaning) but also on relational understanding to apprehend the meanings of mathematical concepts. Van Oers (2001) considers that learning mathematics is about structuring the structures through problem solving with symbolic tools as well as participating in preexistent cultural practices. Sfard (2000) and Dörfler (2000) consider that learners of mathematics should be inducted into mathematical discourse and locate students' activity as acts of participating in communal practices. In contrast, Bauersfeld (1995) tilts our attention towards the actual interpreting activity of the students. He notes that students are usually left alone with their own processes of interpretation and their private meanings are rarely expressed publicly in order to be modified or validated. When students do not share their own processes of interpretations with others, sooner or later they become public in their written work and some of them are labeled as *misconceptions* or *errors*. Such misconceptions appear to be truncated processes of interpretation of what should be a continuous progression gearing toward generalization and abstraction as essential elements of mathematical thinking. The interpretation of mathematical concepts should be a continuous process mediated by the collaborative communication between teacher and students and students themselves.

As Rotman (1988) points out, *communication* is essential to the creative activity of mathematicians and therefore *communication* should be essential to the mathematical activity of teacher and students in the classroom. The teaching-learning of mathematics should be about interpreting mathematical concepts through problem solving and sign use; interpretation that is mediated by intentional and reciprocal communication between the classroom participants. Thus *communication* should be taken as a *teaching-learning tool* to interpret well-established mathematical concepts through continuous, collaborative, and synergistic processes of interpretation that are both intra-subjective and inter-subjective. In these processes, personal interpretations come to be modified, validated, and generalized; and alongside mathematical meanings emerge in the minds of the interpreters.

This intentional and reciprocal classroom communication between teachers and students that allows for *awareness* and *active engagement* in students' processes of interpretation is what I call *interpreting games*. Interpreting games are a transformative semiotic activity that entails both the interpretation and use of conventional mathematical signs and the generation of idiosyncratic signs to record and communicate developing mathematical meanings. Mediated by interpreting games written marks are transformed into mathematical symbols (in Peirce's sense) endowed with mathematical meanings that approximate the meanings of already established mathematical concepts. This transformative process of interpretation and meaning-making is integral to mathematical activity and it constitutes an essential aspect not only of the socio-historical development of mathematical concepts but also of the individual's evolving interpretation of these concepts.

While mathematical concepts to be learned are static, interpreting processes are dynamic. While there are formal ways of speaking about mathematical concepts, there are also idiosyncratic and informal ways of speaking in interpreting processes. While the meanings of mathematical concepts are already abstracted, validated, and generalized, the mathematical meanings in personal interpreting processes are in a state of becoming abstracted, validated, and generalized. While concepts are considered to be well-formed mathematical objects; conceptions in the interpreting processes are in an interim state of metamorphosis. All in all, in interpreting processes private mathematical meanings should approximate established meanings of mathematical concepts; and such approximations should be fostered and sustained by collaborative classroom communication or *interpreting games*.

7. Interpreting Games

At center stage in the classroom communication is not only *what* concepts but also *whose* meanings and *whose* interpretations. When the teacher concentrates only on *what* she is teaching and overlooks (or leaves almost forgotten) *who* is interpreting and *who* is making meaning, the dialogue between teacher and students becomes asymmetric. But when the teacher takes into account students' interpretations and continuously tries to interpret how students' meaning-making processes are taking place, the dialogue becomes symmetric.

Although teaching entails a *directive* role (Dewey, 1916/1997; Freire, 1970/2001), what should be at the center of the teaching practice is the nature and characteristics of this directive role. On the one hand, the teacher could transform the students' thinking into the mere shadows of her thinking; on the other, the teacher could use this directive role to facilitate the constant growth of the interpreting capacity of the students.

Ideally, a teacher who considers *communication* as a *tool for teaching and learning* would be bound to have an asymmetric-symmetric relationship with the students. The relation should be asymmetric in the pre-teaching activity as the teacher uses her knowledge to structure the lessons to be taught and the teaching strategies to be used in order to engage students in dialogue to facilitate their thinking strategies, their interpretations, and their ways of communicating. However, when the teacher invites dialogue in the classroom, questions the students, and asks for explanations and justifications, she consciously transforms the asymmetric authoritative teacher-student relationship into a symmetric one in which both teacher and students collaborate in the teaching – learning experience. In other words, teachers teach-learn and students learn – teach complementing each other's roles. Making an effort to maintain an asymmetric-symmetric relationship with the students constitutes in essence a solution to the teacher – student contradiction posed by Freire (1970/2001). He argues that projecting absolute ignorance onto the students negates their education and knowledge as processes of inquiry. Inviting dialogue is to acknowledge students as *acting subjects* rather than *objects* of the teacher's actions.

In a symmetric teacher – student classroom relationship, the teacher allows students to participate in dialogue and to assume an active role in their own learning as they interpret, explain, justify, and modify their original interpretations. This is to say that the teacher as a more capable individual fosters communicative and interpretive actions that results in the students expressing their own interpretants (i.e., transforming their interpretants into new signs to be interpreted by others). This symmetric relationship between teacher and students allows the teacher to probe her interpretants of students' interpretations (i.e., expressions of students' interpretants), students' conceptualizations, and students' processes of meaning-making.

The conceptualization of objective mathematical meanings embedded and carried out by conventional mathematical signs does not come from receiving those meanings and storing them. On the contrary, they emerge from the learners' efforts to re-create and to re-invent such meanings through their own acts of interpretation. From a semiotic point of view, signs (among them mathematical notation and natural language) play an essential role in the learning of mathematics because mathematical knowledge depends, among other things, on (a) the ability to represent in order to communicate (Otte, 1998, this issue; Sfard, 2001; Ernest, this issue), (b) the ability to deal simultaneously with several semiotic systems (Duval, 1999; Ernest, this issue), (c) the ability to recognize a mathematical object (i.e., concept) embodied in different representations without conflating the object with any of its representations (Otte, 1998), (d) the ability to transform representations of mathematical objects within and between representational systems (Duval, this issue), and (e) the ability to construct and interpret meanings mediated by signs (Dörfler, 2000; Radford, this issue).

The interpretation of mathematical concepts goes beyond the interpretation of a particular sign in a particular mathematical context. As Otte (this issue) points out, a mathematical sign is a *type* (i.e., a general, a class) and therefore it could be materialized in a diversity of *tokens*. To really understand the meaning(s) of a mathematical sign, it needs to be interpreted in a variety of meaning-giving contexts that should be provided by the teacher. In this sense, mathematical signs, like words, could carry different meanings according to the context in which they are used (Sáenz-Ludlow, 2003a). Sign interpretation is greatly facilitated by the communication between teacher and students and students themselves. The Peircean semiotics perspective is most useful to explain *communication as a tool for teaching and learning* because it implies: (1) the presence of utterers/intenders and interpreters (teacher and students will indiscriminately take on one role or the other), (2) the something to be interpreted by utterer/intender and interpreter (i.e., mathematical concepts), and (3) the continuity in the process of sign interpretation and the continuity of meaning-making processes.

Social interactions between teacher and students and among students themselves are *acts of communication* constituted at two levels: acts of communication with *oneself* and acts of communication with *others*. As was argued before, these two levels can only be separated for the purpose of conceptualization because they synergistically co-exist. Furthermore, the acts of communication with *oneself* can only be inferred from the acts of communication of *oneself* with *others*. Since communication is possible only through signs (Peirce, CP 4.7), communication is in essence a continuous process of sign interpretation (unlimited semiosis) in which interpretants are constructed and translated into new signs (see Figure 1). Thus, every *act of interpretation* entails the generation of an interpretant and its translation into a new sign to express it (e.g., mathematical argument, diagram, verbal expression, gesture, or any other visible sign).

It is this co-constructed interpreting process between the classroom participants that I call an *interpreting game*. Interpreting games are in essence tinkering processes with elements of randomness, playfulness, creativity, surprise and unpredictability (Sáenz-Ludlow, 2003a). Let us consider the communication between teacher and students or among students themselves from the perspective of interpreting games. When participants interact and self-reflect, old interpretations are modified and refined and new interpretations emerge. Some interpretations generate *logical interpretants* that translate into arguments that are shared with others. Other interpretations generate *effectual interpretants* that may or may not remain suspended waiting for future refinements and may or may not generate logical intepretants. Still other interpretations generate *intentional interpretants* in order to sustain the communication. *Cominterpretants* are generated when the subjective interpretants of the participants are transformed into new signs that are interpreted by others and agreed upon in communication (see Figure 2).

Interpreting games account for the co-constructed cognitive activity that takes place in the classroom when the participants intentionally engage in interaction through natural language, mathematical signs, and other kinds of signs. At the beginning, mathematical signs may have the status of symbols (in Peirces's sense) for the teacher, but for the students they stand as simple written marks devoid of meaning. Mediated by interpreting games (IG), interpretants are generated to ascribe mathematical meanings to these

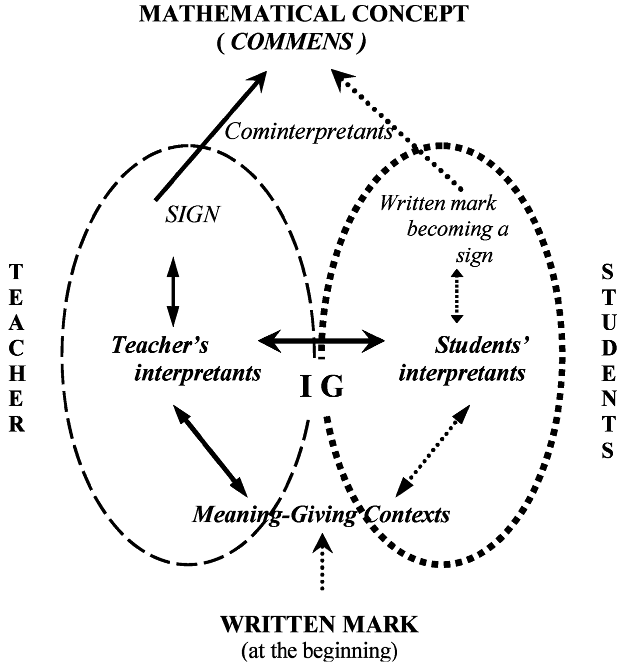

Figure 2. Interpreting games (IG) mediating the transformation of written marks into mathematical signs in the minds of the students.

written marks in the minds of the students (see Figure 2). That is, mediated by interpreting games, written marks become endowed with interim and private meanings subjected to modification and refinement. As the interpreting game continues, interim meanings become less and less subjective, cominterpretants are generated, and they eventually approximate to the objective conventional meanings of mathematical concepts (commens).

The dynamics of the dialogical interaction between teacher and students shapes and is also shaped by the interpretants of the participants that are translated into (valid or invalid) mathematical arguments. These arguments support and sustain the endowment of written marks with generalized and less subjective (i.e., more objective) meanings that approximate (i.e., asymptotically converge) to the objective, standardized, and conventional meanings of mathematical concepts. As a result of these arguments, written marks take on increasingly rich and complex meanings, meanings that evolve directed by the rules of syntax, grammar, and semantics of mathematical notations and language as already constituted semiotic systems.

When written marks are endowed with mathematical meanings, they are elevated to the status of mathematical symbols. The metamorphosis of written marks into symbols, in the mind of the learner, is mediated by interpreting games through natural language (formal and informal), mathematical notation (conventional and idiosyncratic), gestures, and other signs. Interpreting games are considered to be constituted by *interpreting cycles* as their elementary unit of analysis. In these cycles, *interpreted meanings* anchor and give form to *intended meanings* in a synergistic manner. Each cycle mediates the constitution of some kind of meaning that primes the stage for a subsequent cycle in which new meanings emerge and are modified and further refined.

Figure 3 represents the structure of interpreting cycles in the meaning-making process. In this figure, the beginning of a cycle initiated by the teacher is represented by the digits placed next to the arrows; that is, this cycle follows the arrows in the order 1, 2, 3, 4. The beginning of a cycle initiated by the student follows the arrows indicated by the letters a, b, c, d placed next to the digits that indicate the cycle initiated by the teacher. In each interpreting cycle, every intention is grounded on some prior interpretation that generates a particular interpretant of a written mark within a particular mathematical context. That is, each interpreting cycle is constituted through interpretation, intention, translation of interpretants into new signs (i.e., expression of interpretants), and the eventual emergence of some sort of a consensual mathematical meaning or cominterpretant. Cominterpretants evolve and eventually approximate mathematical concepts (commens).

Let us consider a cycle in which the teacher is the first acting participant. The teacher's acts of interpretation give rise to an intentional interpretant (T-i) which is anchored in her own effectual interpretant (T-I) of a mathe-

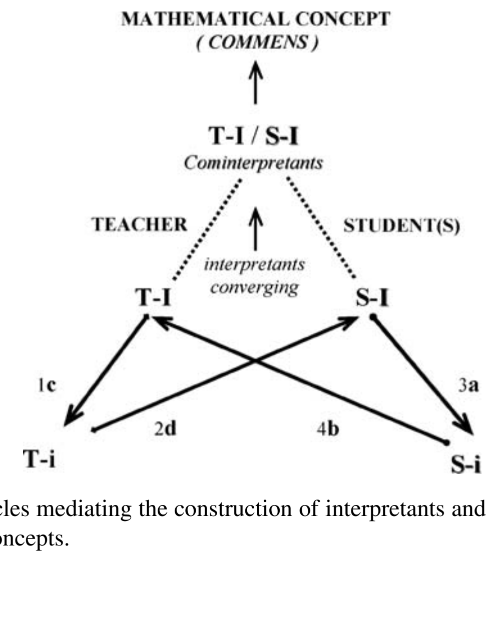

Figure 3. Sign cycles mediating the construction of interpretants and their approximation to mathematical concepts.

matical sign. The teacher expresses her intentional interpretant (T-i), which the student interprets in order to generate an effectual interpretant (S-I). In turn, the student's effectual interpretant (S-I) anchors an intentional interpretant (S-i) that is expressed by the student. Then, the teacher interprets the student's expressed intentional interpretant and generates an effectual interpretant (T-I′) that anchors an intentional interpretant (T-i′) that is again expressed by the teacher. This latter intentional interpretant is interpreted by the student who generates an effectual interpretant (S-I′) that anchors an intentional interpretant (S-i′) that is, again, expressed by the student. So, on the cycle continues while effectual and intentional interpretants (which may or may not generate logical interpretants) are constructed and expressed by teachers and students in their own processes of interpretation until some kind of cominterpretant or consensual meaning is generated and expressed. The relative closeness, or better, the relative openness of each cycle primes the stage for a new cycle that sustains the continuity of the meaning-making processes that approximate the mathematical concept (commens).

Getting students engaged in interpretation (i.e., interpreting games) is one of the major challenges confronting the teaching of mathematics. Interpreting games allow for the continuous and corrective construction of interpretants (intentional, effectual, or logical), the expression of those interpretants into further signs and the constitution of interim meanings that are further refined and eventually will converge to those objective meanings bestowed in mathematical symbols. The interpreting activity of the participants fosters generality and therefore fluency in the use of conventional mathematical signs in order for students to function mathematically.

8. Interpreting Games as Language Games

For Wittgenstein (1991) *language games* are mediating processes for meaning-making that impact how meanings are produced according to the particular circumstances in which words and sentences are used in sociocultural contexts. Language games go beyond the logic and structure of the language, they account for the semantic aspects of linguistic expressions (Harris, 1990). Likewise, *Interpreting games* are also mediating processes of meaning-making that have an effect on how interim mathematical meanings are produced according to the meaning-giving contexts used by the teacher and interpreted by the students. *Interpreting games* account for the semantic aspects of mathematical symbolizations.

In *language games* the grammar, syntax, semantics, and conventions of the language help to generate and to sustain categories of thought that constitute and organize thinking. Likewise, in *interpreting games* the grammar, syntax, semantics, and conventions of the plurality of semiotic systems used in the mathematics classroom help to generate those categories of thought that constitute mathematical thinking. Furthermore, the idiosyncratic signs generated by classroom participants contribute, in great measure, to the constitution of personal ways of thinking mathematically.

Both *language games* and *interpreting games* direct our attention towards the private-public dialectic that characterizes the metamorphic processes of meaning-making in language and in mathematics.

In *language games* some of the meanings produced in social interaction could become socially accepted over time and could constitute part of the language system. Others are interim meanings and will be modified as they approximate already established linguistic meanings. In classroom *interpreting games* all meanings produced in social interaction are transitory and subject to refinement and modification to approximate the already established and historically constituted mathematical concepts.

Language games take into account, primarily, the language system but other semiotic systems are not excluded (e.g., gestures). Likewise, *interpreting games* take into account, primarily, the notational mathematical system but other semiotic systems are not excluded (e.g., language and a variety of representational systems). Both games are rooted in social interaction (i.e., communication taken in its broadest sense). Also both games use semiotics systems constituted by "signs, rules of sign use and production, and underlying meanings; *all of these depend on social practices*, and *human beings as quintessentially sign using and meaning making creatures* can never be eliminated from the picture, even if for some purposes we *foreground signs and rules* and *background people and meanings*" (Ernest, this issue, emphasis added).

Language games are considered to be an explanatory model for the construction of linguistic meanings in different social practices as the participants come to describe the activity in which they participate. Likewise, *interpreting games* are considered to be an explanatory model for the construction of mathematical meanings in the teaching-learning practice as teacher and students come to express and modify their personal interpretations of mathematical concepts.

Language games do not mirror the structure of the language nor *do interpreting games* mirror the structure of mathematical concepts. They only mediate the private constitution of linguistic and mathematical conceptions that in the long run approximate standardized and historically constituted linguistic and mathematical concepts.

In general, *language games* can be considered as essential tools for communicating while *interpreting games* can be considered as essential tools for *teaching and learning*. Both games facilitate and accelerate sign interpreting processes and the achievement of less subjective (i.e., more objective) meanings.

9. Illustration: An interpreting game in a third-grade classroom

9.1. *Classroom setting and teaching method*

The third graders who participated in the episode to be analyzed were collaborating in a year-long teaching experiment. The school in which the teaching experiment took place was considered an at-risk school. The third-grade class consisted of five girls and nine boys, all of them children from families of factory workers.

The teaching experiment considered communication as a tool for teaching and learning. The teaching method used advocated the intellectual and collaborative interaction between teacher and students. This brought to the fore the complementarities between partnership and individuality, inter-subjectivity and intra-subjectivity, conventionality and idiosyncrasy, teaching and learning. Teacher and students working in collaboration created a classroom environment in which all felt active constituents in a community where listening, interpreting, explaining, and justifying were expected. The teacher genuinely valued everyone's presence and recognized that everyone could contribute and influence the thinking of the other classroom members.

This method of teaching put on the shoulders of the teacher a new set of expectations and obligations with herself and the students because she needed to focus on students' emergence and evolution of mathematical meanings and to conjecture continuously their acts of interpretation. In order to do so, the teacher learned to note and to differentiate her own personal actions from those of the students. She learned to hypothesize and to assess what the students interpreted as well as to project lines of questioning. Likewise, students learned to think on their own, to compare, to explain, and to validate their reasoning. This method required (a) for the students to be involved in constructing their mathematical understanding in the interaction with others; and (b) for the teachers to respond to students' interpretations and means of knowing as well as to teach in harmony with students' current understanding. In this way of teaching, the teacher was not only the one who taught but also the one who learned, and the students were not only the ones who learned but also the ones who taught. Teaching and learning were really complementary processes.

The teaching experiment consisted of daily teaching episodes five times a week throughout the school year. Each day, the teacher and the researcher engaged in conversations (in the form of symmetric dialogues) about the nature of the arithmetical interpretations of the students and the teacher, the purpose of the arithmetic tasks in accordance with the students' current understanding, and the mediating role of language (formal and informal) and mathematical notations (conventional or idiosyncratic) in the continuous unfolding of the interpreting process.

To analyze the semiosis of the classroom arithmetical activity, the lessons were videotaped and field notes were kept by the researcher on a daily basis. Task pages, students' scrap papers, and copies of overhead transparencies used by the students were also collected. All data were chronologically organized.

9.2. *About the teacher and her teaching*

The teacher who collaborated in the teaching experiment for two years was a veteran teacher with 21 years of classroom practice and set in her ways of teaching arithmetic to children in an instrumental way (in Skemp's sense). Prior to the first year of the teaching experiment, the collaborating teacher participated in a summer camp with other teachers. In this camp, the researcher used communication as a tool for teaching and learning so that teachers could experience first hand the influence of social interaction and interpretation in the teaching-learning process. The researcher also used, with the teachers, sequences of arithmetical tasks prepared for the teaching experiment with third and fourth graders and tapes of children solving the same activities posed to the teachers. Thus, the teachers found themselves solving novel arithmetical tasks, making their own interpretations, collaborating with each other, explaining and justifying their solutions as well as interpreting students' solutions. With this prior common methodological experience, the collaborating teacher and the researcher team-taught the teacher's fourth-grade arithmetic class during the first year of the teaching experiment. In this first year, the teacher was convinced of the creativity and ingenuity that children have with numbers when they are encouraged to think on their own, to make their own interpretations, and to explain them to others. During the second year, the teacher taught third graders and the researcher was only a participant observer. The episode to be analyzed comes from this second year of the teaching experiment.

9.3. *Analysis*

The task to be analyzed was posed to the students when the teaching experiment was in its fifth month of the second year. By this time, students' conceptualization of numbers in terms of units allowed them the flexibility to add numbers mentally using idiosyncratic strategies. The teacher proposed an open number sentence to the students, gave the students time to think, and then started a dialogue with them. This dialogue is analyzed here as an *interpreting game* between teacher and students. The dialogical interaction began when the students read the number sentence, interpreted it, and tried to find the number that could be placed in the blank space without altering the equality. In this interpreting process, the teacher inferred students' interpretants and their lines of reasoning from signs expressed by them in order to facilitate the construction of a new meaning for the equal sign. It appears that, in the process, students started to anticipate somewhat the commutativity property for addition.

The dialogue lasted 50 min and in the retrospective analysis it was subdivided into five interpreting cycles each accounting for the emergence of some type of meaning and the attainment of some kind of consensus whether it was agreement or disagreement. In the dialogue, T stands for teacher, S for an unidentified student, Ss for more than one unidentified student, and the abbreviations indicated by a capital letter followed by a vowel or a consonant for the name of a student (e.g., Da, Sh). These abbreviations are italicized in the body of the paper to avoid confusion. Furthermore, the mark. . . stands for the suspension of voice.

Cycle	#1	Students' initial interpretations of the equal sign
		Which number will make the number sentence true? $246 + 14 = \cdots + 246$.
1	T	Da, please read the question on the board.
2	Da	Which number will make the number sentence true?
3	T	All right Da. Now read the number sentence for me.
4	Da	Two-hundred forty-six plus fourteen equals. . ..
5	T	. . .. something. . ..
6	Da	Plus two-hundred forty-six.
7	T	Kr, what does that equal sign mean?
8	Kr	Equals. . . it equals something?
9	T	Sh, what does "equals something" mean?
10	Sh	It's. . . it's when you add something. The equal sign is there so you can put the answer by the equal sign.
11	T	So, are you telling me that on the other side of the equal sign you have to have the answer?
12	Sh	Well yeah, because the equal sign is like when you add something up and the equal is there so you can put the answer down.
13	T	Okay. Does anyone else have an explanation?
14	Ka	The equal sign is the sum. It's like if you add two-hundred forty-six plus fourteen the sum is two-hundred sixty.
15	T	Mmm hmm . . . . So, is that what the equal symbol means here?

Through questioning, the teacher interpreted students' constructed meaning of the equal sign. Lines 4-5-6 indicate the teacher's interpretation of Da's difficulty in reading an empty space in the context given. Instead, lines 7–14 show how the teacher guided the dialogue using the student's expressed logical interpretants (i.e., the equal sign as a command to write the answer down). The teacher's intentional questioning led her to hypothesize that the students were far away from seeing the equal sign in the given equality as indicating that the order of the addends was not significant in the result of the addition. In line 15 the teacher expressed the students' interpretation of the equal sign as a command to find "the answer" and the blank space as the place to "put the answer down". The teacher's last statement marked a temporary closure of this first cycle. The dialogical interaction in this cycle indicates that the students expressed a logical interpretant for the equal sign that was consistent with the meaning conveyed by this sign in arithmetical tasks they had encountered up to this moment. At this point in time, the current inferred cominterpretant for this sign was that the equal sign stood for a command to perform the indicated operation.

From the above dialogical interactions between teacher and students two questions come to mind. How will the teacher contribute to the modification of the students' interpretation of the equal sign? Will the students progressively influence each other and finally attain a consensus on the meaning of the equal sign in this context? The following cycle indicates that the students started to modify their interpretants of the equal sign and tried to give an incomplete example. The teacher picked it up and reconstructed it to exemplify an easier and obvious case of the equality given. Using this particular case the teacher engaged the students in the discussion again.

The teacher modified the incomplete example introduced by Da and put it into the context of the initial question (line 18 indicates the expression of T's intentional interpretant). Sh's negative answer and Da's answer in uncertain terms was argued by Ka with her own numerical "proof" (lines 23, 25, 31 indicate Ka's logical interpretant in the form of her numerical argument). Ka was able to see the validity of the equality $6 + 6 = 6 + 6$ keeping in mind the result of the addition on both sides of the equal sign, and without the need to see the result written down. Regardless of Ka's numerical argument (line 23), Sh continued to struggle with her own conception (line 28 indicates the expression of Sh's logical interpretant). For Sh repeating the numbers was not the same as performing the addition and writing the answer down. The teacher interpreted Sh's expressed interpretant and used the argument given by Ka to try to convince Sh of the truth of the equality $6+6 = 6+6$ (line 32). The teacher generated and expressed an intentional interpretant to involve Sh in the re-creation of Ka's numerical

Cycle	#2	Students start to modify their initial interpretations of the equal sign
16	T	Da, you want to say something, what is it?
17	Da	Umm, I think that the equal sign is asking you something like what is six plus six.
18	T	What if I say six plus six equals six plus six? Is this a true sentence?
19	Sh	No.
20	T	So, six plus six does not equal six plus six!
21	Da	Actually it does. It's kind of the same.
22	T	Kind of the same!
23	Ka	It does. I can prove it because that's how much it equals up to. Six plus six equals twelve and you could say that six plus six equals six plus six because they both equal the same amount.
24	T	[Teacher writes on the board] $6 + 6 = 6 + 6$.
25	Ka	And that equals the same thing.
26	Sh	I disagree.
27	T	Tell me why Sh.
28	Sh	Because equal doesn't mean you put six plus six again. You're supposed to add the numbers up and put the answer down. That's what equal means.
29	T	Okay, so are you saying that equals means you have to have an answer on the other side? So, six plus six does not equal six plus six.
30	Sh	Yeah.
31	Ka	Yes, it does because both sides equal the same amount.
32	T	Can I write $6 + 6 = 6 + 6$?
33	Ka	Yes.
34	Mi	Yes
35	Sh	No. Six plus six equals twelve.
36	T	Sh, six plus six is twelve [covering the left side of the equality]. What is this six plus six [covering the right side of the equality]?
37	Sh	Twelve.
38	T	[Teacher writes on the board] $\underset{12}{6 + 6} = \underset{12}{6 + 6}$ Are you telling me that twelve does not equal twelve?
39	Sh	Yes... no... I don't get it... That's equal... But how do you do the six plus six? Six plus six equals six plus six? You can't do that because six plus six equals twelve. If you write six plus six instead of 12 you're just repeating over six plus six again.

argument (lines 35–38). This argument could have been presented by the teacher in a unilateral manner but she decided to keep the dialogue symmetric. As a result, Sh came to doubt her own interpretation although it was not modified at that moment (line 39).

In summary, Sh's expressed logical interpretant of the equal sign (in the context introduced by Da and modified by the teacher to fit the task at hand) continued to be strongly grounded on her interpretation of the equal sign as a command to perform an operation. In contrast, Ka was able to construct and express a new logical interpretant of the equal sign in this new context (compare Ka's interpretants expressed in lines 14 and 23) to account for both the operation of addition and a numerical balance given that "both [sides] equal the same amount". Sh's doubt marked a breaking point in the interpreting game since some consensus was reached.

Will a student come up with an argument to convince Sh (and maybe other students) to modify her logical interpretant of the equal sign in order to determine the missing number that will make the number sentence true? The following cycle indicates that Ka was able to create a numerical argument by contradiction to try to convince Sh that the number in the blank space should be 14 instead of 260.

Cycle	#3	Ka's intuitive numerical argument by contradiction
40	T	What is the number that will make true the equality $246 + 14 = \cdots + 246$?
41	Ka	[Goes to the board] You could put the answer right here [Ka writes 260 on the blank space of the original equality] $246 + 14 = \underline{2}60 + 246$. Now, it would not be the same on both sides of the equal symbol because two-hundred sixty plus two-hundred forty-six is not the same as two-hundred forty-six plus fourteen. But if instead of 260 you write 14 then that would be the same thing.
42	T	So, is two-hundred forty-six plus fourteen equals two-hundred sixty plus two-hundred forty-six a true number sentence?
43	Ss	No. That's not a true statement.
44	T	Well, how can we make this [the equality written by Ka on the board] a true statement? [T erases the 260 that Ka wrote on the blank space].
45	Sh	By putting two-hundred forty-six again or fourteen either one.
46	T	Why?
47	Sh	Six plus six equals six plus six. Two-hundred forty-six plus fourteen equals two-hundred forty-six plus two-hundred forty-six.
48	Ka	But if you put 246 on the blank space, then $246 + 14 = 246 + 246$. If you put these two together (she refers to the numbers on the right side of the equality) then it's going to be four-hundred ninety-two.
49	Sh	You don't add them!
50	T	Yes you do; it says plus. The left side is two-hundred sixty; we know that. Ka says the right side is four-hundred ninety-two. Is this a true number sentence? $\underset{260}{246 + 14} = \underset{492}{246 + 246}$
51	Ke	Can I show you something?
52	T	Uhh huh.
53	Ke	[Ke goes to the board and erases the 246 in the blank space] All you're doing is to equal up to two-hundred sixty to make this one [referring to the right side of the equation] equal up to two-hundred sixty. Then, all you're doing is just putting 14 backwards [Ke writes 14 in the blank space] $246 + 14 = 14 + 246$
54	T	So, now you're trying to tell us that two-hundred forty-six plus fourteen is the same as fourteen plus two-hundred forty-six?
55	Ke	Yes.

The teacher, aware of the students' interpretations, intentionally turned the attention of the class to the initial question (line 40 indicates the expression of T's intentional interpretant). Ka created a numerical argument by contradiction (line 41 indicates the expression of Ka's new logical interpretant). Ka's new intepretant was original because this kind of numerical argument had not been used in this classroom prior to this occasion. It is important to note Ka's corrective sequence of interpretants when we contrast this new interpretant with her initial interpretants of the equal sign (compare lines 14 and 23 with line 41).

In line 42, the teacher used Ka's new numerical argument intentionally to pose questions to the students in order to sustain the dialogue. In this dialogic interaction, the teacher came to hypothesize that Sh had over generalized the number sentence $6 + 6 = 6 + 6$ (lines 45 and 47). However, it was Ka who dealt with Sh's overgeneralization. Ka, with her capacity to tinker with arguments by contradiction, took on Sh's interpretation of repeating one of the numbers and modified her argument by contradiction (line 48). This time Ka was more explicit in her explanation. In line 50, the teacher used Ka's argument with the intention of making the contradiction even more explicit. It is important to note that Sh made no immediate intervention; maybe she constructed an effectual interpretant that was not ready to be expressed. Ka's and Sh's logical interpretants are of a different kind but nonetheless logical within their own conceptual webs. This means that logical interpretants are relative to the conceptual webs of the interpreters although in all cases they are potentially open to modifications.

In line 53, Ke, a student who had not intervened in the discussion up to this point, came up with the idea of using 14 in the blank space to "equal up" the results of the additions on both sides of the equal sign. This, for him, was the same as writing the same addends on the right side of the equality but "backwards". Up to that moment, Ke and Ka appeared to be the only two students who explicitly came to see three things simultaneously: (a) the change in the order of the addends, (b) the actual addition, and (c) the quantitative balance between the two sides of the equal sign. Lines 54 and 55 marked a temporary breaking point in the interpreting game as the teacher verbally summarized Ke's argument and Ke agreed. The expressed and converging consensus of these two students (cominterpretant) marked a breaking point in the dialogical interaction.

In this cycle, we not only observe Ka's argument by contradiction (i.e., a new sign expressing her logical interpretant) but also her willingness to use it to counter Sh's expressed logical interpretant. In addition, Ke also expressed his own logical interpretant. It is also apparent that the sophistication of Ka's logical argument by contradiction and Ke's comprehensive interpretation of the equal sign would not have been possible without the interpretations (valid or invalid) of the other students. That this was the case was indicated by the lack of this type of argument on the part of the students when they first interpreted the equality. In other words, we can say that a progressive sequence of interpretants and the translations of them into new signs were possible because of the intentional dialogical interaction between teacher and students which led the students to the modification of their prior interpretants of the equal sign.

Two questions still beg to be answered. One question is whether or not Sh will be able to modify her logical interpretant of the equal sign as a command to perform an operation in so far as to include the preservation of a quantitative balance when the order of the addends is changed. The other question is whether or not there are other students actively making their own interpretations and constructing their own interpretants although they had not openly expressed them up to this point. In the following cycle the answers to these two questions appear to be in the positive.

One could have assumed that the interpreting game had ended in line 55 given that the answer to the question posed by the teacher was attained. However, the teacher continued the dialogue because other students were willing to participate now and had not participated before. Me, for example, not only agreed with Ke's conclusion but she also made her own hypothesis about the interpretations of other students (lines 57 and 61). Sh, on the other hand, wanted to participate again and what a rewarding surprise it was. Sh modified her initial logical interpretant and now she started to assume that the number in the blank space was 260 and recreated Ka's argument by contradiction. Finally, Sh concluded that the number in the blank space should be 14 (lines 64 and 66 are the expressions of Sh's modified logical interpretants) in order for the number sentence to be true. Sh even went a step further and created a chain of equalities. This chain indicates that Sh had come around to modify her first interpretant. Sh added the numbers to be consistent with her initial interpretant of the equal sign as a command to "find the answer and write it down" but now she also used the equal sign to symbolize a quantitative balance. Sh's logical argument indicates a consensus on the part of the students about the number that should be in the blank space in order for the number sentence to be true. That is, the students have constructed and expressed a consensus or cominterpretant. It is unclear what effect Me's intervention could have had in Sh's thinking. It could have been that Me's translation of her own interpretant into a verbal argument was all that was needed at this point for Sh to modify her interpretant.

The above cycle indicates that the students, through their interpretation efforts, were able to modify their initial logical interpretants of the equal sign as a *token* for a command to perform an operation and to use it as a

Cycle	#4	Me intervenes and Sh recreates Ka's numerical argument by contradiction
56	T	[The teacher sees Me raising her hand] Let's see what Me has to say.
57	Me	I think this. Two-hundred forty-six plus fourteen equals fourteen plus two-hundred forty-six. So, I say the same as Ke. It is fourteen.
58	T	Why do you think it is fourteen? You are the third person who says that. Three people said fourteen and two people said two-hundred sixty.
59	Me	Well other people think that it's two-hundred sixty. Umm. . . . I don't mean to disagree but I disagree.
60	T	Why? Why do you disagree?
61	Me	Well, what I think you guys are thinking is that when you guys put these two together [Me is referring to the numbers on the left side of the equal symbol] it's two-hundred sixty; so you guys think you put two-hundred sixty right here [referring to the blank space on the right side of the equal symbol] and then two-hundred sixty plus two-hundred forty-six will be two-hundred sixty. That's what I think some of you guys are thinking. But I think that fourteen should be in the blank space.
62	Sh	May I say something?
63	T	Huh uhh.
64	Sh	[Sh goes to the board] It's like this. Two-hundred forty-six plus fourteen is two-hundred sixty. If we put two-hundred sixty right here [Sh is referring to the blank space] then we have to plus two-hundred-sixty and two-hundred forty-six and that would be five-hundred six. Like this $\underbrace{246 + 14}_{260} = \underbrace{260 + 246}_{506}$.
65	T	So do you think this is a true statement? Will you put two-hundred sixty in the blank space?
66	Sh	I don't agree with that. It's kind of like [Sh erases 260 and replaces it with 14]. It's kind of like the equal sign is down here and you put it right here. It's kind of like you're just separating this $\underbrace{246 + 14}_{260} = \underbrace{14 + 246}_{260} = 260$.
67	T	All right. She has an interesting idea because she wants to see what she considers to be the answer.
68	Sh	Yeah, I like to see the answer.

token for a quantitative equivalence. It also appears that the students went away with a sense that the addends could be interchanged without affecting such equivalence. Such a sense for the commutative property of addition was only emerging. To consolidate students' latest logical interpretants for the equal sign, new arithmetic tasks were generated so that students could be able to make sense of this property in the context of addition and also in the context of multiplication. I will touch again on this issue in the conclusion.

The following cycle, the last in this interpreting game, came into being as a result of the students' initiative to reflect on their initial interpretation of the equal sign. It was surprising to see Sh (the student who struggled the most) initiating a cycle to reflect on students' diverse interpretations.

Cycle	#5	Students reflect on their interpretations of the equal sign
69	Sh	Why were we stuck with the answer?
70	T	It's the way you almost always see it in your math book, isn't it?
71	Ka	Yeah.
72	T	You almost always see it where there's something you have to do on this side [left side] and on this side [right side] there's only one space for one number.
73	Sh	Why do they make it like that?. . .. That's weird. Because they make you think like that and you. . . stick with it.
74	T	That's very nice Sh because it gets in the kids brains and you stick with it and stays there always the same. But now we are leaning that with our numbers we can do lots with them.
75	S	Oh, out of those books!
76	T	Out of the books yeah. Out of the books are things that you've seen ever since you started school, practically.
77	Sh	Cause that's wrong.
78	Ss	That's not wrong.
79	T	It's not wrong but it's only one way.
80	Sh	Yeah there's lots of ways to do that. You can do that a thousand ways.
81	T	Well, that was a struggle today.

Sh started to question the cause of her interpretation and the teacher, after a swift analysis of the event, tried to explain the cause of the students' first interpretation of the equal sign. The teacher considered that math books (i.e., elementary arithmetic textbooks) use the equal sign to signify the performance of operations and therefore students come to interpret the sign in that sense. One of the students, Ka, approves the teacher explanation although Sh considers that this causes, in a sense, cognitive obstacles that students have to overcome (line 73). Sh's idea was made more explicit by the teacher in line 74 where she paraphrased (i.e., translated into a new sign) Sh's idea. Sh also considered that using the equal sign in only one way was "wrong" although other students disagreed with her judgment. It appears that in this argumentation among the students they were referring to two different meaning of "wrong". Sh seemed to be referring to "wrong" in some kind of moral or ethical sense while the other students seemed to be referring to "wrong" in a logical sense. The teacher appeared to combine the two senses of "wrong" in her intervention (line 79) and her statement prompted Sh to make a broader statement about her new ways of looking at numbers and operations with them. Sh's statement appears to be based not only on this episode but on the way she and the other students had started to conceptualize number as a result of participating in the teaching experiment.

9.4. *Conclusion*

It is important to note that the above episode occurred in a classroom where *communication* was taken as a *tool for teaching and learning* and where teacher and students had built up an environment in which they were equally committed to inquiry and interpretation. These interpretations shaped and were shaped by the classroom communication. The teacher, after a year-and-a-half of collaborating in the teaching experiment, felt comfortable engaging students in symmetric dialogue and making interpretations of students' expressed interpretants to support and sustain their evolving construction of arithmetical meanings. The teacher also understood that students' interpretations of mathematical meanings were subjective and transitory but in the process of becoming more refined and objective. Students, on the other hand, felt comfortable making their own interpretations, generating their own interpretants, translating these interpretants into new signs, inferring the interpretants of others when they were expressed, and generating consensus through the construction of cominterpretants.

The number sentence posed by the teacher triggered an *interpreting game* that mediated the construction of interpretants under her intended yet indirect guidance. The symmetric dialogue analyzed here indicates that teacher and students contributed to the creation of a space for intervention and a progressive construction of interpretants (intentional, effectual, or logical) that yielded a cominterpretant about the number that would make the number sentence true. The semiosis of this interpreting game had the dynamic of a ping-pong ball between teacher and students and the students themselves rooted in their own *acts of interpretation* (i.e., generation of interpretants and the translation into new signs).

It would have been impossible for the participants in the interpreting game to be consciously aware of the cycles and breaking points in the construction of meaning while participating in it. To have done this would have stopped their acts of interpretation in the same way that cyclists or swimmers would have stopped their motions if they were to reflect on the logic of their movements rather than to concentrate on their actual performance. However, in a retrospective analysis it is possible to differentiate such cycles and to observe the interpreting acts of the teacher and the students and their effects.

The analysis of this teaching episode as an interpreting game allows us to "see" the interdependence between thought and communication in *the acts of interpretation* of the students and the teacher. The most active participants in the dialogue were Kr, Da, Sh, Ka, Ke, Me, and T; the other students made only monosyllabic interventions. Those students who expressed their interpretants in the form of arguments (whether valid or invalid) influenced the construction and modification of the interpretants of other students. For example, it could be inferred that Sh's interpretants were influenced by those of Ka, Ke, and probably Me. At the same time Sh's interpretants appear to have influenced those of Ka, Ke, and even Me who appeared to have taken on the challenge of shaping Sh's acts of interpretation of the equal sign in the given number sentence.

From this teaching episode it was clear to the teacher and researcher that the students were ready to continue their experimentation with equalities and inequalities as well as with the commutativity of addition. Subsequent teaching episodes were based on our interpretations of students' expressed interpretants of the equal sign and their emerging understanding of the commutativity property of addition. Based on what we learned from students' expressed interpretants, arithmetical tasks were generated to provide students with meaning-given contexts to foster a quantitative sense for a variety of "equivalences" as well as to contrast quantitative expressions that were "different" (see Table I). Verbal and written expressions like "equal", "equals", "different", "less than", "smaller than", "greater than", and "bigger than" were generously used by teacher and students before mathematical signs were introduced. In addition, students were always expected to give verbal explanations of their interpretations and ways of reasoning. In this way, students had the opportunity to express their interpretants and to transform them into new signs (see Figure 1). Tasks like the ones in Table I provided for the continuity of the teaching episode analyzed here as well as for the continuity of the construction of students' interpretants to endow the written marks "$=$", "$\neq$", "$<$", "$>$" and their corresponding linguistic expressions with mathematical meanings.

This episode and subsequent teaching episodes contributed to the construction of students' notions of equality and inequality and, in the process, opportunities were also provided for constructing the commutativity property for addition and multiplication.

10. Final remarks

Classroom interpreting games constitute an explanatory model, based on the semiotics of Charles Sanders Peirce, to analyze classroom communication (taken in its broadest sense) as a tool for teaching and learning. Interpreting games draw together the social and the individual dimensions of communication and mathematical activity as well as the public and private dimensions of meaning-making processes. That is, interpreting games bring to the fore processes of collaborative construction of mathematical meanings.

TABLE I
Sample of arithmetical tasks solved by the students in subsequent teaching episodes

(A) Choose a number that will make both sides *equal*	(B) Is the left side *equal* or *different* from the right side? Why? Explain it to your friends
9 + 6 ··· ··· + 15	54 + 6 ··· ··· 83
9 + 6 ··· 14 + ···	99 − 11 ··· ··· 88
9 + 6 ··· ··· + 13	75 − 6 ··· ··· 49
9 + 6 ··· ··· + 9	134 + 10 ··· ··· 120 + 24
38 + 3 ··· ··· + 41	221 + 14 ··· ··· 250 − 15
38 + 3 ··· 20 + ···	200 − 145 ··· ··· 200 − 120
38 + 3 ··· ··· + 11	10 ÷ 2 ··· ··· 55÷11
38 + 3 ··· ··· + 38	15 × 4 ··· ··· 5 × 9
25 + 4 ··· ··· + 29	5 × 5 ··· ··· 125 ÷ 5
25 + 4 ··· 22 + ···	25 × 2 ··· ··· 5 × 10
25 + 4 ··· ··· + 9	15 × 3 ··· ··· 5 × 9
25 + 4 ··· ··· + 25	25 × 5 ··· ··· 5 × 25
You have written number sentences. There are three number sentences that are similar in some special way. These number sentences are	15 × 3 ··· ··· 3 × 15 45 × 1 ··· 1 × 45 6 × 4 ··· 8 × 3 12 × 2 ··· 24 × 1
...	
...	
Could you tell why these number sentences are similar? Explain your reasoning to your friends	8 × 3 ··· 3 × 8 6 × 4 ··· 4 × 6 12 × 2 ··· 2 × 12
(C) Is the left side *greater than* or *smaller than* the right side? Explain your reasoning to your friends	(D) *Choose the operation* that makes both sides *equal*. Explain your reasoning to your friends
213 − 145 ··· ··· 83	93 ( ) 25 = 68
98 + 105 ··· ··· 183	19 ( ) 13 = 32
43 − 33 ··· ··· 85	51 ( ) 33 = 18
39 − 12 ··· ··· 30 +20	49 ( ) 1 = 7 × 7
86 + 24 ··· ··· 120 − 15	54 ( ) 9 = 9 × 5
148 − 28 ··· ··· 48 + 28	8 ( ) 4 = 20 − 18
5 × 4 ··· ··· 140 ÷ 20	23 ( ) 46 = 75 ( ) 6
48 ÷ 2 ··· ··· 6 × 5	41 ( ) 12 = 25 ( ) 4
8 × 25 ··· ··· 400 ÷ 4	103 ( ) 28 = 50 ( ) 25

For interpreting games to become teaching-learning tools they should also become inquiring tools. For this to happen, it is necessary to build up classroom environments in which a communicative relationship between teacher and students naturally emerges with immediate and mediated intellectual results. That is, teacher and students should be equally committed to the establishment of a communicative relationship in which sign use, sign interpretation, and inquiry become a continuous state of affairs.

To use interpreting games as a teaching-learning tool the teacher should consider seriously her teaching activity as constituted in two parts – the pre-teaching and the actual teaching activity. The pre-teaching activity should be based on an asymmetric relationship with the students as the teacher reflects on her own understanding of mathematical concepts and prepares mathematical tasks and teaching strategies to engage students' inquisitive capacity. In the actual teaching activity, the teacher (for the most part) should keep a symmetric dialogue with the students without forgetting her directive role but without transforming students' thinking into the shadows of her own thinking (cf. Freire, 1970/2001). Therefore, it is necessary that the teacher learn to differentiate her interpretants from those she infers from the students' expressions of theirs in order to support and sustain the continuous transformation of students' interpretants into more refined signs.

Successful interpreting games foster sign use and sign interpretation and therefore the transformation of written marks into more developed signs endowed with less subjective (i.e., more objective) mathematical meanings. Thus, *interpreting games* explain the possibility for students to identify meanings constructed in collaboration with others as "*theirs*" and therefore take those meanings as *shared* and *shared* in a communicative fashion (cf. Cobb and Bauersfeld, 1995). In these games, teacher and students constitute themselves as intentional and interpreting subjects, and teaching and learning become complementary activities in which teachers teach-learn while students learn-teach and in doing so, teacher and students are empowered in the process.

When the construction of interpretants and the translation of interpretants into new signs occur in communication (i.e., social interaction), the *self* and the *other* bring to the fore higher levels of interpretation, construction, and translation. This is due to the fact that in an *interpreting act* two concomitant interpretants appear: the interpretant of the *self* who intends and expresses, and the interpretant of the *other* who interprets and constructs. To communicate, the *self* and the *other* have to transform their interpretants into new signs standing for some sort of common meaning. These interpretants (transformed into new signs) enter an animated interaction bringing with them the public and private aspects of meaning-making processes. Through converging interpretants, the personal and transitory construction of mathematical meanings tends to be generalized and encapsulated in cominterpretants that converge toward the conventional mathematical meanings of mathematical concepts or commens. Thus, interpreting games mediate the constitution of personal mathematical meanings and the transformations of these meanings into more objective ones. Interpreting games provide a lens for understanding the constant dialectic relationship between thinking, communicating, and meaning-making processes.

Usually the effects of interpreting games are not necessarily felt in one teaching episode like the one analyzed here. More often than not, the continuous nature of their effects is felt over time. When interpreting games happen over time, one can "see" the emergence and constitution of children's metaphors and their effect on the constitution of their mathematical thinking (Sáenz-Ludlow, 2004) as well as the emergence of chains of signification in children's elaboration of arithmetical concepts (Sáenz-Ludlow, 2003b).

Because of the immaterial nature of mathematical concepts, being aware of and engaging students in their own processes of sign use and sign interpretation is a necessary but challenging process in the teaching and learning of mathematics. Becoming aware of the continuous and evolutionary constructions of students' interpretants and personal mathematical meanings as well as the constitution of cominterpretants that asymptotically converge to the established mathematical concepts is to understand the thought-communication interdependence and the teaching-learning process as a collaborative social and conceptual experience. This understanding could encourage teachers (like the one in the teaching episode) to engage students in interpreting games as much as possible. Because the curriculum in elementary schools tends to be less intense and children tend to be more open to dialogue due to their age, classrooms environments in which interpreting games flourish tend to be easier to establish in these schools. It might also be possible to establish this type of classroom environments in secondary schools but more research is needed in this direction.

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